An Introduction to Computational Finance

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Some Facts?

- Populations in the developed world are living longer

→ More people will need to live on invested capital in their retirement years.

- Conventional wisdom

→ Stocks are good investments over the long term

- Recent study (Financial Analysts Journal, 2004)

→ An investor holding a diversified equity portfolio has a 14% chance of a negative real return over a twenty year period

Financial Insurance

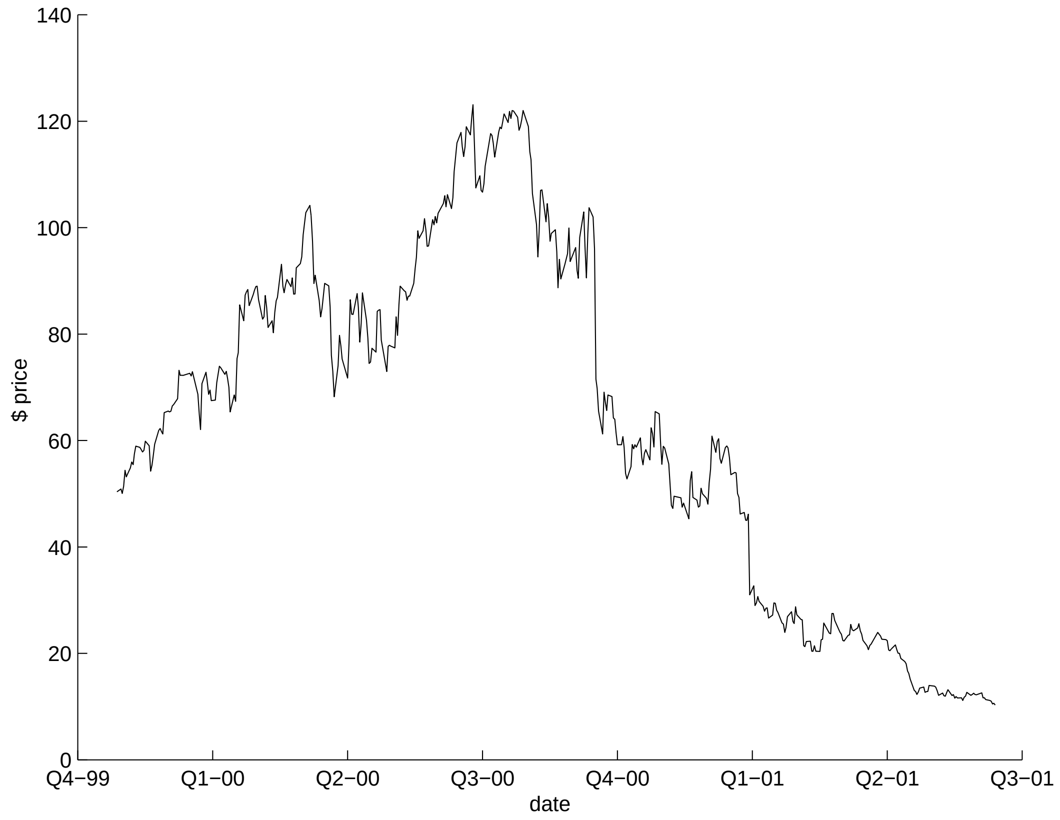
- Derivative securities (options, futures, forwards) are tools which can be used to manage risk.
- Any investment which includes some kind of protection contains an *embedded option*.
- Derivative securities are used by financial institutions to hedge risk such as
  - Currency fluctuations
  - Uncertain energy costs
  - Changes in interest rates

Individual Investors?

- Individual investors are often unaware that they buy/sell options
- Contracts with embedded options
  - Mortgage prepayment privileges
  - Fixed rate natural gas home heating contracts
  - Equity linked GICs
- As well, many pensions plans use derivative securities in their investment portfolio

Do we need financial insurance?

Nortel Share Price



Example: A call option

- Suppose I have decided that I want to buy IBM stock for $\$K$ in 1 month's time
- But the price of IBM stock will fluctuate during the next month
- I don't want to pay more than $\$K$ for the stock
- But if the stock falls below $\$K$, then I will be happy to pay less than $\$K$ to own IBM
- Can I take out some sort of insurance to ensure that I will have to pay at most $\$K$ for the stock?

A call option

- A *call* option gives me the right, but *not* the obligation to purchase the stock for a specified price (the strike price K) at some time in the future (the expiry date T).

$$S = \text{ value of stock}$$

- By purchasing a call option, at strike K, expiry 1 month
  - If $S > K$, I exercise the option, and buy the IBM stock for $\$K$.
  - If $S < \$K$, I let the option expire, and buy the IBM stock on the open market

The option value

- In one month, we know for sure what the option is worth
- If $S > K$, I can buy the stock for K and immediately sell it for S
- If $S < K$, I will not exercise the option (why pay K for something worth S?)
- More mathematically ($T =$ one month)

$$\text{Value of call option} = V(T = 1 \text{ month}) = \max(S - K, 0)$$

- What is a fair price for this option today?

A put option

Our previous example was for a *call option*

- Protection against rising prices
- Protection against falling prices can be obtained using a *put option*
- A put option is the right but not the obligation to sell an asset for the strike K at time T.
- More mathematically $(t = T)$

$$\text{Value of put option} = V(t = T) = \max(K - S, 0)$$

Some Jargon

The value of the option at maturity is also called the *payoff*

$$\begin{aligned}\text{Payoff of call} &= \max(S-K,0)\\ \text{Payoff of put} &= \max(K-S,0)\end{aligned}$$

A *European Option* can only be exercised at maturity T.

An *American Option* can be exercised at any time in $[0,T]$.
$\hookrightarrow$ The holder can decide to receive the payoff at any time.
$\hookrightarrow$ Most options traded on exchanges are American style.

We will consider a European call option in this example.

