CS 240 – Data Structures and Data Management

Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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References: Sedgewick 9.1-9.4

Outline

1. Priority Queues
   - Abstract Data Types
   - ADT Priority Queue
   - Binary Heaps
   - Operations in Binary Heaps
   - *PQ-sort* and *Heapsort*
   - Towards the Selection Problem

Outline

Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various **realizations** of an ADT, which specify:

- How the information is stored (**data structure**)
- How the operations are performed (**algorithms**)

Stack ADT

Stack: an ADT consisting of a collection of items with operations:

- *push*: inserting an item
- *pop*: removing the most recently inserted item

Items are removed in LIFO (*last-in first-out*) order.
Items enter the stack at the *top* and are removed from the *top*.
We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited web sites, procedure calls

Realizations of Stack ADT

- using arrays
- using linked lists

Queue ADT

Queue: an ADT consisting of a collection of items with operations:

- *enqueue*: inserting an item
- *dequeue*: removing the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order.
Items enter the queue at the *rear* and are removed from the *front*.
We can have extra operations: *size*, *isEmpty*, and *front*

Applications: Waiting lines, printer queues

Realizations of Queue ADT

- using (circular) arrays
- using linked lists

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
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Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a **priority**) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing the item of *highest* priority

deleteMax is also called *extractMax* or *getmax*.
The priority is also called *key*.

The above definition is for a **maximum-oriented** priority queue. A **minimum-oriented** priority queue is defined in the natural way, replacing operation *deleteMax* by *deleteMin*,

Applications: typical "todo" list, simulation systems, sorting

Using a Priority Queue to Sort

> *PQ-Sort*($A[0..n-1]$)
> 1.   initialize PQ to an empty priority queue
> 2.   **for** $k \leftarrow 0$ **to** $n-1$ **do**
> 3. $PQ.insert(A[k])$
> 4.   **for** $k \leftarrow n-1$ **down to** 0 **do**
> 5. $A[k] \leftarrow PQ.deleteMax()$

- Note: Run-time depends on how we implement the priority queue.
- Sometimes written as: $O(n + n \cdot insert + n \cdot deleteMax)$

Realizations of Priority Queues

Realization 1: unsorted arrays

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- *insert*: $O(1)$
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Note: We assume **dynamic arrays**, i. e., expand by doubling as needed.
(Amortized over all insertions this takes $O(1)$ extra time.)

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PQ-sort with this realization yields *selection sort*.

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Realization 2: sorted arrays

- *insert*: $O(n)$
- *deleteMax*: $O(1)$

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Outline

Realization 3: Heaps
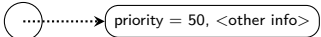
A **(binary) heap** is a certain type of binary tree.

You should know:

- A **binary tree** is either
  - empty, or
  - consists of three parts:
    a node and two binary trees (left subtree and right subtree).
- Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
- Any binary tree with n nodes has height at least $\log(n + 1) - 1 \in \Omega(\log n)$.

Example Heap
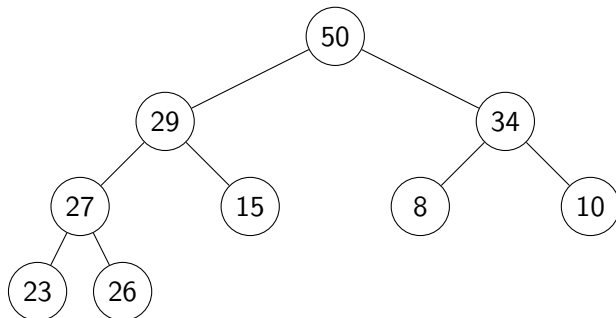
$$\left(\begin{array}{l} \text{In our examples we only show the priorities, and we show them directly in} \\ \text{the node. A more accurate picture would be} \quad \bigcirc \cdots\cdots\cdots \rightarrow \boxed{\text{priority} = 50, \text{<other info>}} \end{array} \right)$$

Heaps – Definition

A **heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property:** For any node i, the key of the parent of i is larger than or equal to key of i.

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The full name for this is *max-oriented binary heap*.

