CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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Fall 2020

References: Sedgewick 12.4, 15.2-15.4
Goodrich & Tamassia 23.5.1-23.5.2

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Outline

1. Lower bound

2. Interpolation Search

3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

Lower bound for search

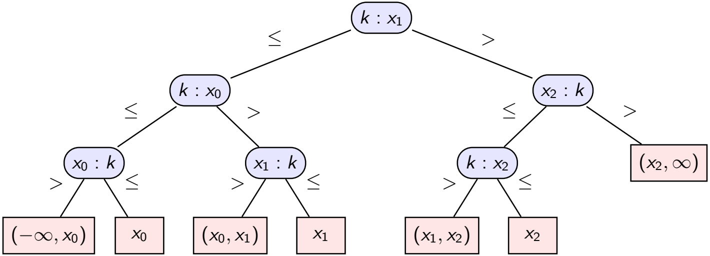
The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Lower bound for search

The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Theorem: In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof: via decision tree



But can we beat the lower bound for special keys?

Binary Search

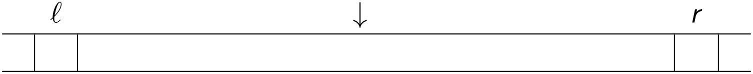
Recall the run-times in a *sorted array*:

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

```
Binary-search(A, n, k)
A: Sorted array of size n, k: key
    $\ell \leftarrow 0$
    $r \leftarrow n-1$
    while ($\ell < r$)
        $m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
        if ($A[m] < k$) then $\ell = m+1$
        else if ($k < A[m]$) then $r = m-1$
        else return $m$
    if ($k = A[\ell]$) return $\ell$
    else return "not found, but would be between $\ell-1$ and $\ell$"
```

Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$



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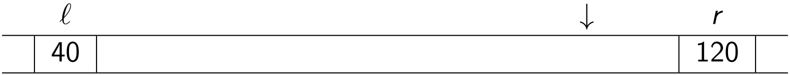
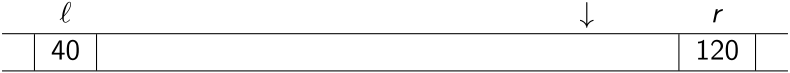


Question: If keys are numbers, where would you expect key $k = 100$?

Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$



Question: If keys are numbers, where would you expect key $k = 100$?

interpolation-search$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]} (r - \ell) \right\rfloor$

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

interpolation-search(A[0..10],449):

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500
ℓ		$\uparrow$								r

interpolation-search(A[0..10],449):

- Initially $\ell = 0,\ r = n - 1 = 10,\ m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$

Interpolation Search Example

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interpolation-search(A[0..10],449):

- Initially $\ell = 0,\ r = n - 1 = 10,\ m = \ell + \lfloor \frac{449-0}{1500-0}(10 - 0) \rfloor = \ell + 2 = 2$
- $\ell = 3,\ r = 10,\ m = \ell + \lfloor \frac{449-3}{1500-3}(10 - 3) \rfloor = \ell + 2 = 5$

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
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interpolation-search(A[0..10],449):

- Initially $\ell = 0,\ r = n-1 = 10,\ m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3,\ r = 10,\ m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3,\ r = 4,\ m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$, found at $A[4]$

Interpolation Search Example

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interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, $m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$, found at $A[4]$

Works well if keys are *uniformly* distributed:

- Can show: the array in which we recurse into has size $\sqrt{n}$ on average.
- Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

But: Worst case performance $\Theta(n)$

Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash due to $A[\ell] = A[r]$

```
*interpolation-search*$(A, n, k)$
$A$: Sorted array of size $n$, $k$: key
    $\ell \leftarrow 0$
    $r \leftarrow n - 1$
    **while** $(\ell < r) \&\& (A[r] != A[\ell]) \&\& (k \geq A[\ell]) \&\& (k \leq A[r])$
        $m \leftarrow \ell + \lfloor \frac{k - A[\ell])}{A[r] - A[\ell]} \cdot (r - \ell) \rfloor$
        **if** $(A[m] < k)$ **then** $\ell = m + 1$
        **else if** $(k < A[m])$ **then** $r = m - 1$
        **else return** $m$
    **if** $(k = A[\ell])$ **return** $\ell$
    **else return** "not found, but would be between $\ell{-}1$ and $\ell$"
```