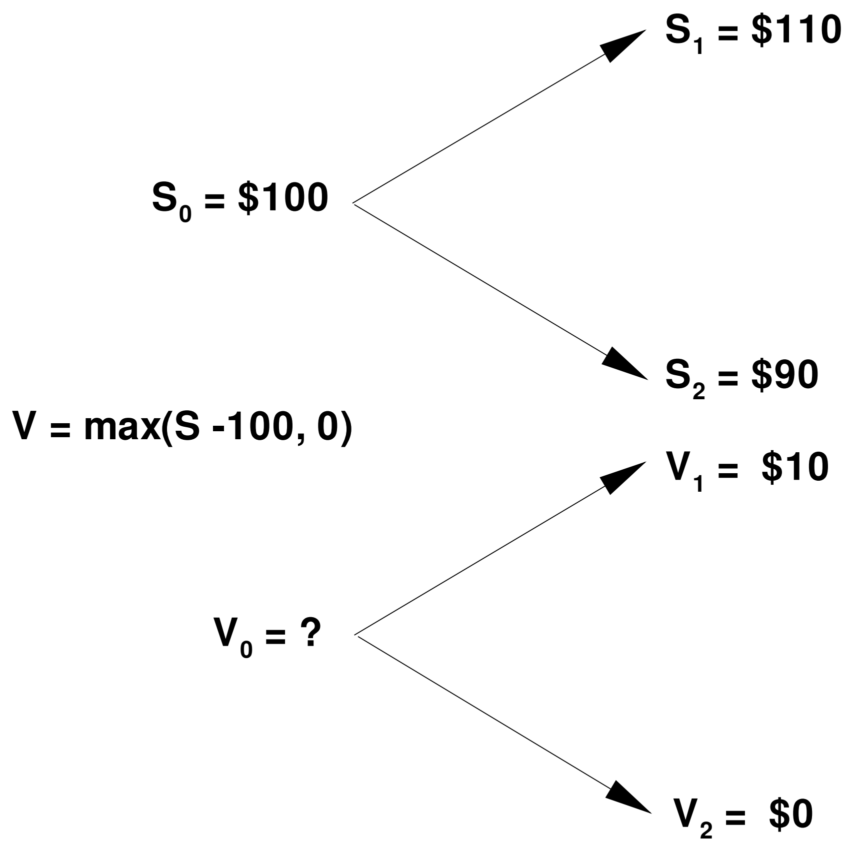
A simple model: call option

- Suppose the strike price is \$100, and the stock is trading today at $S_0 = \$100$.
- Let's assume a very simple model: in one month's time, the stock price can have only two possible values

$$S_0 \;\; \longrightarrow \;\; S_1 = 110$$
$$S_0 \;\; \longrightarrow \;\; S_2 = 90$$

A two state tree

What is the value of the option today? V_0

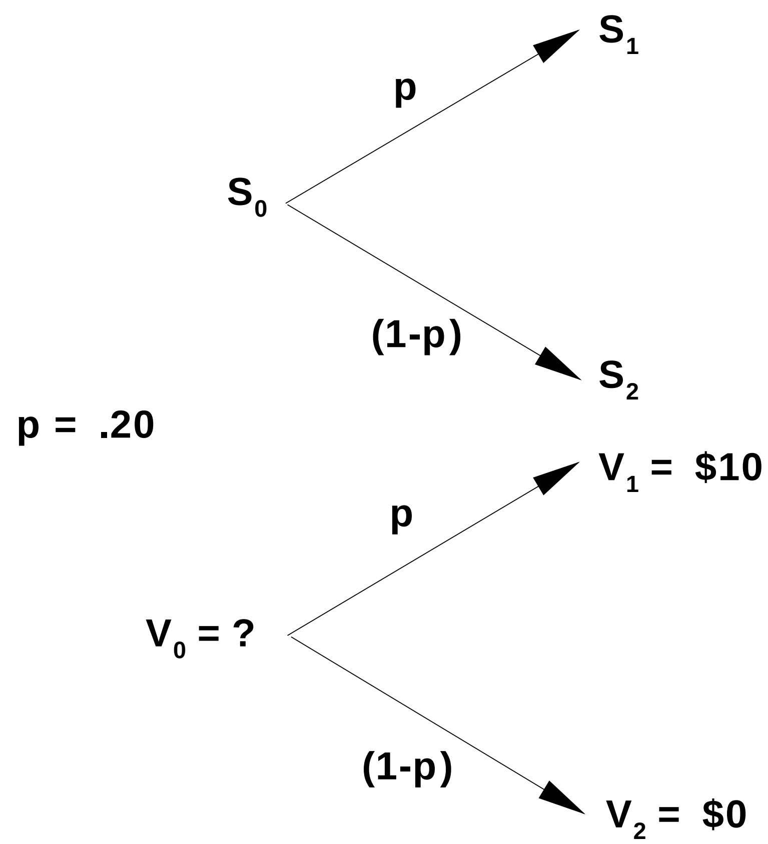


Some extra information

Probability

$$S_0 \longrightarrow S_1 \;=\; p$$

$$S_0 \longrightarrow S_2 \;=\; (1-p)$$



Option value?

- At this point, most people would value the option today as the discounted, expected value of the payoff

$$V_0 = e^{-r\Delta t}(pV_1 + (1-p)V_2)$$

$$r = \text{ interest rate}$$

$$\Delta t = \text{ One month}$$

- To keep things simple here, let's ignore the discounting effects ($r\Delta t \simeq 0$)

Option value

- This gives (in our example: $p = .2$)

$$\begin{aligned} V_0 &= .2 \times \$10 + .8 \times \$0 \\ &= \$2 \end{aligned}$$

- Suppose I offer to buy this option from you for $3, will you accept my offer?
- On each option, you will make an expected profit of (price - expected payout $= 3 - 2$) $1

An arbitrage opportunity

- I will be very happy to buy the option from you for $\$3$, since I will immediately exploit the *arbitrage opportunity*
- I can devise a trading strategy, such that I will make a profit of \$2, *regardless of whether* $S_0 \to 110$, or $S_0 \to 90$.
- How do I do this? Sounds like magic.