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The full name for this is *max-oriented binary heap*.

Lemma: The height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

Heaps should *not* be stored as binary trees!

Let H be a heap of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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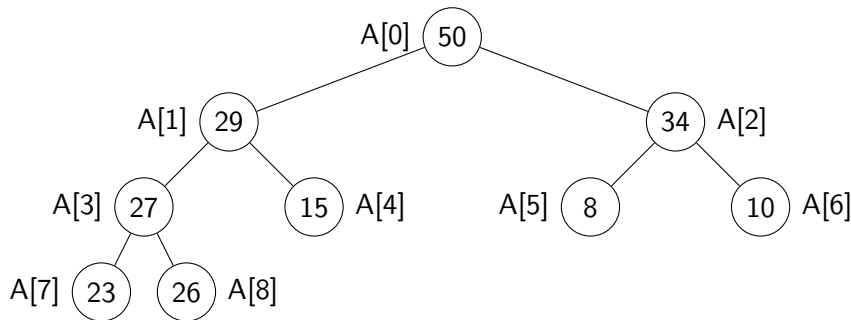
Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the *root* node is at index 0
  (We use "node" and "index" interchangeably in this implementation.)
- the *left child* of node i (if it exists) is node $2i + 1$
- the *right child* of node i (if it exists) is node $2i + 2$
- the *parent* of node i (if it exists) is node $\lfloor \frac{i-1}{2} \rfloor$
- the *last* node is $n - 1$

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We should hide implementation details using helper-functions!

- functions *root()*, *parent(i)*, *last(n)*, etc.

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> 1. **while** *parent*(k) exists **and** $A[parent(k)] < A[k]$ **do**
> 2.   swap $A[k]$ and $A[parent(k)]$
> 3. $k \leftarrow parent(k)$

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Time: $O(\text{height of heap}) = O(\log n)$.

