CS 497: Computational Finance Assignment

Fall 2007

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Due Thursday, November 8

Hand in Assignment by 4:30pm (3rd floor assignment box in MC).

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Assignment Questions

This assignment concerns the no-arbitrage lattice algorithm as described in the slides and the *Agonizing Pain* notes on my web site.

`www.scicom.uwaterloo.ca/~paforsyt/agon.pdf`

These notes are 80 pages. Probably you only want to print out the sections which are specific to Lattice methods. The relevant sections of the notes are Sections: 1, 2.1, 2.2, 2.3, 2.4, 5. You should read the slides first.

Given an option which expires at $t = T$, written on an underlying asset with price $S(t)$, with payoff function $PAYOFF(S(T)))$, then recall that the basic binomial tree algorithm is

- Choose $\Delta t = T/N$. Construct tree of prices

$$\begin{aligned} S_j^n &= S_0^0 e^{(2j-n)\sigma\sqrt{\Delta t}} \\ & \quad n = 0, ..., N \\ & \qquad j = 0, .., n \end{aligned}$$

- Determine risk neutral probability

$$p^* = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

- Initialize payoff

For $j = 0, ..., N$
  $V_j^N = \text{Payoff}(S_j^N)$
EndFor

- Backward recursion

For $n = N - 1, ..., 0$
  For $j = 0, ..., n$
    $V_j^n = e^{-r\Delta t}(p^* V_{j+1}^{n+1} + (1 - p^*) V_j^{n+1})$
    If (AMERICAN) Then
      $V_j^n := \max(V_j^n, \text{Payoff}(S_j^n))$
    Endif
  EndFor
EndFor

The payoff function for a put is

$$\begin{aligned} \text{Payoff}(S) &= \max(K - S, 0) \\ K &= \text{ strike} \end{aligned}$$

1. (10 marks) (Binomial Lattice)

Develop code for pricing European and American puts/calls using a Binomial lattice in either C or MATLAB. Be sure you are using double precision arithmetic. If $N = T/\Delta t$, then your code should require $O(N)$ storage NOT $O(N^2)$. Note that there is a Matlab function *binprice* for binomial tree pricing, but it is only for American options, and it stores the entire tree.

Test your code for European options by using the data in Table 1.

Table 1: European Test Case. Exact solution correct to seven figures 14.45191.

σ	.80
r	.10
Time to expiry	.25 years
Strike Price	\$100
Initial asset price S_0^0	\$100
Option Type	European Put

Show tables with the option value ($V(S_0, t = 0)$) as a function of Δt. Start off with a timestep $(\Delta t)^0 = .05$, and show the option value for $(\Delta t)^0/2, (\Delta t)^0/4, \ldots$. You should see your results converging to the *exact* value. Your tables should look like Table 2.

Table 2: Convergence Test

Δt	Value	Change	Ratio
.05	$(V_0^0)_1$		
.025	$(V_0^0)_2$	$(V_0^0)_2 - (V_0^0)_1$	
.0125	$(V_0^0)_3$	$(V_0^0)_3 - (V_0^0)_2$	$\frac{(V_0^0)_2-(V_0^0)_1}{(V_0^0)_3-(V_0^0)_2}$
...	...	...	...

If V^{exact} is the exact price (from the solution to the Black-Scholes PDE), and V^{tree} is the price from a lattice pricer, then it can be shown that

$$V^{tree}(S,t) = V^{exact}(S,t) + C\Delta t + O((\Delta t)^{3/2}) \quad (1)$$

$$\Delta t \to 0 \quad (2)$$

where C is a constant independent of Δt. Does your convergence table agree with theory, in terms of rate of convergence?

Next, repeat the above tests for American puts (no analytic solution available). Show a convergence table as in Table 2. Use the data in Table 3.

Submit your code listings, tables, and explain what you see.

Table 3: American Test Case

σ	.30
r	.05
Time to expiry	.25 years
Strike Price	\$100
Initial asset price S_0^0	\$100
Option Type	American Put

Something to think about if you have time (do not hand in)

Given the no-arbitrage lattice as described in the course notes, note that

$$
\begin{aligned}
S_N^N &\rightarrow \infty \\
&\Delta t \rightarrow 0
\end{aligned}
$$

This means that for a European call,

$$
\begin{aligned}
V_N^N &\rightarrow \infty \\
&\Delta t \rightarrow 0 \ .
\end{aligned}
$$

It is perhaps not immediately obvious that the binomial lattice algorithm will generate a value for V_0^0 that is bounded as $\Delta t \rightarrow 0$. Prove that the binomial lattice method for a European Call option is stable, i.e.

$$
|V_0^0| \leq C \quad (3)
$$

$$
\Delta t \rightarrow 0 \quad (4)
$$

where C is a constant independent of Δt.

Hint: Assume that Δt is sufficently small so that $0 \leq p^* \leq 1$. First, use induction to show that

$$
V_j^n \geq 0 \ ; \ \forall j, n
$$

Then develop a recursive equation for

$$
E_j^n = S_j^n - V_j^n
$$

and then use induction again to show that

$$
E_j^n \geq 0 \ ; \ \forall j, n
$$