The Hedging Portfolio

- To exploit this arbitrage opportunity, I will construct a portfolio Π which is long the option, and short α shares

$$\Pi = V - \alpha S$$

  - A short position means I have borrowed the security, sold it, but have to give it back at some future time.
- I will choose α so that there is no uncertainty about the value of the portfolio at the expiry time of the option

The Hedging Portfolio

- The possible portfolio values are (in 1 month)

$$\begin{aligned} \Pi_1 &= V_1 - \alpha S_1 \\ \Pi_2 &= V_2 - \alpha S_2 \end{aligned}$$

- Setting $\Pi_1 = \Pi_2$, and solving for α

$$\begin{aligned} \alpha &= \frac{V_1 - V_2}{S_1 - S_2} \\ &= \frac{10 - 0}{110 - 90} = \frac{1}{2} \end{aligned}$$

The Hedging Portfolio

- Today, I buy the option from you for \$3.

  $\longrightarrow$ I have to pay \$3.

- I borrow $1/2$ share of the stock, and sell it ($S_0 = \$100$). I will have to return this in one month's time.

  $\longrightarrow$ This gives me \$50 in cash.

- The total value of my cash is C_0

$$C_0 = \$50 - \$3 = \$47$$

$Case : S_0 \to S_1 = 110$

Now, in one month's time, suppose

- $S_0 \to S_1$, which means that

$$\begin{aligned} V_1 &= \$10 \\ -\alpha S_1 &= -\frac{1}{2}(110) = -\$55 \end{aligned}$$

- (I have to buy $1/2$ shares at $\$110$, and return to broker)
- So the value of the portfolio is $\Pi_1 = V_1 - \alpha S_1 = -\$45$.
- But I have $\$47$ in cash, so I gain $2.

$Case : S_0 \longrightarrow S_2 = 90$

Similarly

- $S_0 \longrightarrow S_2$, which means that

$$\begin{aligned} V_2 &= \$0 \\ -\alpha S_2 &= -\frac{1}{2}(90) = -\$45 \end{aligned}$$

- So the value of the portfolio is $\Pi_2 = V_2 - \alpha S_2 = -\$45$.
- But I have $\$47$ in cash, so I gain \$2.

Arbitrage opportunity

- So, no matter what happens to the stock (it can go up or down, *I don't care*), I make a riskless profit of $2.
- What would happen in reality?
  - Arbitrageurs would buy up as many options as possible
  - This would drive up the price of the option, until the option price was the *no-arbitrage* price.
- The observed market price should be the no-arbitrage price, **not** the expected payoff

No-Arbitrage Price

- In our example, the no-arbitrage price is given by solving $V_0 - \alpha S_0 = \Pi_1$ for V_0, giving $V_0 = \$5$.
- Note that we do not care what the probabilities of an up or down movement in the stock price are!
- A bank can sell me the call option for \$5($+$ some profit), and construct the hedging portfolio.
- At expiry, the portfolio can be liquidated to give exactly enough to pay off the option *regardless of whether* $S_0 \rightarrow S_1, S_2$ (plus a locked in profit)

More Realistic Models

- Assume more complex *stochastic models* for stock prices
- But use same basic *no-arbitrage* idea
- Black-Scholes differential equation for no-arbitrage price (Scholes-Merton, Nobel Prize in Economics, 1997)
  - Can be solved using numerical algorithms for no-arbitrage price
- Note that we do not care about precise path taken by stock (we can't predict it)
- Only gross statistical properties (volatility)

Conclusions

- Many financial products contain embedded options
- These options are *financial insurance*, which are used to minimize risk
- Even though stock prices are unpredictable
- We can determine the no-arbitrage price of an option
- We can construct a hedging strategy to payout option (insurance) no matter what happens to the stock price!
- Modern finance is now a very technical discipline (Mathematics, Statistics, Computer Science)

More Reading

- Peter Bernstein, *Capital Ideas: the improbable origins of modern Wall street*
- Burton Malkeil, *A random walk down Wall Street*
- N. Taleb, *Fooled by Randomness*
- N. Taleb, *The Black Swan*
- *An Introduction to Computational Finance without Agonizing Pain* (wwww.scicom.uwaterloo.ca/ paforsyt/agon.pdf)

A Model for Stock Prices

- Observation: For every quoted price we see in the stock market: there is one buyer for every seller
- In each transaction
  - Buyer thinks price is going up
  - Seller thinks price is going down
- Conclusion: Stock prices follow a random walk (verified by statistical tests)
  - No observable patterns in prices