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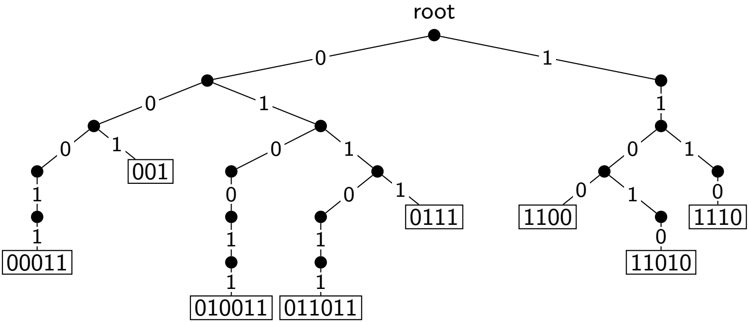
3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

Tries: Introduction

Trie (also know as **radix tree**): A dictionary for bitstrings.
(Should know: string, word, $|w|$, alphabet, prefix, suffix, comparing words,....)

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key

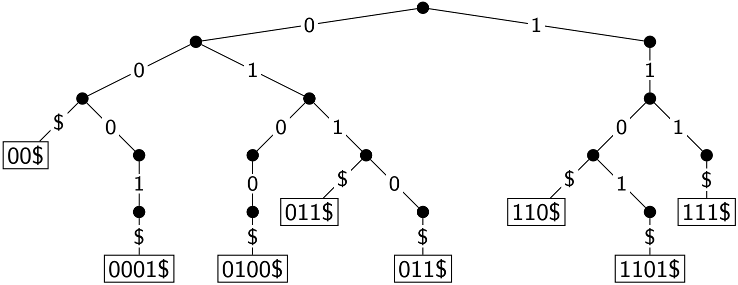


More on tries

Assumption: Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with 'end-of-word' character \$.

Example: A trie for $\{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}$



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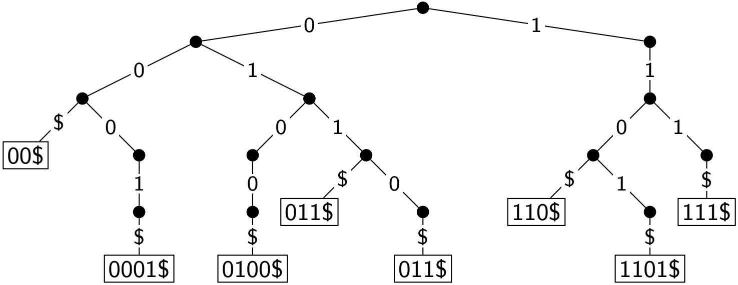


Then items (keys) are stored *only* in the leaf nodes

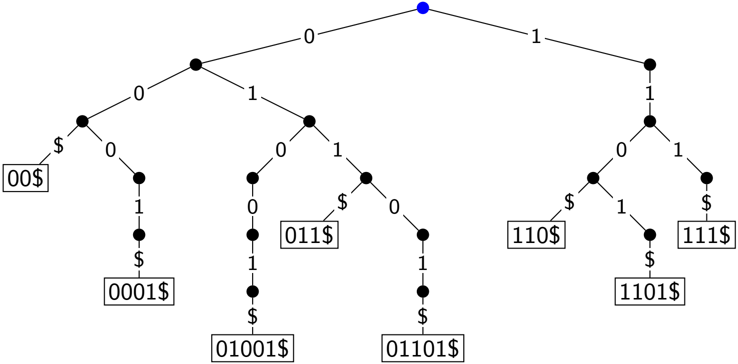
Tries: Search

- start from the root and the most significant bit of x
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- return success if we reach a leaf (it must store x)
- else recurse on the new node and the next bit of x

```
Trie::search($v \leftarrow$ root, $d \leftarrow 0, x$)
$v$: node of trie; $d$: level of $v$, $x$: word stored as array of chars
    if $v$ is a leaf
        return $v$
    else
        let $v'$ be child of $v$ labelled with $x[d]$
        if there is no such child
            return "not found"
        else Trie::search($v', d+1, x$)
```