fix-up example
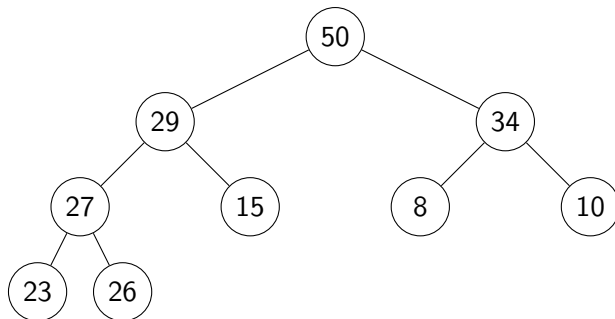
fix-up example
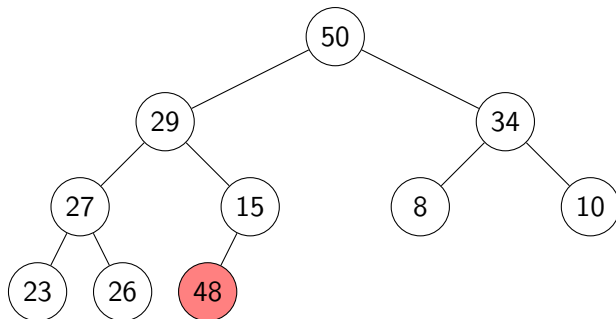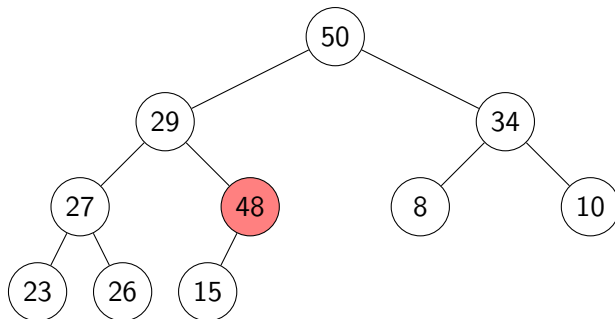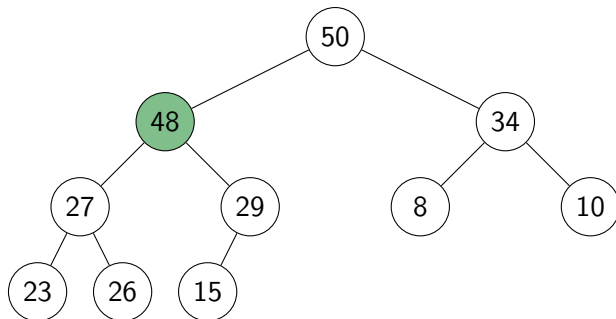
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deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
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```
fix-down(A, n, k)
A: an array that stores a heap of size n
k: an index corresponding to a node of the heap
1.    while k is not a leaf do
2.        // Find the child with the larger key
3.        j ← left child of k
4.        if (j is not last(n) and A[j + 1] > A[j])
5.            j ← j + 1
6.        if A[k] ≥ A[j] break
7.        swap A[j] and A[k]
8.        k ← j
```

Time: O(height of heap) $= O(\log n)$.

deleteMax example
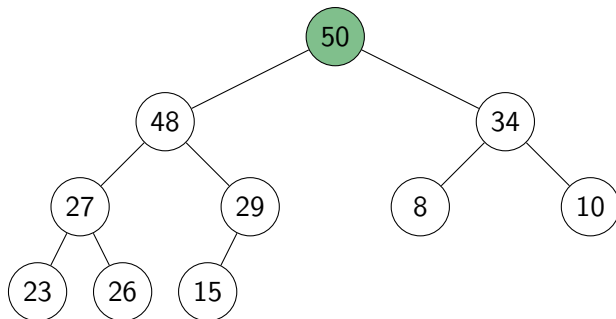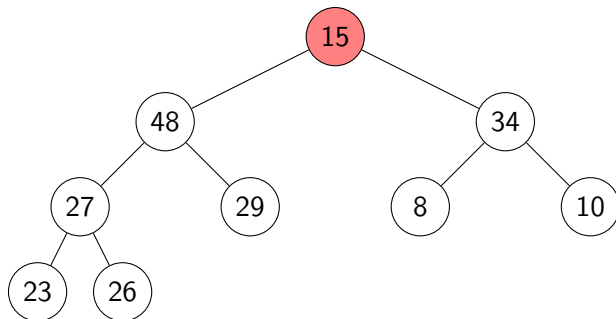
deleteMax example
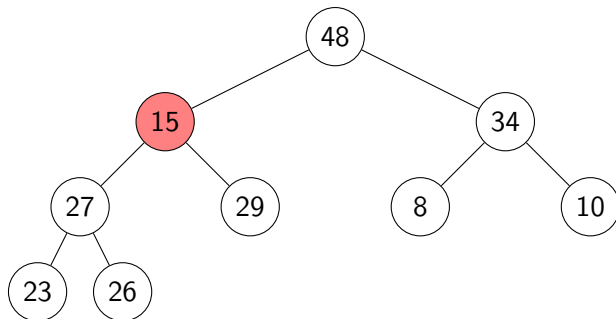
deleteMax example

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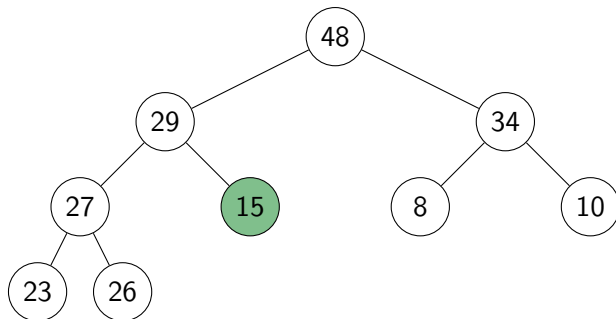
Priority Queue Realization Using Heaps

- Store items in array A and globally keep track of *size*

```
insert(x)
1.    increase size
2.    ℓ ← last(size)
3.    A[ℓ] ← x
4.    fix-up(A, ℓ)
```

