

CS 240 – Data Structures and Data Management

Module 8: Range-Searching in Dictionaries for Points

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Based on lecture notes by many previous cs240 instructors

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Outline

① Range-Searching in Dictionaries for Points

- Range Searches
- Multi-Dimensional Data
- Quadtrees
- kd-Trees
- Range Trees
- Conclusion

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1 Range-Searching in Dictionaries for Points

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Range searches

- So far: *search*(k) looks for *one* specific item.
- New operation **RangeSearch**: look for *all* items that fall within a given range.
 - ▶ Input: A **range**, i.e., an interval $I = (x, x')$
It may be open or closed at the ends.
 - ▶ Want: Report all KVPs in the dictionary whose key k satisfies $k \in I$

Example:

5	10	11	17	19	33	45	51	55	59
---	----	----	----	----	----	----	----	----	----

RangeSearch((18,45]) should return {19, 33, 45}

- Let s be the **output-size**, i.e., the number of items in the range.
- We need $\Omega(s)$ time simply to report the items.
- Note that sometimes $s = 0$ and sometimes $s = n$; we therefore keep it as a separate parameter when analyzing the run-time.

Outline

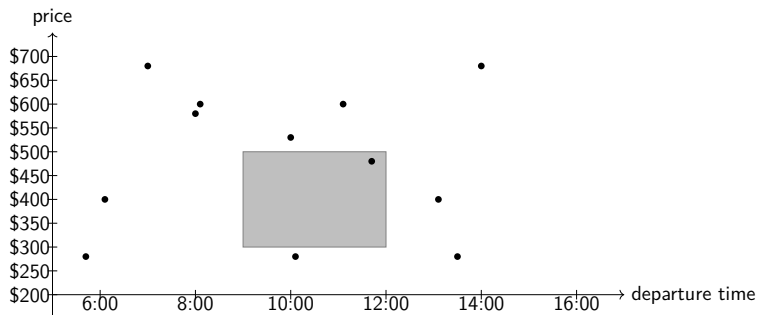
1 Range-Searching in Dictionaries for Points

- Range Searches
- **Multi-Dimensional Data**
- Quadtrees
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Multi-Dimensional Data

Range searches are of special interest for **multi-dimensional data**.

Example: flights that leave between 9am and noon, and cost \$300-\$500



- Each item has d **aspects** (coordinates): $(x_0, x_1, \dots, x_{d-1})$
- Aspect values (x_i) are numbers
- Each item corresponds to a point in d -dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane

Multi-dimensional Range Search

(Orthogonal) d -**dimensional range search**: Given a **query rectangle** A , find all points that lie within A .

The time for range searches depends on how the points are stored.

- Could store a 1-dimensional dictionary (where the key is some combination of the aspects.)
Problem: Range search on one aspect is not straightforward
- Could use one dictionary for each aspect
Problem: inefficient, wastes space
- **Better idea**: Design new data structures specifically for points.
 - ▶ Quadtrees
 - ▶ kd-trees
 - ▶ range-trees
- **Assumption**: Point are in **general position**:
No two x -coordinates or y -coordinates are the same.
 - ▶ Simplifies presentation; data structures can be generalized.

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Quadtrees

We have n points $S = \{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$ in the plane.

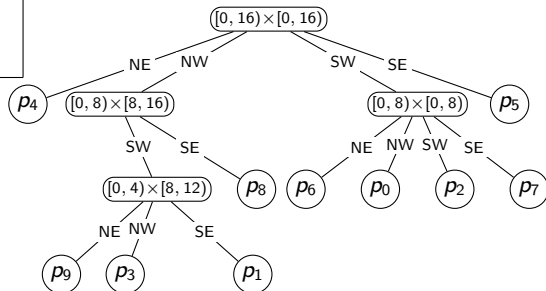
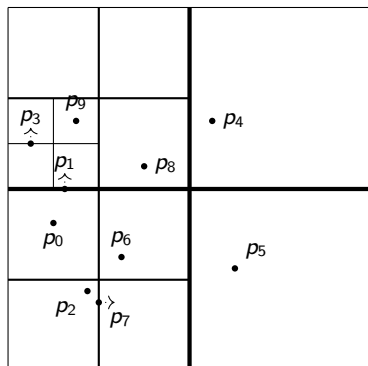
We need a **bounding box** R : a square containing all points.