Tries: Search Example

Example: Trie::search(011$)



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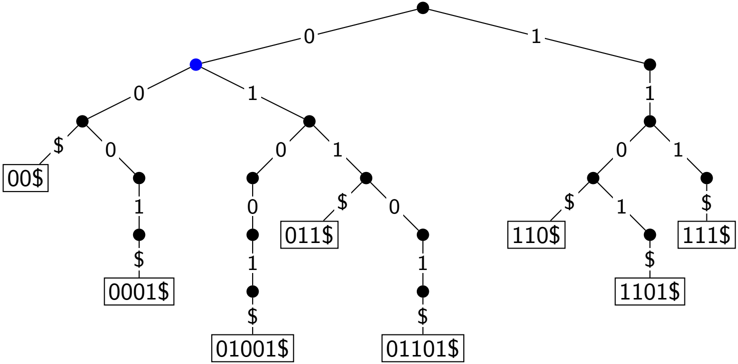

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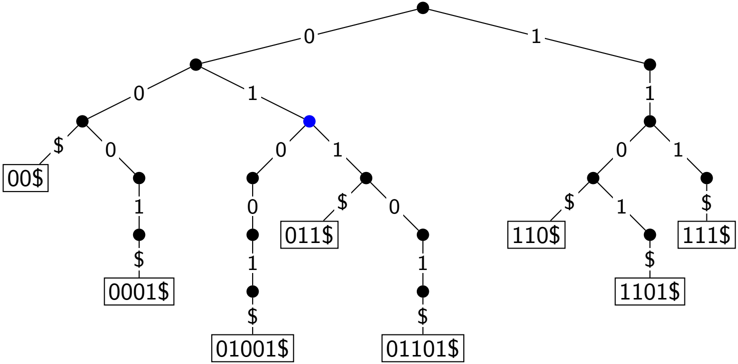

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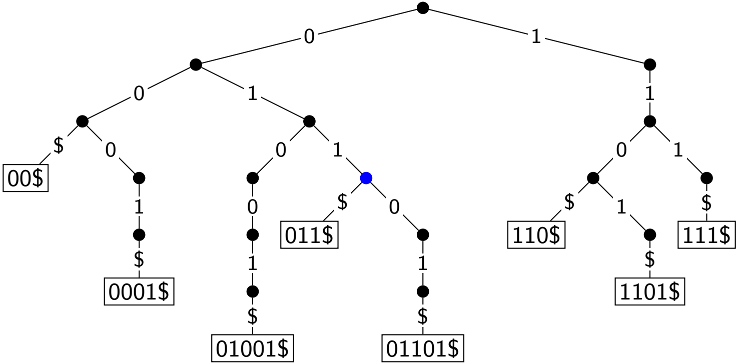
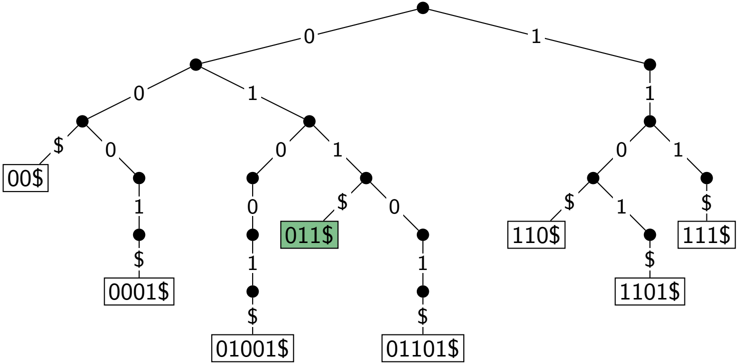


