CS 240 – Data Structures and Data Management

Module 5: Other Dictionary Implementations

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Based on lecture notes by many previous cs240 instructors

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References: Sedgewick 9.1-9.4

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Outline

1. Dictionaries with Lists revisited
   - Dictionary ADT: Implementations thus far
   - Skip Lists
   - Re-ordering Items

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Dictionary ADT: Implementations thus far

A *dictionary* is a collection of key-value pairs (KVPs), supporting operations *search*, *insert*, and *delete*.

Realizations we have seen so far:

- **Unordered array or linked list**: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- **Ordered array**: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- **Binary search trees**: $\Theta(\mathit{height})$ search, insert and delete
- **Balanced BST** (AVL trees):
  $\Theta(\log n)$ search, insert, and delete

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Improvements/Simplifications?

- **Can show:** The average-case height of binary search trees (over all possible insertion sequences) is $O(\log n)$.
- How can we shift the average-case to expected height via randomization?

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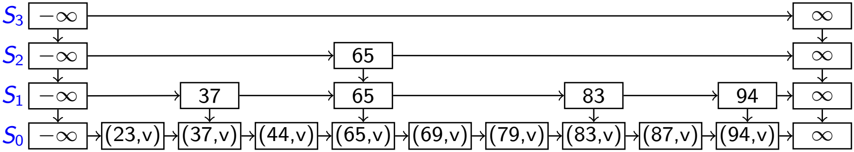
1. Dictionaries with Lists revisited
   - Dictionary ADT: Implementations thus far
   - Skip Lists
   - Re-ordering Items

Skip Lists

- A hierarchy S of ordered linked lists (*levels*) $S_0, S_1, \cdots, S_h$:
  - Each list S_i contains the special keys $-\infty$ and $+\infty$ (sentinels)
  - List S_0 contains the KVPs of S in non-decreasing order. (The other lists store only keys, or links to nodes in S_0.)
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List S_h contains only the sentinels; the left sentinel is the *root*



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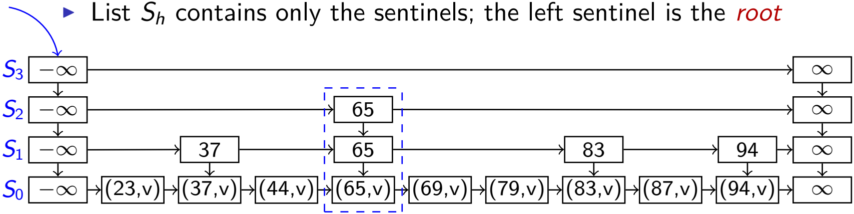


- Each KVP belongs to a **tower** of nodes
- There are (usually) more *nodes* than *keys*
- The skip list consists of a reference to the topmost left node.
- Each node p has references *p.after* and *p.below*

Search in Skip Lists

For each level, find **predecessor** (node before where k would be).
This will also be useful for *insert*/*delete*.

getPredecessors (k)

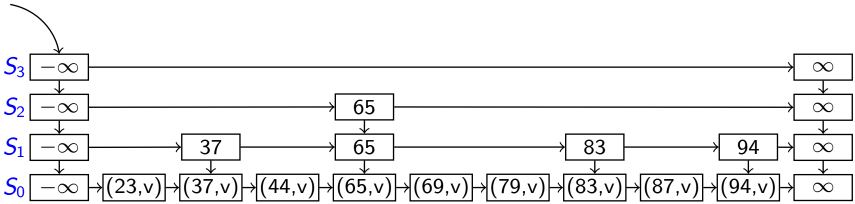
$p \leftarrow$ topmost left sentinel
$P \leftarrow$ stack of nodes, initially containing p
while $p.below \neq \texttt{NIL}$ **do**
    $p \leftarrow p.below$
    while $p.after.key < k$ **do** $p \leftarrow p.after$
    $P.push(p)$
return P

skipList::search (k)

$P \leftarrow$ *getPredecessors*(k)
$p_0 \leftarrow P.top()$ // predecessor of k in S_0
if $p_0.after.key = k$ **return** $p_0.after$
else return "not found, but would be after p_0"

Example: Search in Skip Lists

Example: *search*(87)