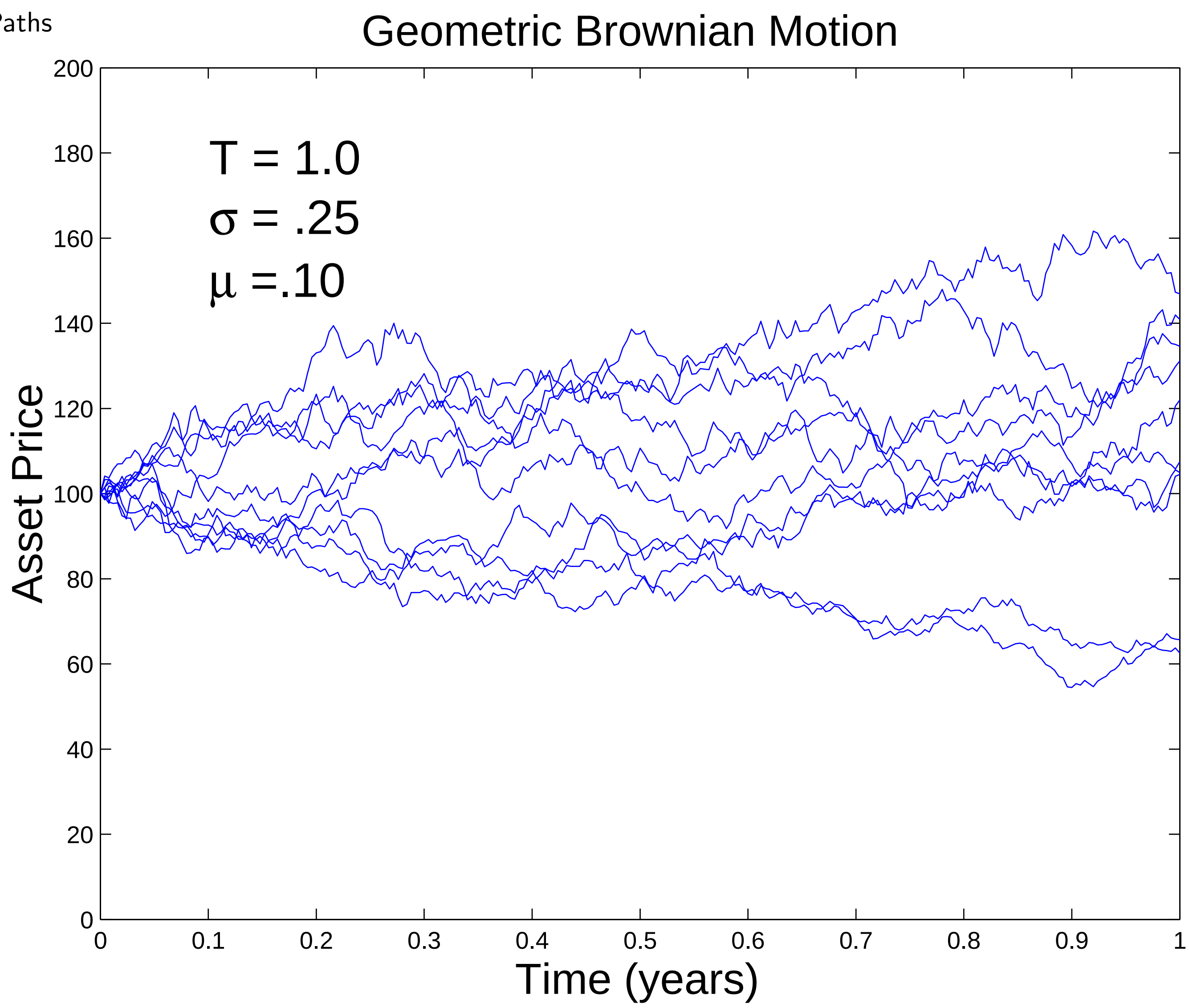
Stochastic Differential Equation for Price

• Let S be the price of an underlying asset (i.e. TSX index).

• A basic model for the evolution of S through time is Geometric Brownian Motion (GBM)

• In $t \rightarrow t + dt$, $S \rightarrow S + dS$

$$
\begin{aligned}
\frac{dS}{S} &= \mu dt + \sigma \phi \sqrt{dt} \\
\mu &= \text{drift rate}, \\
\sigma &= \text{volatility}, \\
\phi &= \text{random draw from a} \\
&\quad \text{standard normal distribution}
\end{aligned}
$$


Geometric Brownian Motion
T = 1.0
σ = .25
μ =.10
Asset Price
Time (years)
200
180
160
140
120
100
80
60
40
20
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

A lattice model

In order to price an option, don't want to deal with the SDE directly.

We will develop a discrete lattice model of GBM.

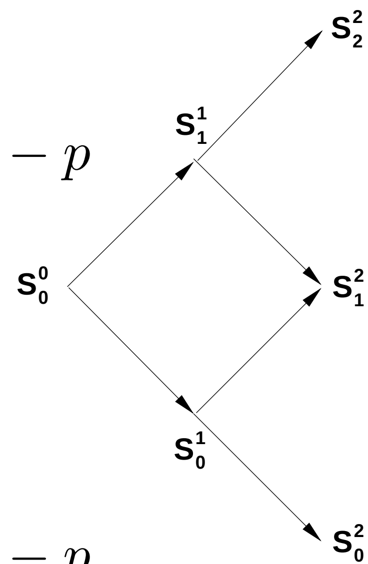
Denote today's stock price ($t = t^0$) by S_0^0. At $t^1 = t^0 + \Delta t$,

$$
\begin{aligned}
S_0^0 &\rightarrow S_1^1 \;;\; \text{up with probability } p \\
S_0^0 &\rightarrow S_0^1 \;;\; \text{down with probability } q = 1 - p
\end{aligned}
$$

A lattice model

At $t^2 = t^1 + \Delta t$,

$$S_1^1 \;\rightarrow\; S_2^2 \;;\quad \text{up with probability } p$$
$$S_1^1 \;\rightarrow\; S_1^2 \;;\quad \text{down with probability } q = 1 - p$$

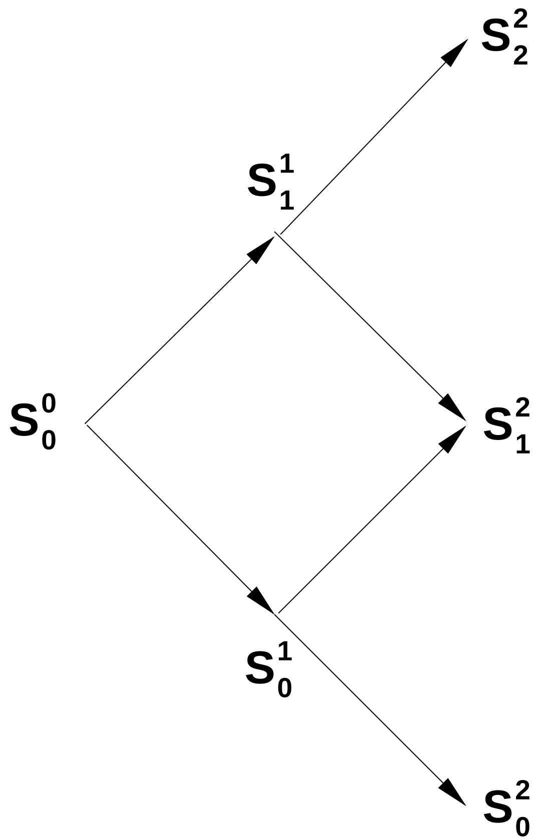


$$S_0^1 \;\rightarrow\; S_1^2 \;;\quad \text{up with probability } p$$
$$S_0^1 \;\rightarrow\; S_0^2 \;;\quad \text{down with probability } q = 1 - p$$