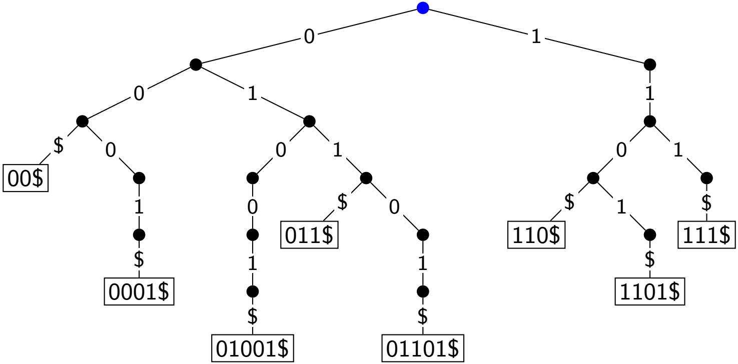
Tries: Search Example

Example: Trie::search(011$) successful



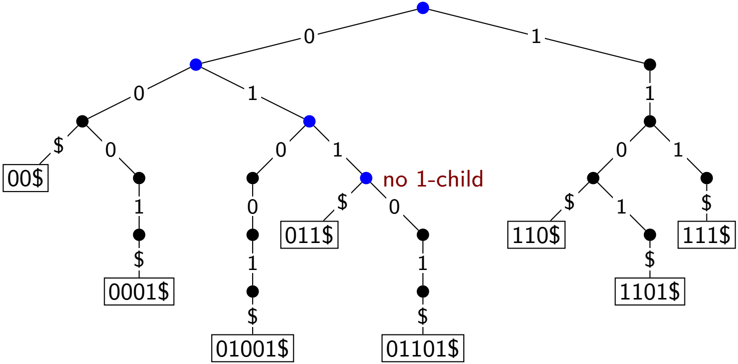
Tries: Search Example

Example: Trie::search(0111$)



Tries: Search Example

Example: Trie::search(0111$) unsuccessful



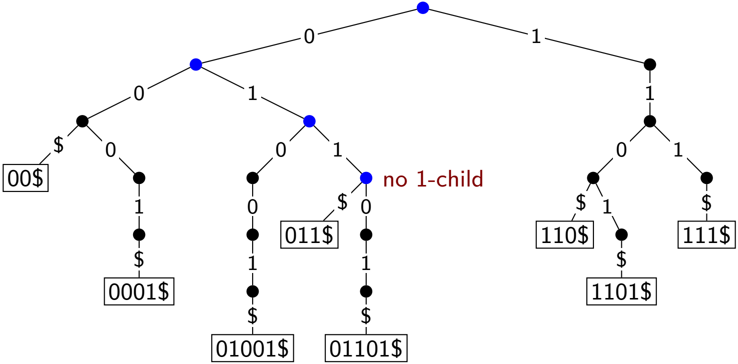
Tries: Insert & Delete

- *Trie::insert(x)*
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child. Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.
- *Trie::delete*(x)
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.
- **Time Complexity** of all operations: $\Theta(|x|)$

  $|x|$: length of binary string x, i.e., the number of bits in x

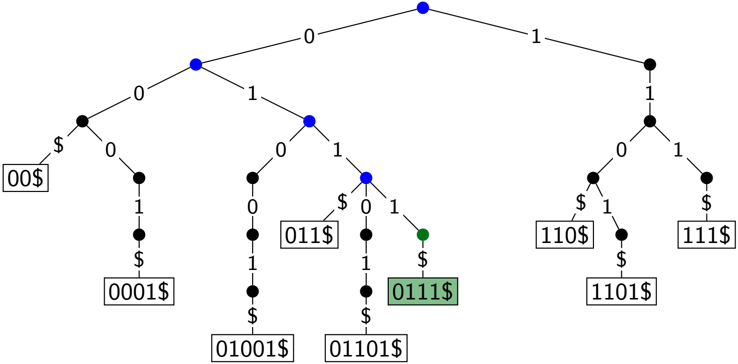
Tries: Insert Example

Example: *Trie::insert*(0111$)