- Can find R by computing minimum and maximum x and y values in S
- The width/height of R should be a power of 2

Structure (and also how to *build* the quadtree that stores S):

- Root r of the quadtree is associated with region R
- If R contains 0 or 1 points, then root r is a leaf that stores point.
- Else *split*: Partition R into four equal subsquares (**quadrants**)
 $R_{NE}, R_{NW}, R_{SW}, R_{SE}$
- Partition S into sets $S_{NE}, S_{NW}, S_{SW}, S_{SE}$ of points in these regions.
 - ▶ **Convention**: Points on split lines belong to right/top side
- Recursively build tree T_i for points S_i in region R_i and make them children of the root.

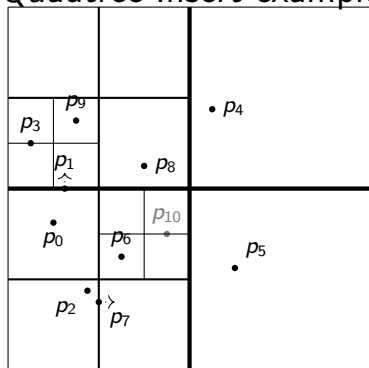
Quadtrees example



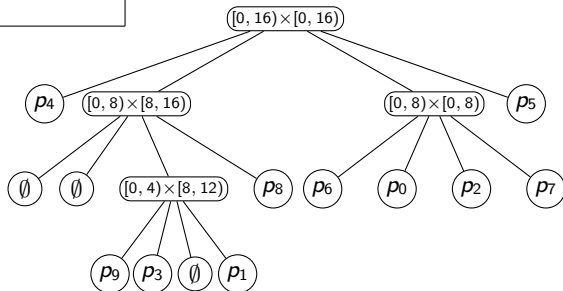
Quadtree Dictionary Operations

- *search*: Analogous to binary search trees and tries
- *insert*:
 - ▶ Search for the point
 - ▶ Split the leaf while there are two points in one region
- *delete*:
 - ▶ Search for the point
 - ▶ Remove the point
 - ▶ If its parent has only one point left: delete parent (and recursively all ancestors that have only one point left)

Quadtree Insert example



insert(p_{10})

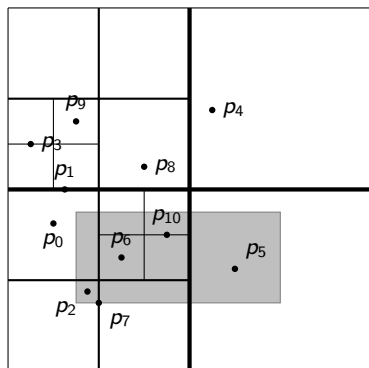


Quadtree Range Search

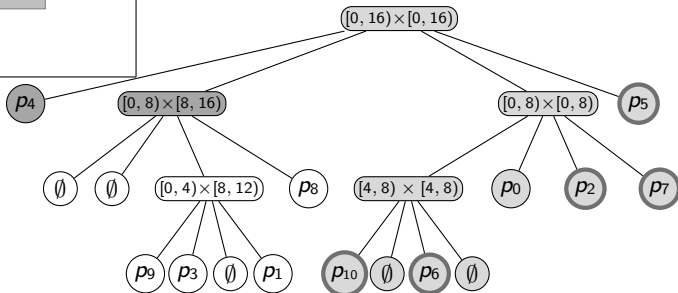
```
QTree::RangeSearch( $r \leftarrow \text{root}, A$ )
 $r$ : The root of a quadtree,  $A$ : Query-rectangle
1.    $R \leftarrow$  region associated with node  $r$ 
2.   if ( $R \subseteq A$ ) then           // inside node
3.       report all points below  $r$ ; return
4.   if ( $R \cap A$  is empty) then // outside node
5.       return
           // The node is a boundary node, recurse
6.   if ( $r$  is a leaf) then
7.        $p \leftarrow$  point stored at  $r$ 
8.       if  $p$  is in  $A$  return  $p$ 
9.       else return
10.  for each child  $v$  of  $r$  do
11.      QTree::RangeSearch( $v, A$ )
```