A Recombining Lattice

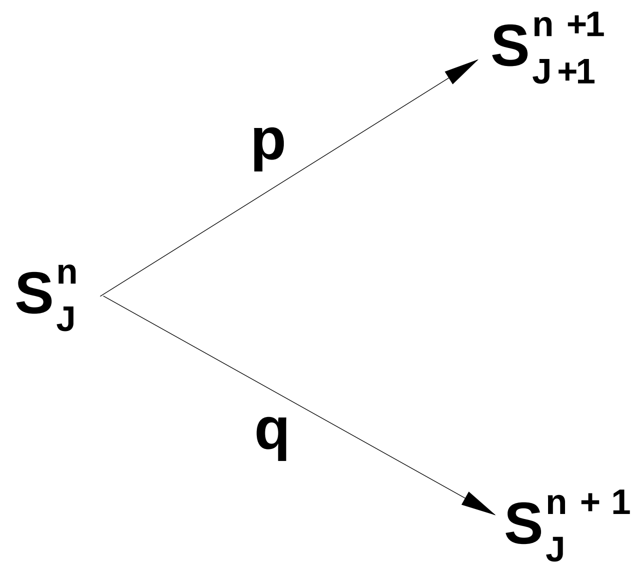
Note: S_j^n
Asset at timestep
n,
node j.

n is a superscript
not a power



General Case

At node j, timestep $t^n = n\Delta t$, asset price is denoted by S_j^n.



Consistent with GBM

If we choose:

$$\begin{aligned} S_{j+1}^{n+1} &= S_j^n e^{\sigma\sqrt{\Delta t}} \\ S_j^{n+1} &= S_j^n e^{-\sigma\sqrt{\Delta t}} \\ p &= \frac{1}{2}[1 + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}] \\ q &= 1 - p \end{aligned}$$

then, as $\Delta t \rightarrow 0$, random walks on this discrete lattice *converge* to the solution of the GBM SDE.

Convergence to GBM

In other words, if we take many random walks on the lattice with these parameters, and record a histogram of the outcomes (an approximate probability density function).

Then, as $\Delta t \rightarrow 0$, this approximate probability density converges to the probability density function of GBM.

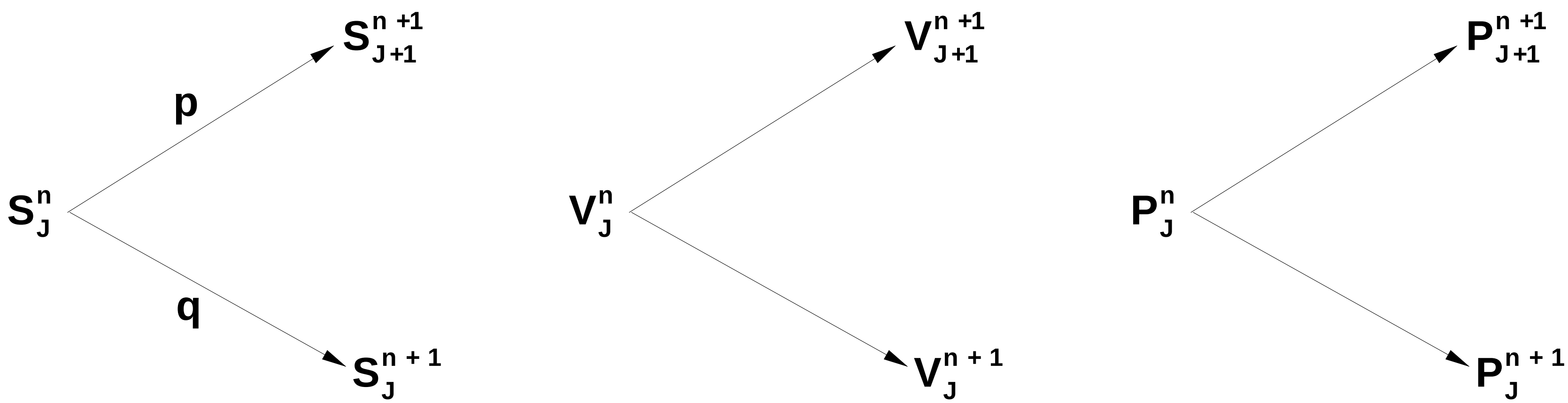
See notes for argument (not proof).

Idea: develop discrete no-arbitrage pricing model on the lattice, and then as $\Delta t \rightarrow 0$, this pricing model should converge to the correct solution for GBM.

No-arbitrage Lattice

We are going to use the same idea as in our simple example. At node S_j^n, associate an option value V_j^n and a hedging portfolio

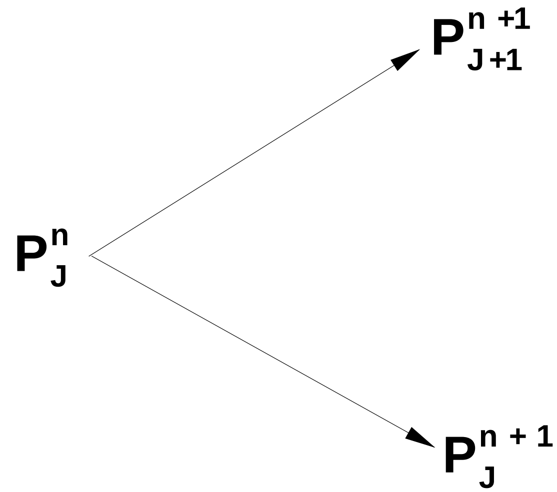
$$P_j^n = V_j^n - \alpha S_j^n$$



No-arbitrage Lattice

Value of hedging portfolio at $t = t^{n+1}$

$$
\begin{aligned}
P_{j+1}^{n+1} &= V_{j+1}^{n+1} - \alpha S_{j+1}^{n+1} \\
P_{j}^{n+1} &= V_{j}^{n+1} - \alpha S_{j}^{n+1}
\end{aligned}
$$