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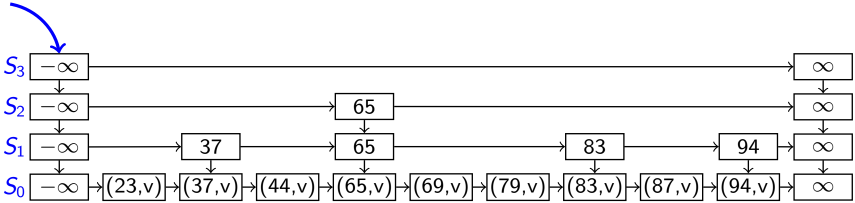

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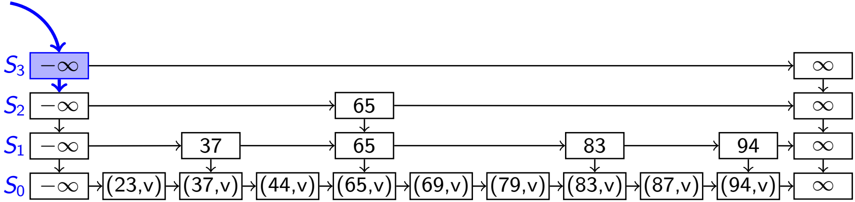

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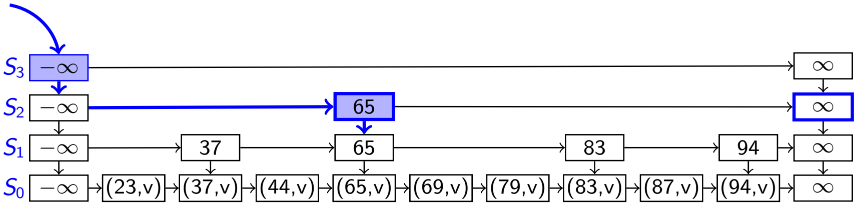

key compared with k

added to P

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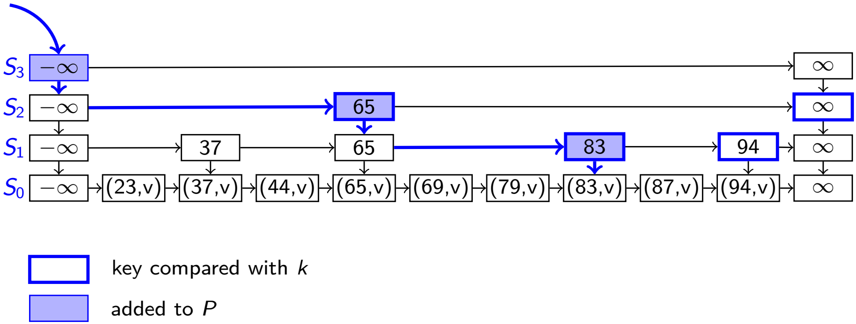

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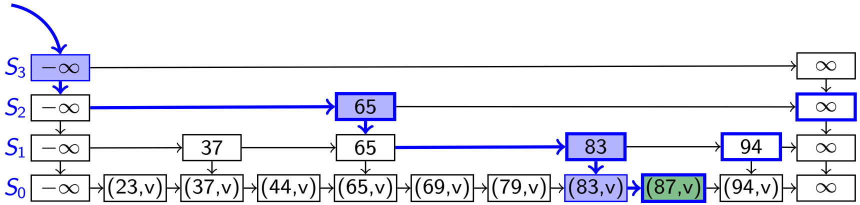


key compared with k

added to P

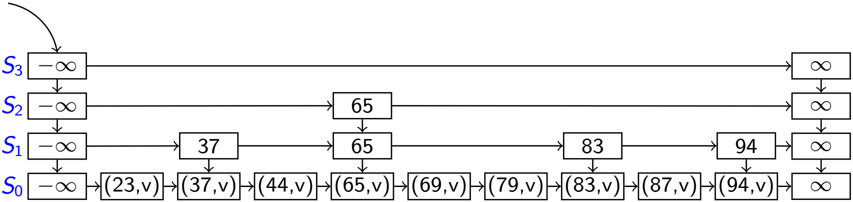
Insert in Skip Lists

skipList::insert(k, v)

- Randomly repeatedly toss a coin until you get tails
- Let i the number of times the coin came up heads
  - we want k to be in lists $S_0, \ldots, S_i$.
  - $i \rightarrow$ *height* of tower of k
  - $P(\text{tower of key } k \text{ has height} \geq i) = \left(\frac{1}{2}\right)^i$
- Increase height of skip list, if needed, to have $h > i$ levels.
- Use *getPredecessors*(k) to get stack P.
  The top i items of P are the predecessors $p_0, p_1, \cdots, p_i$ of where k should be in each list $S_0, S_1, \cdots, S_i$
- Insert (k, v) after p_0 in S_0, and k after p_j in S_j for $1 \leq j \leq i$