Note: We assume here that each node of the quadtree stores the associated square. Alternatively, these could be re-computed during the search (space-time tradeoff).

Quadtree range search example

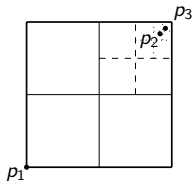


- Red: Search stopped due to $R \cap A = \emptyset$.
- Green: Search stopped due to $R \subseteq A$.
- Blue: Must continue search in children / evaluate.



Quadtree Analysis

- Crucial for analysis: what is the height of a quadtree?
 - ▶ Can have very large height for bad distributions of points



- ▶ **spread factor** of points S :

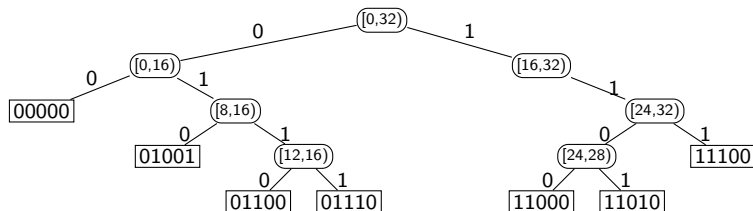
$$\beta(S) = \frac{\text{sidelength of } R}{\text{minimum distance between points in } S}$$

- ▶ Can show: height h of quadtree is in $\Theta(\log \beta(S))$
- Complexity to build initial tree: $\Theta(nh)$ worst-case
- Complexity of range search: $\Theta(nh)$ worst-case even if the answer is \emptyset
- But in practice much faster.

Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

“Points:” 0 9 12 14 24 26 28
(in base-2) 00000 01001 01100 01110 11000 11010 11100



Same as a trie (with splitting stopped once key is unique)

- Quadtrees also easily generalize to higher dimensions (octrees, *etc.*) but are rarely used beyond dimension 3.

Quadtree summary

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (bit-shift!) if the width/height of R is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation: We could stop splitting earlier and allow up to S points in a leaf (for some fixed bound S).
- Variation: Store pixelated images by splitting until each region has the same color.

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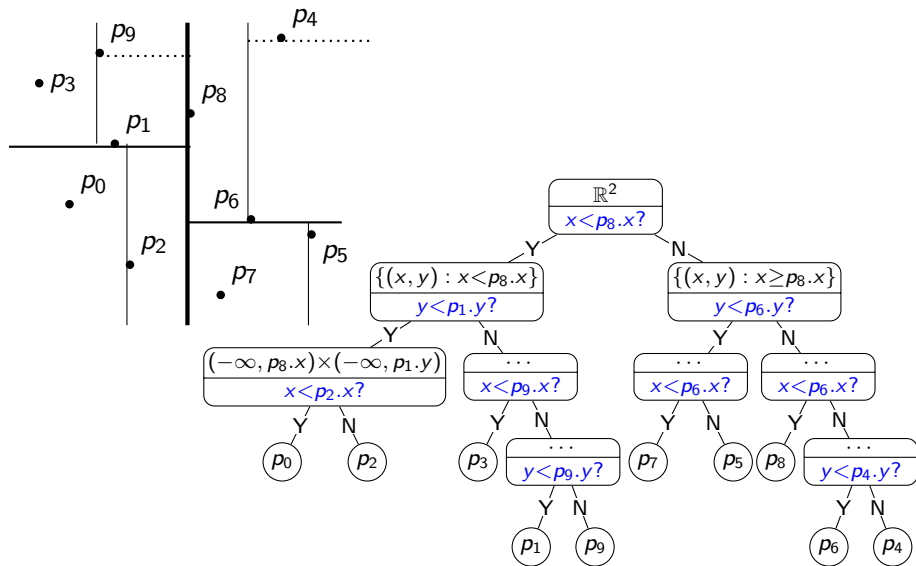
- Range Searches
- Multi-Dimensional Data
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- **kd-Trees**
- Range Trees
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kd-trees