Now, determine α so that $P_{j+1}^{n+1} = P_{j}^{n+1}$

No-arbitrage Lattice

$$V_{j+1}^{n+1} - \alpha S_{j+1}^{n+1} = V_j^{n+1} - \alpha S_j^{n+1} \quad (1)$$

So that

$$\alpha = \frac{V_{j+1}^{n+1} - V_j^{n+1}}{S_{j+1}^{n+1} - S_j^{n+1}} \quad (2)$$

But, this portfolio is risk free (no uncertainty about its value), so that

$$\begin{aligned} P_j^n &= e^{-r\Delta t} P_{j+1}^{n+1} \\ &\rightarrow V_j^n - \alpha S_j^n = e^{-r\Delta t}(V_{j+1}^{n+1} - \alpha S_{j+1}^{n+1}) \quad (3) \end{aligned}$$

Substitute (2) into (3)

No-arbitrage Lattice

$$
\begin{aligned}
V_j^n &= e^{-r\Delta t}\left(p^* V_{j+1}^{n+1} + (1-p^*)V_j^{n+1}\right) \\
p^* &= \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}
\end{aligned} \qquad (4)
$$

Note that the real probabilities of an up/down move do not appear in (4) (p^* does not depend on the drift μ).

We have determined the no-arbitrage value of V_j^n in terms of V_{j+1}^{n+1}, V_j^{n+1}.

No-arbitrage Lattice

Recall that the no-arbitrage value is not the expected value.

But, for $\Delta t \rightarrow 0$, then

$$0 \leq \quad p^* \quad \leq 1$$

so that

$$\left(p^* V_{j+1}^{n+1} + (1 - p^*) V_j^{n+1}\right)$$

looks like an expectation.

But its not the real expected value $\rightarrow$ termed the expectation in the risk neutral world.

Delta Hedging

Since our hedging portfolio is

$$P_j^n = V_j^n - \alpha S_j^n$$

$$\alpha = \frac{V_{j+1}^{n+1} - V_j^{n+1}}{S_{j+1}^{n+1} - S_j^{n+1}}$$

Note that

$$\alpha \simeq \frac{\partial V}{\partial S} = V_S$$

$$(S_{j+1}^{n+1} - S_j^{n+1}) \to 0$$

V_S is called the option delta.

This hedging strategy is called *delta hedging*

Full Lattice Algorithm

Choose $\Delta t = T/N$. Construct tree of prices

$$
\begin{aligned}
S_j^n &= S_0^0 e^{(2j-n)\sigma\sqrt{\Delta t}} \\
& \quad n = 0, ..., N \\
& \qquad j = 0, .., n
\end{aligned}
$$

We know the value of the option at $t = T = t^N$

For $j = 0, ..., N$

$V_j^N = \text{Payoff}(S_j^N)$

EndFor

Backward Recursion: European Option

For $n = N-1, ..., 0$

For $j = 0, ..., n$

$$V_j^n = e^{-r\Delta t}(p^* V_{j+1}^{n+1} + (1-p^*) V_j^{n+1})$$

EndFor

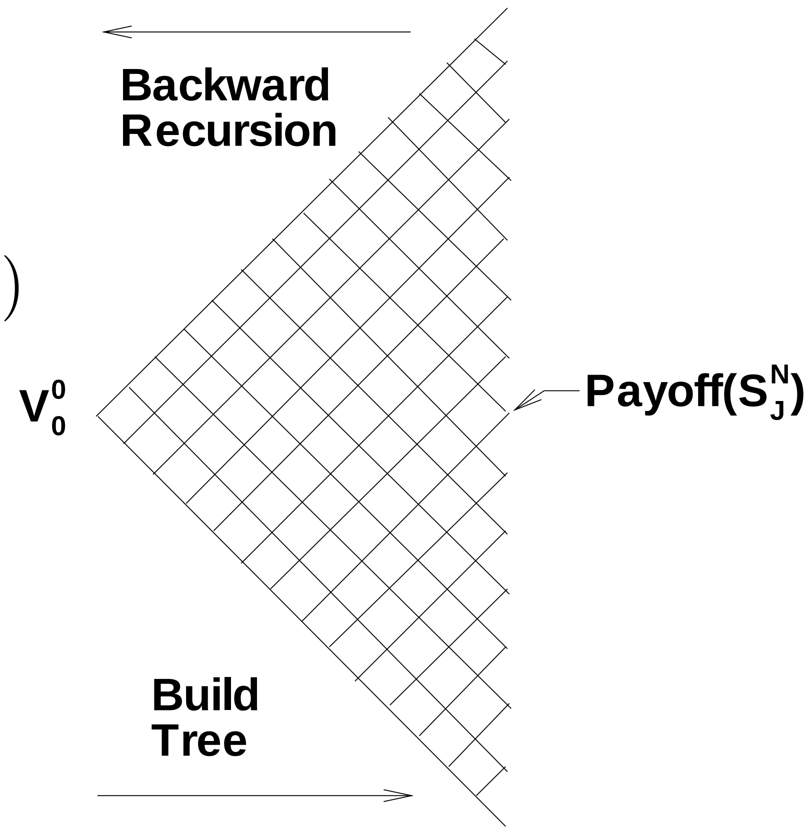
EndFor

V_0^0 is the no-arbitrage value of the option at $t = t^0, S = S_0^0$.

We also get an approximate value of the option delta $= V_S$ at each node in the tree. This is the hedging parameter.

Backward Recursion: European Option

$$V_j^n = e^{-r\Delta t} \left(p^* V_{j+1}^{n+1} + (1-p^*) V_j^{n+1}\right)$$



American Options?

Recall that an American option can be exercised at any time. and the holder can receive the payoff.

So, the holder must decide, at each instant in time

- Continue to hold the option
- Exercise immediately

A rational investor will exercise if the value of exercising is larger than the value of continuing to hold.

Backward Recursion: American Option

For $n = N-1, ..., 0$

For $j = 0, ..., n$

$V_j^n := e^{-r\Delta t}(p^* V_{j+1}^{n+1} + (1-p^*)V_j^{n+1})$

$V_j^n := \max(V_j^n, \text{Payoff}(S_j^n))$

EndFor

EndFor

This is a dynamic programming solution to the American option optimal exercise problem.