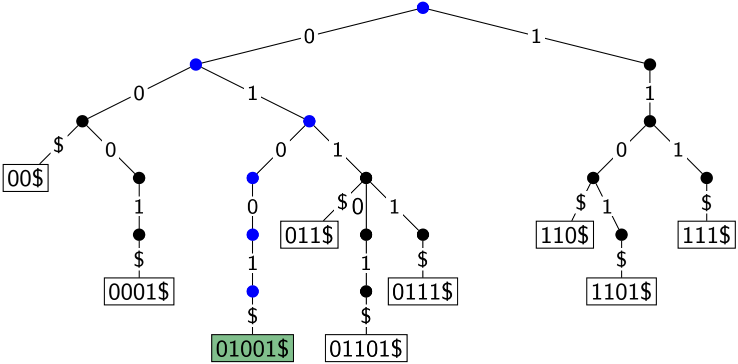
Tries: Insert Example

Example: *Trie::insert*(0111$)



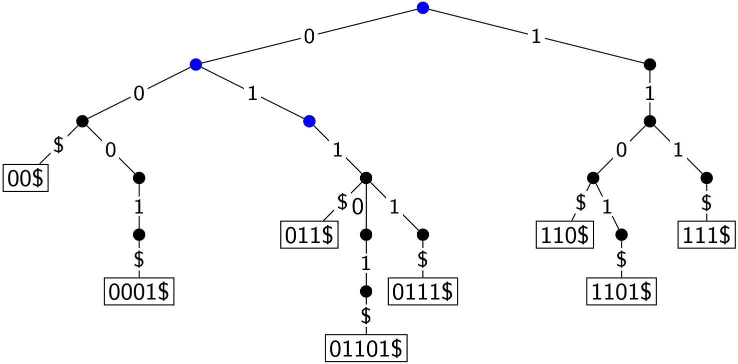
Tries: Delete Example

Example: *Trie::delete*(01001$)



Tries: Delete Example

Example: *Trie::delete*(01001$)



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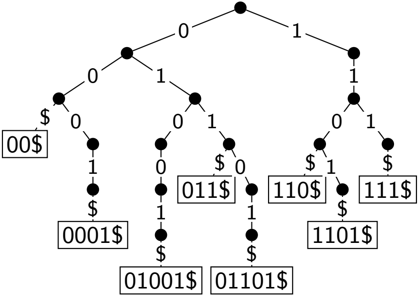
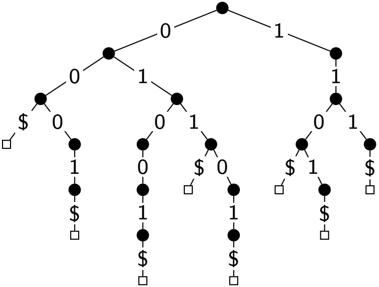
Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.

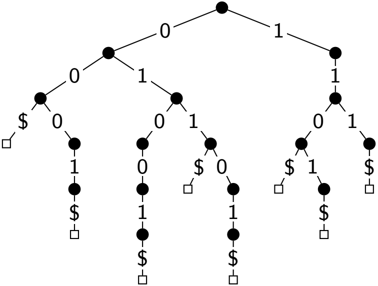


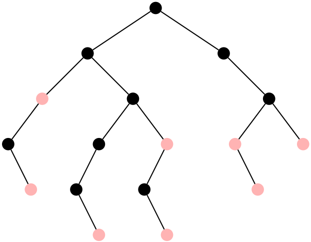


Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys.
  Use a *flag* to indicate such nodes.
- No need for end-of-word character $
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.

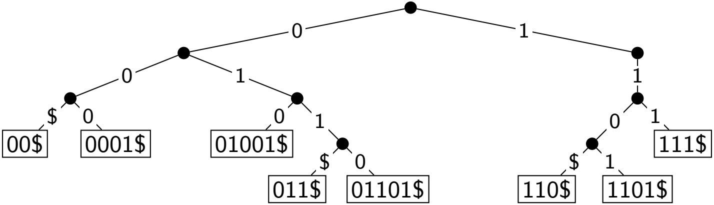




Variations 3 of Tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)



This is in practice the most efficient version of tries, but the operations get a bit more complicated.