- We have n points $S = \{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$
- Quadtrees split square into quadrants regardless of where points are
- (Point-based) kd-tree idea: Split the region such that (roughly) half the point are in each subtree
- Each node of the kd-tree keeps track of a **splitting line** in one dimension (2D: either vertical or horizontal)
- **Convention:** Points on split lines belong to right/top side
- Continue splitting, switching between vertical and horizontal lines, until every point is in a separate region

(There are alternatives, e.g., split by the dimension that has better aspect ratios for the resulting regions. No details.)

kd-tree example



For ease of drawing, we will usually not show the associated regions.

Constructing kd-trees

Build kd-tree with initial split by x on points S :

- If $|S| \leq 1$ create a leaf and return.
- Else $X := \text{quick-select}(S, \lfloor \frac{n}{2} \rfloor)$ (select by x -coordinate)
- Partition S by x -coordinate into $S_{x < X}$ and $S_{x \geq X}$
 - ▶ $\lfloor \frac{n}{2} \rfloor$ points on one side and $\lceil \frac{n}{2} \rceil$ points on the other.
(Recall: Points in general position.)
- Create left subtree recursively (splitting by y) for points $S_{x < X}$.
- Create right subtree recursively (splitting by y) for points $S_{x \geq X}$.

Building with initial y -split symmetric.

Constructing kd-trees

Run-time:

- Find X and partition S in $\Theta(n)$ expected time using *randomized-quick-select*.
- Both subtrees have $\approx n/2$ points.

$$T^{\text{exp}}(n) = 2T^{\text{exp}}(n/2) + O(n) \quad (\text{sloppy recurrence})$$

This resolves to $\Theta(n \log n)$ expected time.

- This can be reduced to $\Theta(n \log n)$ *worst-case* time by pre-sorting (no details).

Height: $h(1) = 0$, $h(n) \leq h(\lceil n/2 \rceil) + 1$.

- This resolves to $O(\log n)$ (specifically $\lceil \log n \rceil$).

kd-tree Dictionary Operations

- *search* (for single point): as in binary search tree using indicated coordinate
- *insert*: search, insert as new leaf.
- *delete*: search, remove leaf.

Problem: After insert or delete, the split might no longer be at exact median and the height is no longer guaranteed to be $\lceil \log_2 n \rceil$.

We can maintain $O(\log n)$ height by occasionally re-building entire subtrees. (No details.) But *rangeSearch* will be slower.

kd-trees do not handle insertion/deletion well.

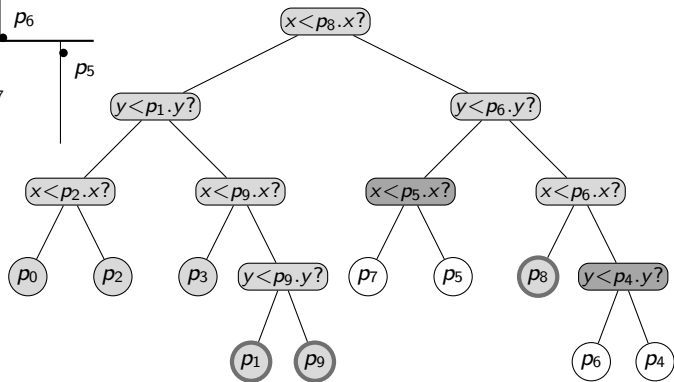
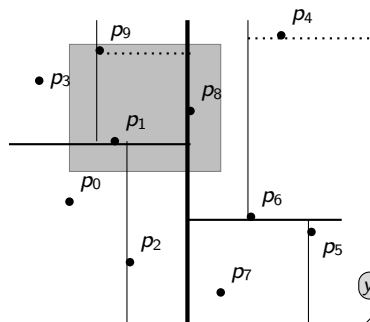
kd-tree Range Search