Dynamic Programming

- Note that the optimal exercise of an American option requires solution of a global optimization problem
- But, since we work backwards from the end state N, we examine all possible outcomes, and choose the optimal choice at all nodes at state $N-1$, and so on
- This reduces the global optimization to a set of trivial one step optimal choices

Black-Scholes Equation

As $\Delta t \to 0$, the solution from the lattice algorithm converges to the solution of the Black-Scholes partial differential equation (B-S PDE)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

It can be shown that

$$\begin{aligned}(V^{lattice})_j^n &= (V^{exact})_j^n + O(\Delta t) \\ & \quad \Delta t \to 0\end{aligned}$$

$\hookrightarrow$ The lattice method is simply a numerical method for solving the B-S PDE.

Hedging

• Let $V(S,t)$ be the value at any time of the option (computed from our lattice).

• The bank will sell the option to me for $V(S,t=0)$ today, and construct the following portfolio Π ($-tive \ \rightarrow$ short)

$$\Pi = -V + \alpha S + B$$

$V =$ value of option

$S =$ price of underlying

$B =$ cash in risk free money market account

$\alpha =$ units of underlying

Note we have included a bank account B in our total portfolio Π.

Hedging

So, what does the bank do?

- Sell the option today for $V(S, t = 0)$ (lattice price).
- Construct the portfolio Π, by buying $\alpha(S, t = 0)$ units at price S, and depositing B in the money market account
- As $t \rightarrow t + \Delta t$, $S \rightarrow S + \Delta S$, bank rebalances the hedge, by buying/selling underlying so that $\alpha(S + \Delta S, t + \Delta t) = V_S$
- Hedging portfolio is *Delta Neutral*

Delta Hedging

- This strategy is called *Delta Hedging*
- Note that this is a dynamic strategy (rebalanced at finite intervals)
- It is self-financing, i.e. once the bank collects cash from selling option, no further injection of cash into Π is required.
- At time T in the future, the bank liquidates Π, pays off short option position, at zero gain/loss, *regardless of random path followed by* S.

No-arbitrage Price

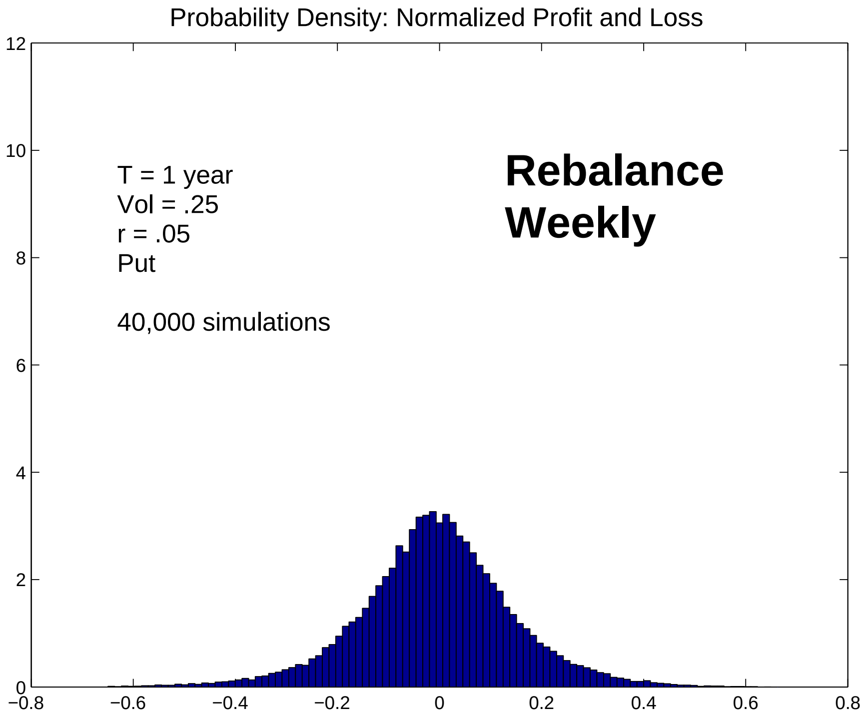
- The value of the option $V(S,t)$ is the *no-arbitrage* value
- $V(S,t=0)$ is the cost of setting up the portfolio Π at $t=0$
- The value of the option is *not* the discounted expected payoff

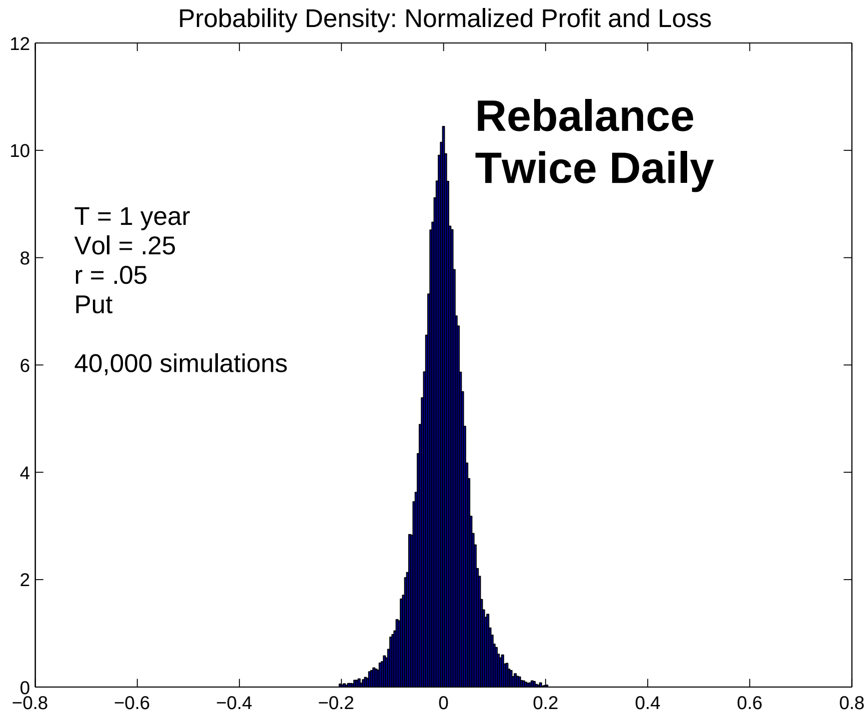
Does this actually work? Can we construct a hedge so we can't lose, regardless of the random path followed by S?