Example: Insert in Skip Lists

Example: *skipList::insert*$(52, v)$
Coin tosses: H,T $\Rightarrow i = 1$



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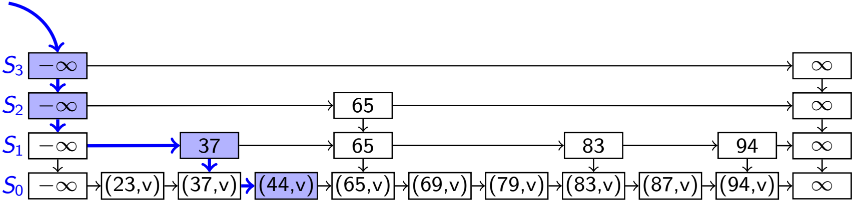
getPredecessors(52)



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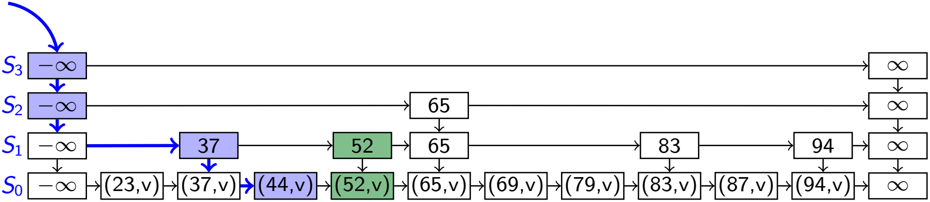
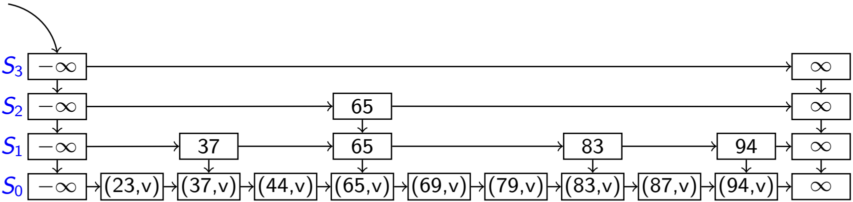


Example 2: Insert in Skip Lists

Example: *skipList::insert*$(100, v)$
Coin tosses: H,H,H,T $\Rightarrow i = 3$



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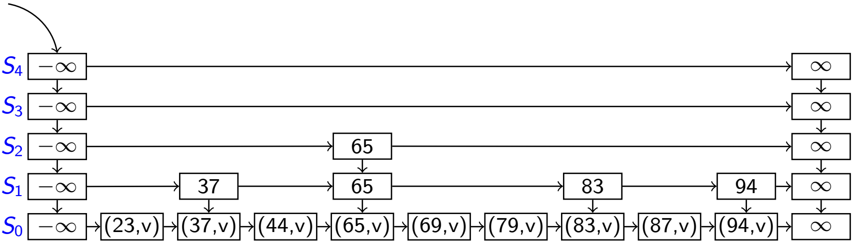
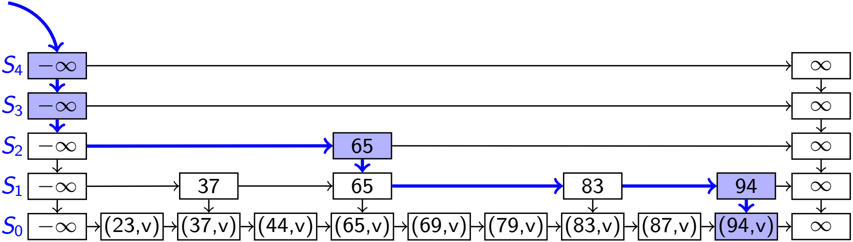
Example: *skipList::insert*$(100, v)$
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Height increase



Example 2: Insert in Skip Lists

Example: *skipList::insert*(100, v)
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Height increase
getPredecessors(100)