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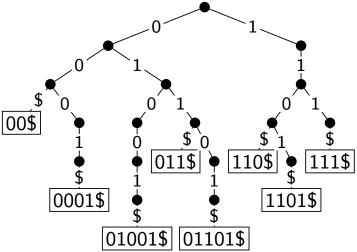
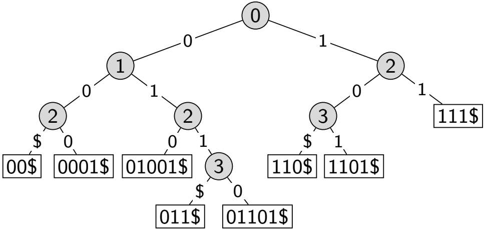
Variation 4 of Tries

Compressed Trie: compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
    - This gives the next bit to be tested during a search
- A compressed trie with n keys has at most $n-1$ internal nodes





Also known as **Patricia-Tries**:

Practical Algorithm to Retrieve Information Coded in Alphanumeric

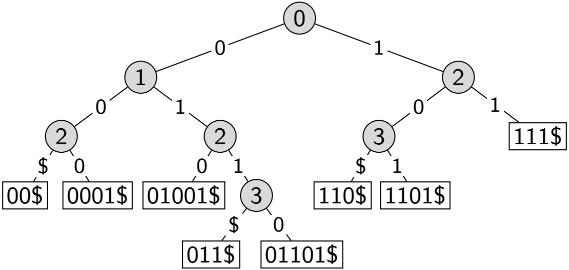
Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in x;
  return failure if the link is missing
- if we reach a leaf, expicitly check whether word stored at leaf is x
- else recurse on the new node and the next bit of x

```
CompressedTrie::search($v \leftarrow$ root, $x$)
$v$: node of trie; $x$: word
    if $v$ is a leaf
        return strcmp($x$, $v.key$)
    $d \leftarrow$ index stored at $v$
    if $x$ has at most $d$ bits
        return "not found"
    $v' \leftarrow$ child of $v$ labelled with $x[d]$
    if there is no such child
        return "not found"
    CompressedTrie::search($v'$, $x$)
```

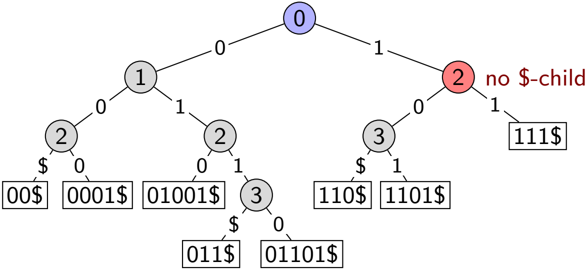
Compressed Tries: Search Example

Example: CompressedTrie::search(10$)



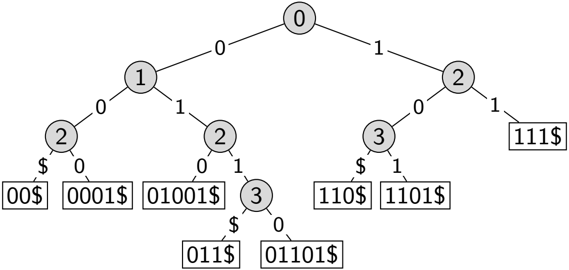
Compressed Tries: Search Example

Example: CompressedTrie::search(10$) unsuccessful



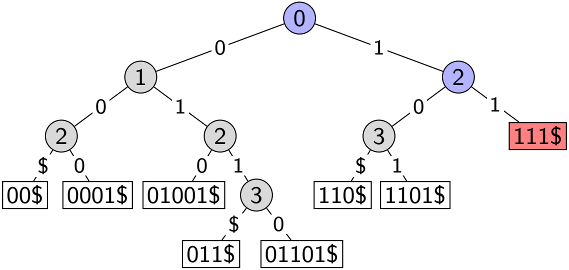
Compressed Tries: Search Example

Example: CompressedTrie::search(101$)