• Simulate a random price path, along path, carry out delta hedge at finite rebalancing times (not a perfect hedge)

• Liquidate portfolio at expiry, pay off option holder, record profit and loss

Monte Carlo Delta Hedge Simulation: Discounted Relative Profit and Loss





Reality

- Nobody hedges at infinitesimal intervals, volatility $\neq const.$, GBM not a perfect model
- Bank wants to make a profit

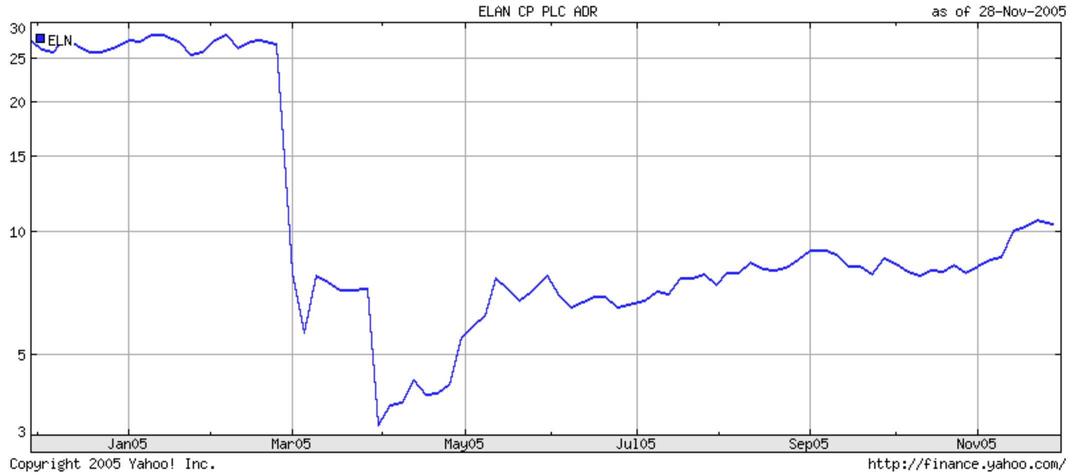
$$
\begin{aligned}
V_{buy}^{market} &= V(S,t)^{model} + \epsilon_1 + \epsilon_2 \\
V_{sell}^{market} &= V(S,t)^{model} - \epsilon_1 - \epsilon_2
\end{aligned}
$$

$\epsilon_1 =$ profit $\quad \epsilon_2 =$ compensation for imperfect hedge

$$V_{buy}^{market} - V_{sell}^{market} = \text{bid-ask spread}$$

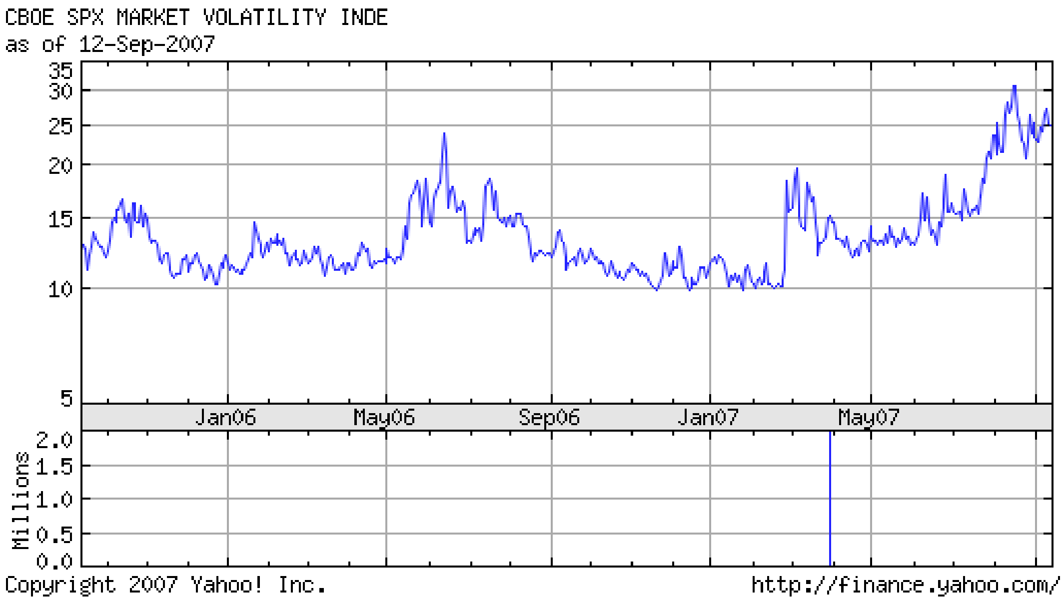
What's Wrong with GBM?

- Equity return data suggests market has *jumps* in addition to GBM
  - Sudden discontinuous changes in price



What's Wrong with GBM?

- Volatility not constant
- VIX index is a measure of instantaneous volatility ($S\&P500$)
- Volatility is itself stochastic



Research Challenges

- Pricing and hedging options under jump processes and stochastic volatility (Monte Carlo, PDE methods)
- Pricing exotic options (Numerical soln of PDEs)
- Optimal trade execution (algorithmic trading)
  - Optimal stochastic control
- Model calibration (optimization)

Notes Reading

An Introduction to Computational Finance without Agonizing Pain (www.scicom.uwaterloo.ca/ paforsyt/agon.pdf)

Sections: 1, 2.1, 2.2, 2.3, 2.4, 5

If you have time: 2.5, 2.6, 8.1, 8.2

And more if you like!

CS476, Winter 2008 *Introduction to numeric computation for financial modelling*