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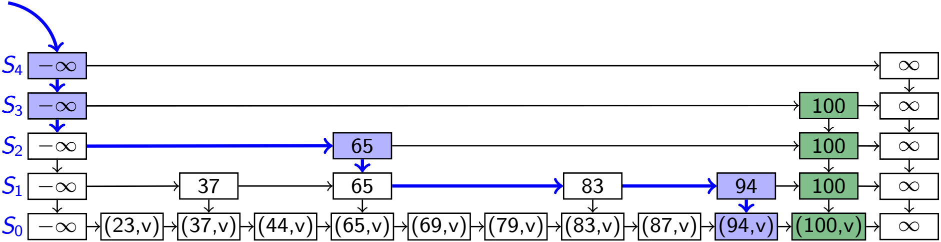
Height increase

getPredecessors(100)



Insert in Skip Lists

```
skipList::insert(k, v)
    P ← getPredecessors(k)
    for (i ← 0; random(2) = 1; i ← i+1) {}        // random tower height
    while i ≥ P.size()                             // increase skip-list height?
        root ← new sentinel-only list,
            linked to previous root-list appropriately
        P.append(left sentinel of root)
    p ← P.pop()                                    // insert (k, v) in S_0
    k_below ← new node with (k, v), inserted after p
    while i > 0                                    // insert k in S_1, ..., S_i
        p ← P.pop()
        k_below ← new node with k,
            inserted after p with below-reference to k_below
        i ← i − 1
```

Delete in Skip Lists

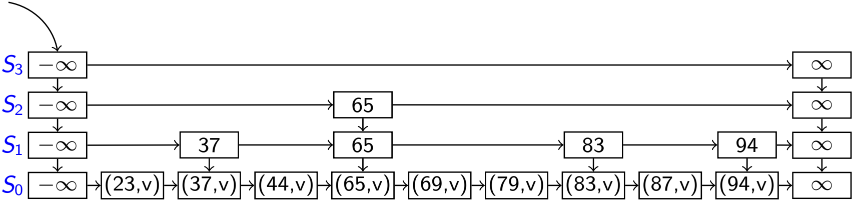
It is easy to remove a key since we can find all predecessors.
Then eliminate layers if there are multiple ones with only sentinels.

```
skipList::delete(k)
    P ← getPredecessors(k)
    while P is non-empty
        p ← P.pop()          // predecessor of k in some layer
        if p.after.key = k
            p.after ← p.after.after
        else break           // no more copies of k

    p ← left sentinel of the root-list
    while p.below.after is the ∞-sentinel
        // the two top lists are both only sentinels, remove one
        p.below ← p.below.below
        p.after.below ← p.after.below.below
```

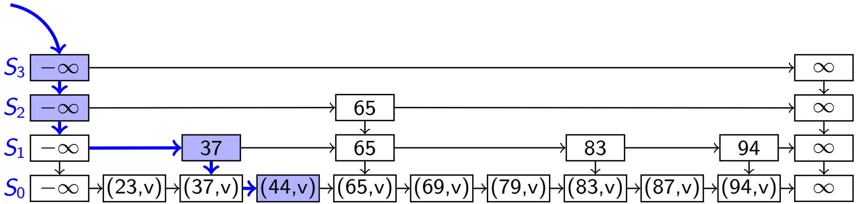
Example: Delete in Skip Lists

Example: *skipList::delete*(65)



Example: Delete in Skip Lists

Example: *skipList::delete*(65)
getPredecessors(65)



Example: Delete in Skip Lists

Example: *skipList::delete*(65)
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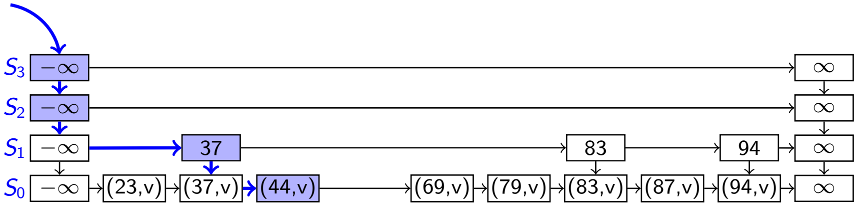
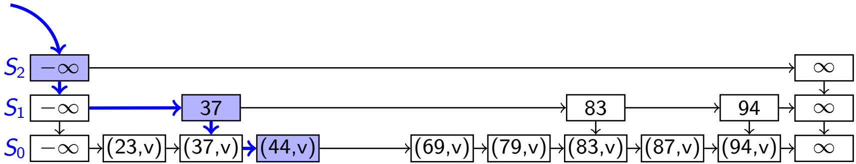