```
deleteMax()
1.    ℓ ← last(size)
2.    swap A[root()] and A[ℓ]
3.    decrease size
4.    fix-down(A, size, root())
5.    return A[ℓ]
```

insert and *deleteMax*: $O(\log n)$

Outline

Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(n + n \cdot insert + n \cdot deleteMax)$$

- Using the binary-heaps implementation of PQs, we obtain:

PQsortWithHeaps(A)
1.   initialize H to an empty heap
2.   **for** $k \leftarrow 0$ **to** $n - 1$ **do**
3. $H.insert(A[k])$
4.   **for** $k \leftarrow n - 1$ **down to** 0 **do**
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- both operations run in $O(\log n)$ time for heaps

$\rightsquigarrow$ *PQ-Ssrt* using heaps takes $O(n \log n)$ time.

- Can improve this with two simple tricks $\rightarrow$ **Heapsort**
  1. Heaps can be built faster if we know all input in advance.
  2. Can use the same array for input and heap. $\rightsquigarrow O(1)$ *auxiliary space!*

Building Heaps with Fix-up

Problem: Given *n* items all at once (in $A[0 \cdots n-1]$) build a heap containing all of them.

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Solution 1: Start with an empty heap and insert items one at a time:

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simpleHeapBuilding(A)
A: an array
1.     initialize H as an empty heap
2.     for i ← 0 to size(A) − 1 do
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```

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This corresponds to doing *fix-ups*
Worst-case running time: $\Theta(n \log n)$.

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Solution 2: Using *fix-downs* instead:

```
heapify(A)
A: an array
1.    n ← A.size()
2.    for i ← parent(last(n)) downto 0 do
3.        fix-down(A, n, i)
```

Building Heaps with Fix-down

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A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

heapify example
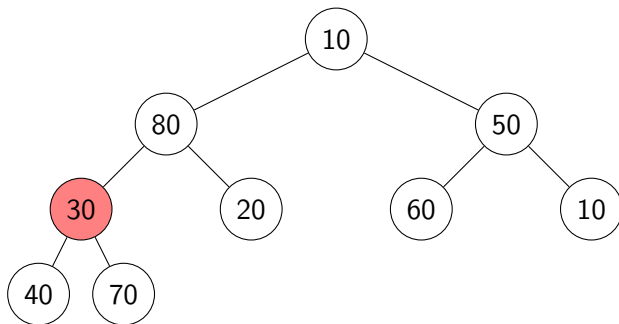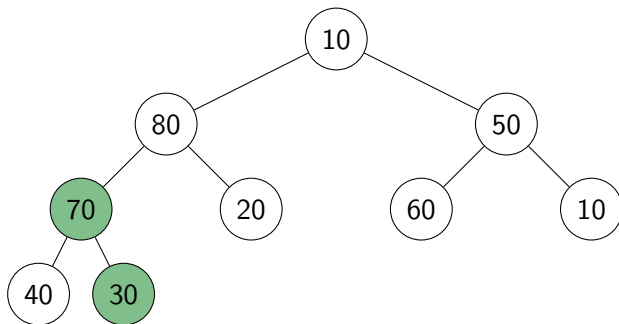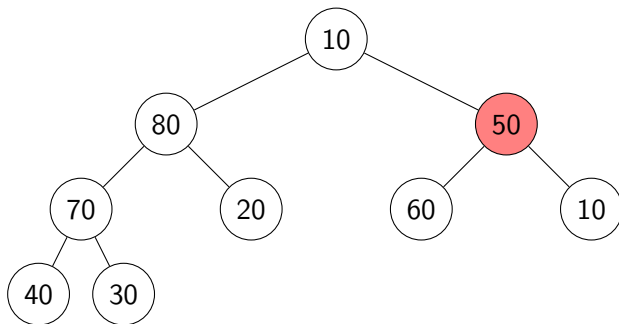
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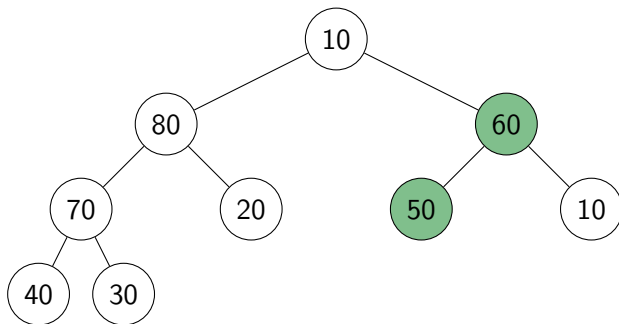
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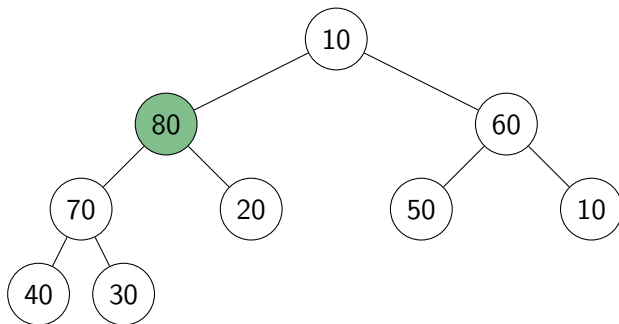
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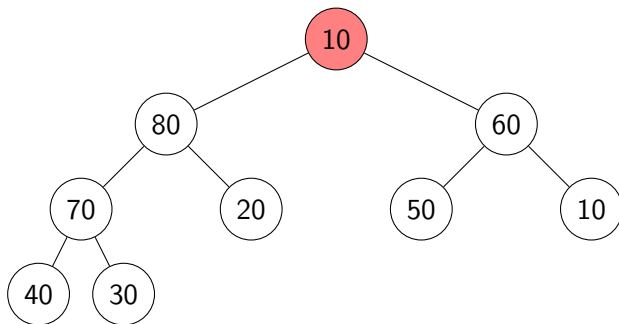
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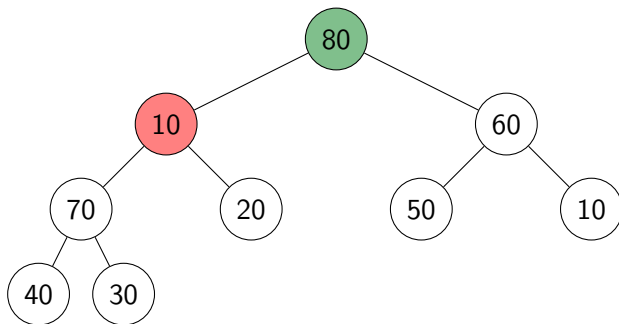
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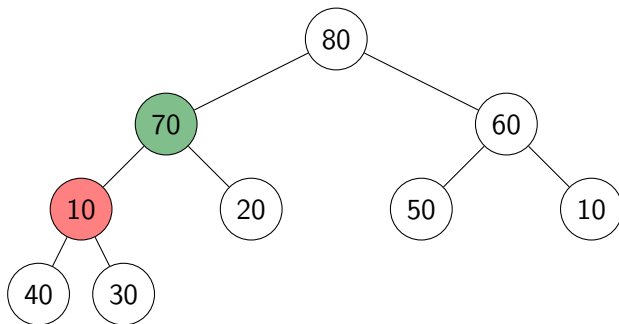
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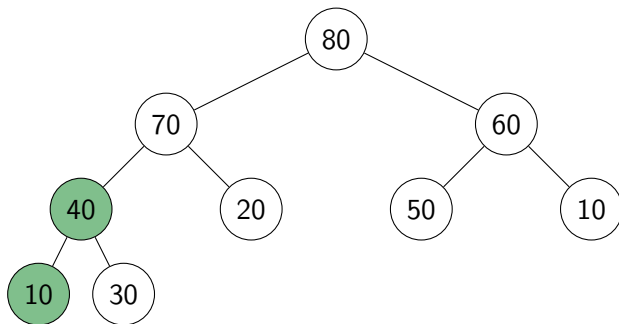
HeapSort

- Idea: *PQ-sort* with heaps.
- But: Use same input-array A for storing heap.