- Range search is *exactly* as for quad-trees, except that there are only two children.

```
kdTree::RangeSearch( $r \leftarrow \text{root}, A$ )  
 $r$ : The root of a kd-tree,  $A$ : Query-rectangle  
1.  $R \leftarrow$  region associated with node  $r$   
2. if ( $R \subseteq A$ ) then report all points below  $r$ ; return  
3. if ( $R \cap A$  is empty) then return  
4. if ( $r$  is a leaf) then  
5.      $p \leftarrow$  point stored at  $r$   
6.     if  $p$  is in  $A$  return  $p$   
7.     else return  
8. for each child  $v$  of  $r$  do  
9.     kdTree::RangeSearch( $v, A$ )
```

- We assume again that each node stores its associated region.
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line.

kd-tree: Range Search Example



Red: Search stopped due to $R \cap A = \emptyset$. Green: Search stopped due to $R \subseteq A$.

kd-tree: Range Search Complexity

- The complexity is $O(s + Q(n))$ where
 - ▶ s is the output-size
 - ▶ $Q(n)$ is the number of “boundary” nodes (blue):
 - ★ *kdTree::RangeSearch* was called.
 - ★ Neither $R \subseteq A$ nor $R \cap A = \emptyset$
- **Can show:** $Q(n)$ satisfies the following recurrence relation (no details):

$$Q(n) \leq 2Q(n/4) + O(1)$$

- This solves to $Q(n) \in O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(s + \sqrt{n})$

kd-tree: Higher Dimensions

- kd-trees for d -dimensional space:
 - ▶ At the root the point set is partitioned based on the first coordinate
 - ▶ At the subtrees of the root the partition is based on the second coordinate
 - ▶ At depth $d - 1$ the partition is based on the last coordinate
 - ▶ At depth d we start all over again, partitioning on first coordinate
- **Storage:** $O(n)$
- **Height:** $O(\log n)$
- **Construction time:** $O(n \log n)$
- **Range search time:** $O(s + n^{1-1/d})$

This assumes that d is a constant.

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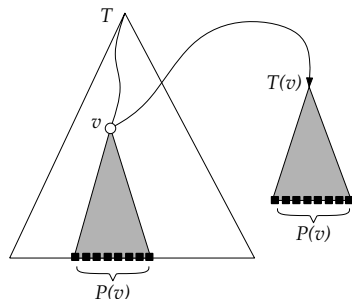
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Towards Range Trees

- Both Quadtrees and kd-trees are intuitive and simple.
- But: both may be very slow for range searches.
- Quadtrees are also potentially wasteful in space.

New idea: **Range trees**

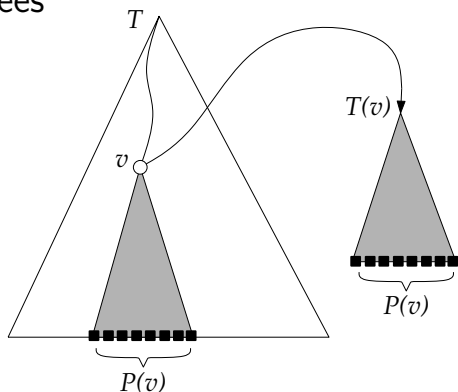


- Somewhat wasteful in space, but much faster range search.
- **Tree of trees** (a *multi-level* data structure)

2-dimensional Range Trees

Primary structure:

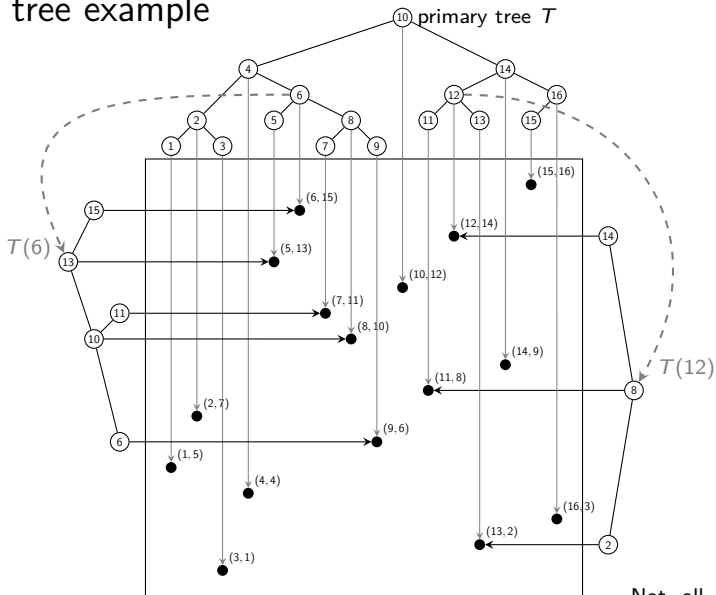
Balanced binary search tree T that stores P and uses x -coordinates as keys.



Each node v of T stores an **associate structure** $T(v)$:

- Let $P(v)$ be all points in subtree of v in T (including point at v)
- $T(v)$ stores $P(v)$ in a balanced binary search tree, using the y -coordinates as key
- Note: v is not necessarily the root of $T(v)$

Range tree example



Range Tree Space Analysis

- Primary tree uses $O(n)$ space.
- Associate tree $T(v)$ uses $O(|P(v)|)$ space
(where $P(v)$ are the points at descendants of v in T)
- **Key insight:** $w \in P(v)$ means that v is an ancestor of w in T
 - ▶ Every node w has $O(\log n)$ ancestors in T
(Recall that we assume T to be balanced.)
 - ▶ Every node w belongs to $O(\log n)$ sets $P(v)$
 - ▶ So $\sum_v |P(v)| \leq \sum_w \#\{\text{ancestors of } w\} \in O(n \log n)$

Therefore: A range-tree with n points uses $O(n \log n)$ space.

Range Trees Operations

- *search*: search by x -coordinate in T
- *insert*: First, insert point by x -coordinate into T .
Then, walk back up to the root and insert the point by y -coordinate in *all* associate trees $T(v)$ of nodes v on path to the root.
- *delete*: analogous to insertion
- **Problem**: We want the binary search trees to be balanced.
 - ▶ This makes *insert/delete* very slow if we use AVL-trees.
(A rotation at v changes $P(v)$ and hence requires a re-build of $T(v)$.)
 - ▶ **Solution**: Completely rebuild highly unbalanced subtrees (no details)
- *range-search*: search by x -range in T .
Among found points, search by y -range in some associated trees.
- Must understand first: How to do (1-dimensional) range search in binary search tree?

BST Range Search

BST::RangeSearch($r \leftarrow \text{root}, x_1, x_2$)

r : root of a binary search tree, x_1, x_2 : search keys

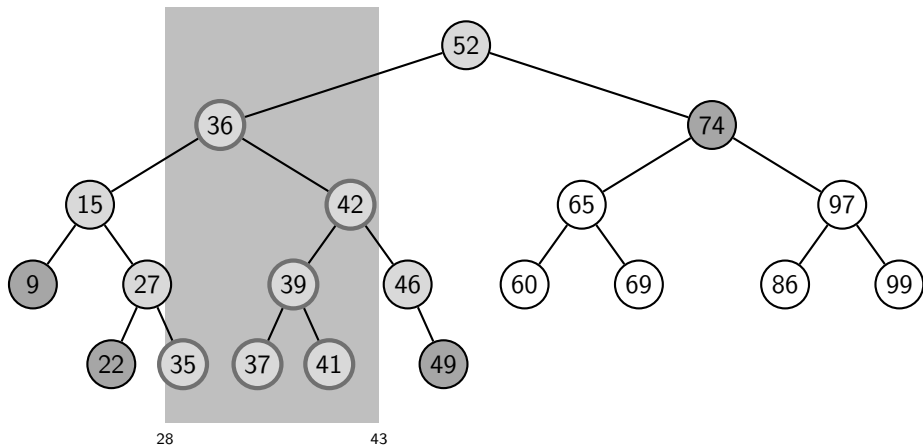
Returns keys in subtree at r that are in range $[x_1, x_2]$