Example: Delete in Skip Lists

Example: *skipList::delete*(65)
getPredecessors(65)
Height decrease

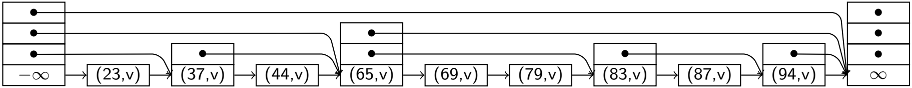


Analysis of Skip Lists

- Expected **space** usage: $O(n)$
- Expected **height**: $O(\log n)$
  A skip list with n items has height at most $3 \log n$ with probability at least $1 - 1/n^2$
- Crucial for all operations:
  - How often do we *drop down* (execute $p \leftarrow p.below$)?
  - How often do we *scan forward* (execute $p \leftarrow p.after$)?
- *skipList::search*: $O(\log n)$ expected time
  - $\#$ drop-downs $=$ height
  - expected $\#$ scan-forwards is ≤ 1 in each level
- *skipList::insert*: $O(\log n)$ expected time
- *skipList::delete*: $O(\log n)$ expected time

Summary of Skip Lists

- $O(n)$ expected space, all operations take $O(\log n)$ expected time.
- As described they are no faster than randomized binary search trees.
- Can show: A biased coin-flip to determine tower-height gives smaller expected run-times.
- Can save links (hence space) by implementing towers as array.



- Then skip lists are fast in practice and simple to implement.

Outline

Re-ordering Items

- Recall: Unordered list/array implementation of ADT Dictionary
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Lists/arrays are a very simple and popular implementation. Can we do something to make search more effective in practice?

Re-ordering Items

- Recall: Unordered list/array implementation of ADT Dictionary
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Lists/arrays are a very simple and popular implementation. Can we do something to make search more effective in practice?
- No: if items are accessed equally likely
- Yes: otherwise (we have a probability distribution of the items)
  - Intuition: Frequently accessed items should be in the front.
  - Two cases: Do we know the access distribution beforehand or not?
  - For short lists or extremely unbalanced distributions this may be faster than AVL trees or Skip Lists, and much easier to implement.

Optimal Static Ordering

Example:

key	A	B	C	D	E
frequency of access	2	8	1	10	5
access-probability	$\frac{2}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{5}{26}$

- We count cost i for accessing the key in the ith position.
- Order A, B, C, D, E has expected access cost
  $\frac{2}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{1}{26} \cdot 3 + \frac{10}{26} \cdot 4 + \frac{5}{26} \cdot 5 = \frac{86}{26} \approx 3.31$
- Order D, B, E, A, C has expected access cost
  $\frac{10}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{5}{26} \cdot 3 + \frac{2}{26} \cdot 4 + \frac{1}{26} \cdot 5 = \frac{66}{26} \approx 2.54$

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- **Claim:** Over all possible static orderings, the one that sorts items by non-increasing access-probability minimizes the expected access cost.
- **Proof Idea:** For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Dynamic Ordering: MTF

- What if we do *not know the access probabilities* ahead of time?
- Rule of thumb (**temporal locality**): A recently accessed item is likely to be used soon again.
- In list: Always insert at the front
- **Move-To-Front heuristic** (MTF): Upon a successful search, move the accessed item to the front of the list

$$A \to B \to C \to D \to E$$
$$\downarrow \text{search(D)}$$
$$D \to A \to B \to C \to E$$
$$\downarrow \text{insert(F)}$$
$$F \to D \to A \to B \to C \to E$$

- We can also do MTF on an array, but should then insert and search from the *back* so that we have room to grow.

Dynamic Ordering: Transpose

Transpose heuristic: Upon a successful search, swap the accessed item with the item immediately preceding it

A → B → C → D → E

$\downarrow$ search(D)

A → B → D → C → E

$\downarrow$ insert(F)

F → A → B → D → C → E

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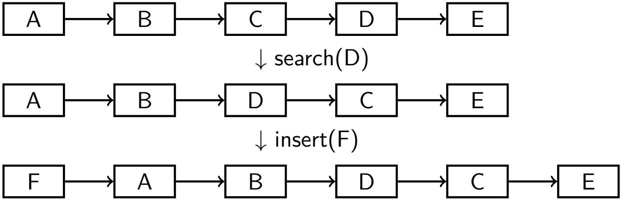

Performance of dynamic ordering:

- Transpose does not adapt quickly to changing access patterns.
- MTF works well in practice.
- **Can show:** MTF is "2-competitive":
  No more than twice as bad as the optimal static ordering.