```
HeapSort(A, n)
1.    // heapify
2.    n ← A.size()
3.    for i ← parent(last(n)) downto 0 do
4.        fix-down(A, n, i)

5.    // repeatedly find maximum
6.    while n > 1
7.        // delete the maximum
8.        swap items at A[root()] and A[last(n)])
9.        decrease n
10.       fix-down(A, n, root())
```

The for-loop takes $\Theta(n)$ time and the while-loop takes $O(n \log n)$ time.
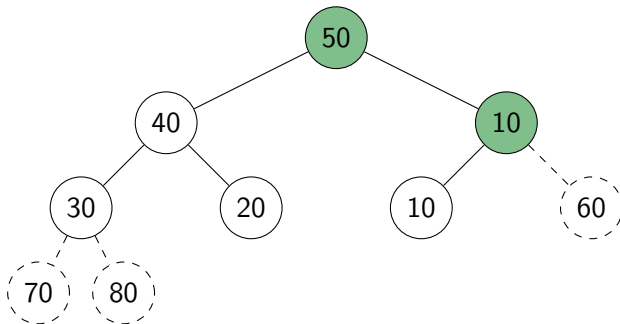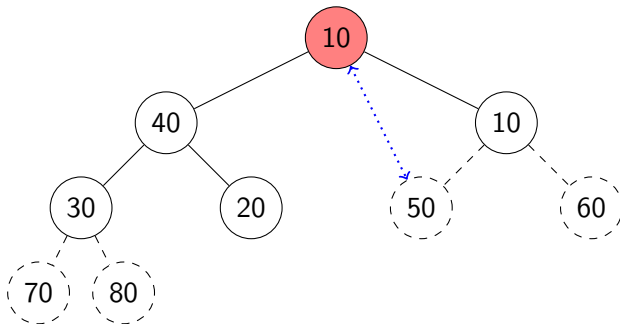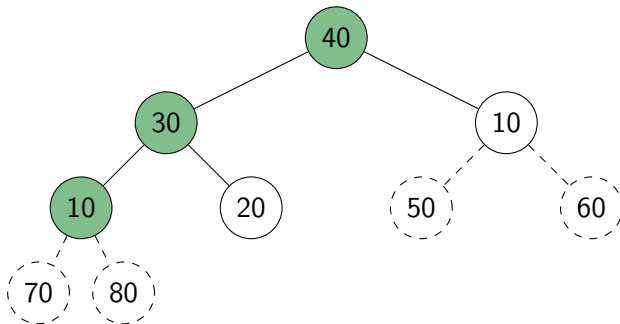
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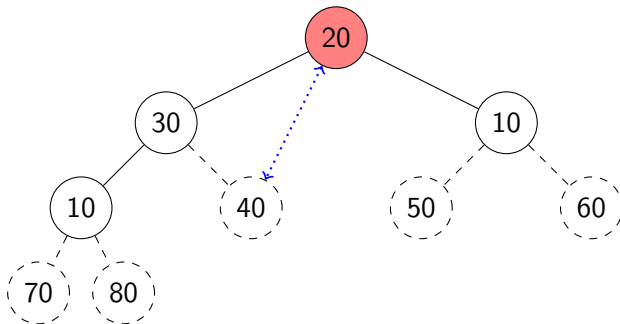
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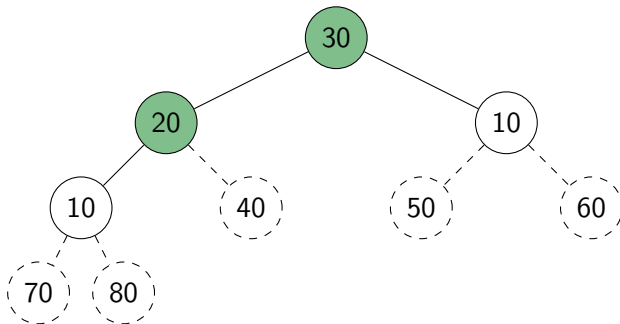
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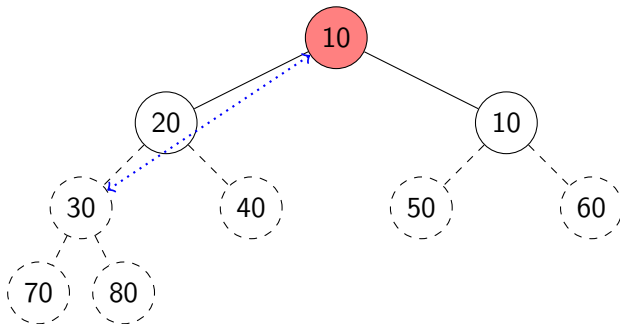Continue with the example from heapify:

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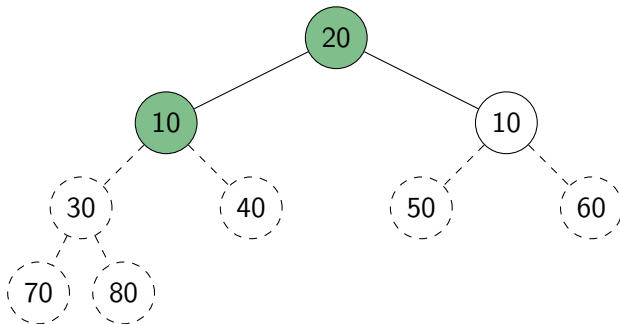# Heapsort example

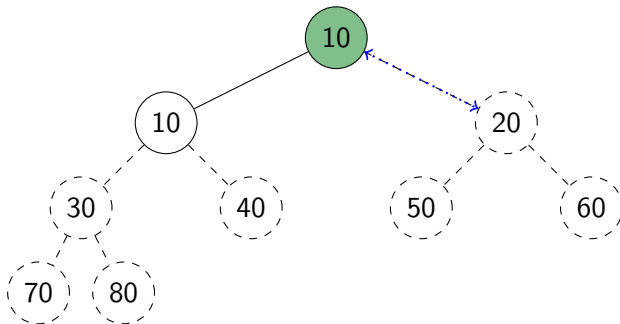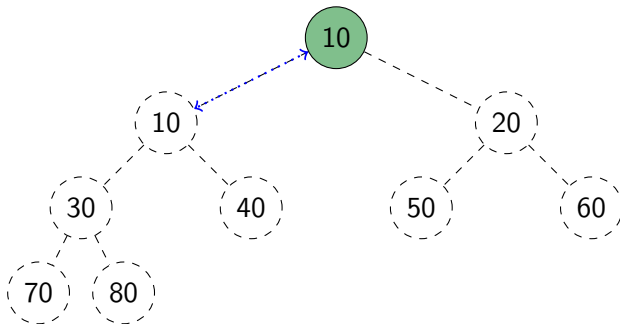Continue with the example from heapify:

Heapsort example

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The array (i.e., the heap in level-by-level order) is now in sorted order.

Heap summary

- **Binary heap**: A binary tree that satisfies structural property and heap-order property.
- Heaps are one possible realization of ADT PriorityQueue:
  - *insert* takes time $O(\log n)$
  - *deleteMax* takes time $O(\log n)$
  - Also supports *findMax* in time $O(1)$
- A binary heap can be built in linear time.
- *PQ-sort* with binary heaps leads to a sorting algorithm with $O(n \log n)$ worst-case run-time ($\rightsquigarrow$ *HeapSort*)
- We have seen here the *max-oriented version* of heaps (the maximum priority is at the root).
- There exists a symmetric *min-oriented version* that supports *insert* and *deleteMin* with the same run-times.

Outline

Finding the largest items

Problem: Find the *kth largest item* in an array A of n distinct numbers.

Solution 1: Make k passes through the array, deleting the maximum number each time.
Complexity: $\Theta(kn)$.

Solution 2: Sort A, then return $A[n-k]$.
Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the k largest numbers seen so far in a min-heap
Complexity: $\Theta(n \log k)$.

Solution 4: Create a max-heap with *heapify*(A). Call *deleteMax*(A) k times.
Complexity: $\Theta(n + k \log n)$.