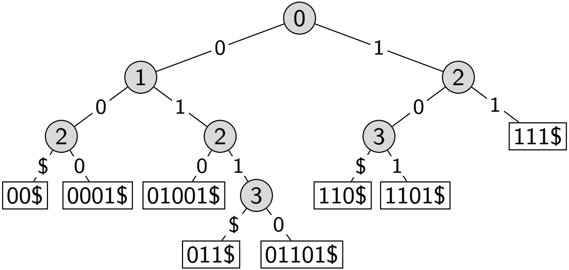
Compressed Tries: Search Example

Example: CompressedTrie::search(101$) unsuccessful



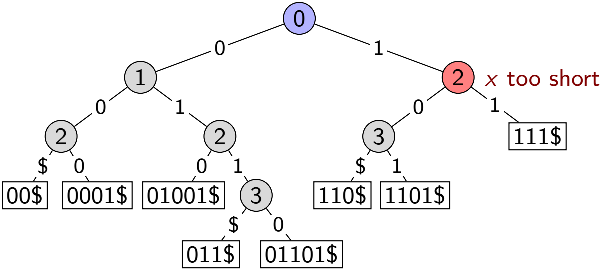
Compressed Tries: Search Example

Example: CompressedTrie::search(1$)



Compressed Tries: Search Example

Example: CompressedTrie::search(1$) unsuccessful



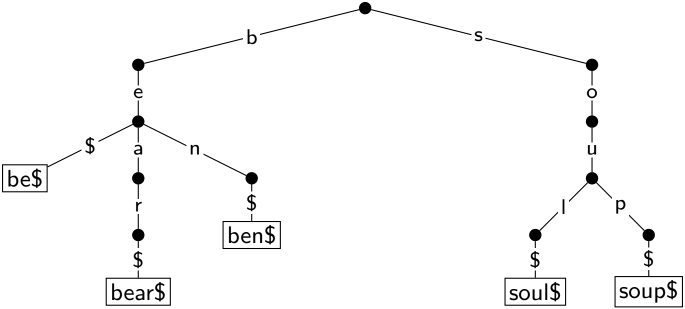
Compressed Tries: Insert & Delete

- *CompressedTrie::delete*(x):
  - Perform *search*(x)
  - Remove the node v that stored x
  - Compress along path to v whenever possible.
- *CompressedTrie::insert*(x):
  - Perform *search*(x)
  - Let v be the node where the search ended.
  - Conceptually simplest approach:
    - Uncompress path from root to v.
    - Insert x as in an uncompressed trie.
    - Compress paths from root to v and from root to x.

    But it can also be done by only adding those nodes that are needed, see the textbook for details.
- All operations take $O(|x|)$ time.

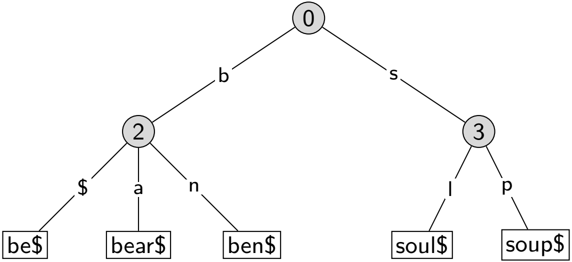
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings $\{$bear\$, ben\$, be\$, soul\$, soup\$$\}$



Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



Multiway Tries: Summary

- Operations *search*(x), *insert*(x) and *delete*(x) are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot \text{(time to find the appropriate child)})$

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- Operations *search*(x), *insert*(x) and *delete*(x) are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot (\text{time to find the appropriate child}))$

Each node now has up to $|\Sigma| + 1$ children. How should they be stored?

Solution 1: Array of size $|\Sigma| + 1$ for each node.
Complexity: $O(1)$ time to find child, $O(|\Sigma| n)$ space.

Solution 2: List of children for each node.
Complexity: $O(|\Sigma|)$ time to find child, $O(\#\text{children})$ space.

Solution 3: Dictionary (AVL-tree?) of children for each node.
Complexity: $O(\log(\#\text{children}))$ time, $O(\#\text{children})$ space.

Best in theory, but not worth it in practice unless $|\Sigma|$ is huge.

In practice, use *hashing* (keys are in (typically small) range Σ).