1. **if** $r = \text{NIL}$ **then return**
2. **if** $x_1 \leq r.\text{key} \leq x_2$ **then**
 3. $L \leftarrow \text{BST::RangeSearch}(r.\text{left}, x_1, x_2)$
 4. $R \leftarrow \text{BST::RangeSearch}(r.\text{right}, x_1, x_2)$
 5. **return** $L \cup r.\{\text{key}\} \cup R$
6. **if** $r.\text{key} < x_1$ **then**
 7. **return** $\text{BST::RangeSearch}(r.\text{right}, x_1, x_2)$
8. **if** $r.\text{key} > x_2$ **then**
 9. **return** $\text{BST::RangeSearch}(r.\text{left}, x_1, x_2)$

Keys are reported in in-order, i. e., in sorted order.

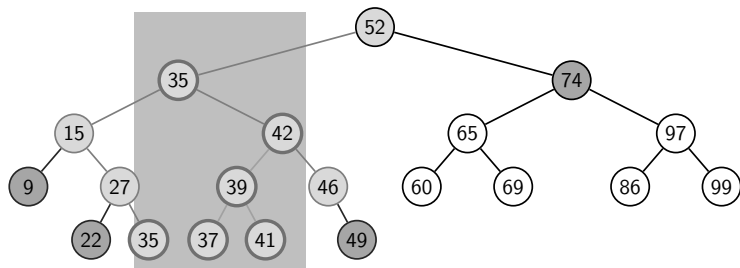
BST Range Search example

$BST::RangeSearch(T, 28, 43)$



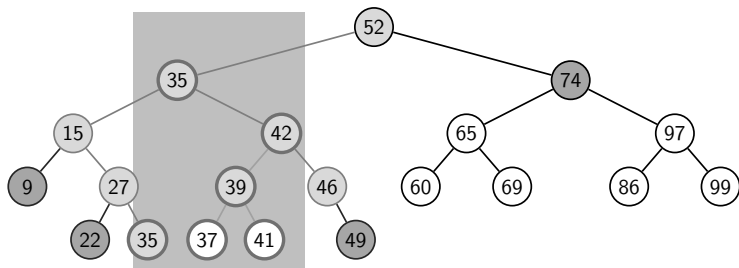
Note: Search from 39 was unnecessary: *all* its descendants are in range.

BST Range Search re-phrased



- Search for left boundary x_1 : this gives path P_1
- Search for right boundary x_2 : this gives path P_2
- This partitions T into three groups: outside, on, or between the paths.

BST Range Search re-phrased

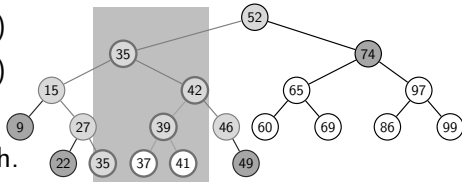


- boundary nodes: nodes in P_1 or P_2
 - ▶ For each boundary node, test whether it is in the range.
- outside nodes: nodes that are left of P_1 or right of P_2
 - ▶ These are *not* in the range, we stop the search at the topmost.
- inside nodes: nodes that are right of P_1 and left of P_2
 - ▶ We stop the search at the topmost inside node.
 - ▶ All descendants of such a node are *in* the range.
For a 1d range search, report them.

BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path P_1 : $O(\log n)$
- Search for path P_2 : $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.



- We spend $O(1)$ time per topmost outside node.
 - ▶ They are children of boundary nodes, so this takes $O(\log n)$ time.
- We spend $O(1)$ time per topmost inside node v .
 - ▶ They are children of boundary nodes, so this takes $O(\log n)$ time.
- For 1d range search, also report the descendants of v .
 - ▶ We have $\sum_v \text{topmost inside } \#\{\text{descendants of } v\} \leq s$ since subtrees of topmost inside nodes are disjoint. So this takes time $O(s)$ overall.

Run-time for 1d range search: $O(\log n + s)$. This is no faster overall, but topmost inside nodes will be important for 2d range search.

Range Trees: Range Search

Range search for $A = [x_1, x_2] \times [y_1, y_2]$ is a two stage process:

- Perform a range search (on the x -coordinates) for the interval $[x_1, x_2]$ in primary tree T ($BST::RangeSearch(T, x_1, x_2)$)
- Get boundary, topmost outside and topmost inside nodes as before.
- For every boundary node, test to see if the corresponding point is within the region A .
- For every topmost inside node v :
 - ▶ Let $P(v)$ be the points in the subtree of v in T .
 - ▶ We know that all x -coordinates of points in $P(v)$ are within range.
 - ▶ Recall: $P(v)$ is stored in $T(v)$.
 - ▶ To find points in $P(v)$ where the y -coordinates are within range as well, perform a range search in $T(v)$: $BST::RangeSearch(T(v), y_1, y_2)$

Range Trees: Range Search Run-time

- $O(\log n)$ time to find boundary and topmost inside nodes in primary tree.
- There are $O(\log n)$ such nodes.
- $O(\log n + s_v)$ time for each topmost inside node v , where s_v is the number of points in $T(v)$ that are reported
- Two topmost inside nodes have no common point in their trees
⇒ every point is reported in at most one associate structure
⇒ $\sum_v \text{topmost inside } s_v \leq s$

Time for range search in range-tree is proportional to

$$\sum_{v \text{ topmost inside}} (\log n + s_v) \in O(\log^2 n + s)$$

(There are ways to make this even faster. No details.)

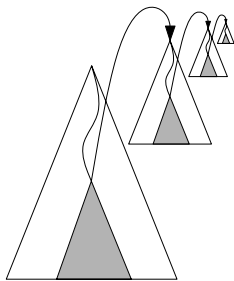
Range Trees: Higher Dimensions

- Range trees can be generalized to d -dimensional space.

Space	$O(n(\log n)^{d-1})$	kd-trees: $O(n)$
Construction time	$O(n(\log n)^d)$	kd-trees: $O(n \log n)$
Range search time	$O(s + (\log n)^d)$	kd-trees: $O(s + n^{1-1/d})$

(Note: d is considered to be a constant.)

- Space/time trade-off compared to kd-trees.



Outline

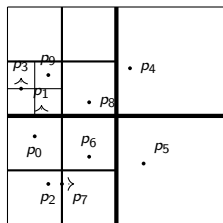
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Range search data structures summary

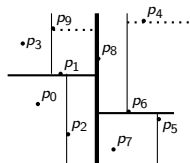
- Quadtrees

- ▶ simple (also for dynamic set of points)
- ▶ work well only if points evenly distributed
- ▶ wastes space for higher dimensions



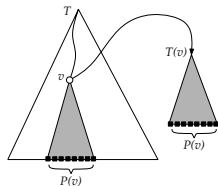
- kd-trees

- ▶ linear space
- ▶ range search time $O(\sqrt{n} + s)$
- ▶ inserts/deletes destroy balance and range search time (no simple fix)



- range-trees

- ▶ range search time $O(\log^2 n + s)$
- ▶ wastes some space
- ▶ inserts/deletes destroy balance (can fix this with occasional rebuild)



Convention: Points on split lines belong to right/top side.