

CS 240 – Data Structures and Data Management

Module 2: Priority Queues

M. Petrick V. Sakhnini O. Veksler

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Spring 2021

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

Outline

1 Priority Queues

- Abstract Data Types
 - ADT Priority Queue
 - Binary Heaps
 - Operations in Binary Heaps
 - *PQ-sort* and *Heapsort*
 - Towards the Selection Problem

Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various **realizations** of an ADT, which specify:

- How the information is stored (**data structure**)
- How the operations are performed (**algorithms**)

Stack ADT

Stack: an ADT consisting of a collection of items with operations:

- *push*: inserting an item
- *pop*: removing (and typically returning) the most recently inserted item

Items are removed in LIFO (*last-in first-out*) order.

Items enter the stack at the *top* and are removed from the *top*.

We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited web sites, procedure calls

Realizations of Stack ADT

- using arrays
- using linked lists

Queue ADT

Queue: an ADT consisting of a collection of items with operations:

- *enqueue*: inserting an item
- *dequeue*: removing (and typically returning) the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order.

Items enter the queue at the *rear* and are removed from the *front*.

We can have extra operations: *size*, *isEmpty*, and *front*

Applications: Waiting lines, printer queues

Realizations of Queue ADT

- using (circular) arrays
- using linked lists

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a **priority**) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing and returning the item of *highest* priority

deleteMax is also called *extractMax* or *getmax*.

The priority is also called *key*.

The above definition is for a **maximum-oriented** priority queue. A **minimum-oriented** priority queue is defined in the natural way, replacing operation *deleteMax* by *deleteMin*,

Applications: typical “todo” list, simulation systems, sorting

Using a Priority Queue to Sort

PQ-Sort($A[0..n-1]$)

1. initialize *PQ* to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. *PQ.insert*($A[i]$)
4. **for** $i \leftarrow n - 1$ **down to** 0 **do**
5. $A[i] \leftarrow$ *PQ.deleteMax*()

- Note: Run-time depends on how we implement the priority queue.
- Sometimes written as: $O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$

Realizations of Priority Queues

Realization 1: unsorted arrays

Realizations of Priority Queues

Realization 1: unsorted arrays

- *insert*: $O(1)$
- *deleteMax*: $O(n)$

Note: We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes $O(1)$ extra time.)

Realizations of Priority Queues

Realization 1: unsorted arrays

- *insert*: $O(1)$
- *deleteMax*: $O(n)$

Note: We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes $O(1)$ extra time.)

Using unsorted linked lists is identical.

PQ-sort with this realization yields *selection sort*.

Realizations of Priority Queues

Realization 1: unsorted arrays

- *insert*: $O(1)$
- *deleteMax*: $O(n)$

Note: We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes $O(1)$ extra time.)

Using unsorted linked lists is identical.

PQ-sort with this realization yields *selection sort*.

Realization 2: sorted arrays

Realizations of Priority Queues

Realization 1: unsorted arrays

- *insert*: $O(1)$
- *deleteMax*: $O(n)$

Note: We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes $O(1)$ extra time.)

Using unsorted linked lists is identical.

PQ-sort with this realization yields *selection sort*.

Realization 2: sorted arrays

- *insert*: $O(n)$
- *deleteMax*: $O(1)$

Using sorted linked lists is identical.

PQ-sort with this realization yields *insertion sort*.

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

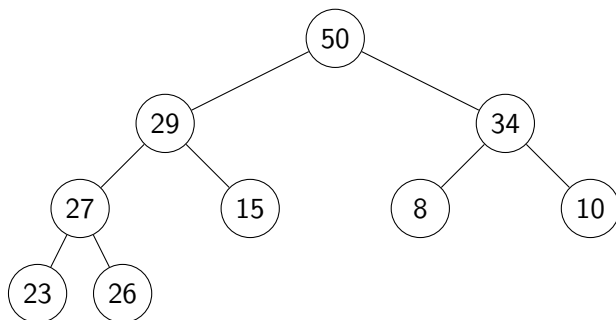
Realization 3: Heaps

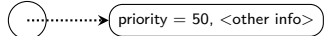
A **(binary) heap** is a certain type of binary tree.

You should know:

- A **binary tree** is either
 - ▶ empty, or
 - ▶ consists of three parts:
a node and two binary trees (left subtree and right subtree).
- Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
- Any binary tree with n nodes has height at least $\log(n + 1) - 1 \in \Omega(\log n)$.

Example Heap



(In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be )

Heaps – Definition

A **heap** is a binary tree with the following two properties:

- 1 **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
- 2 **Heap-order Property:** For any node i , the key of the parent of i is larger than or equal to key of i .

Heaps – Definition

A **heap** is a binary tree with the following two properties:

- 1 **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
- 2 **Heap-order Property:** For any node i , the key of the parent of i is larger than or equal to key of i .

The full name for this is *max-oriented binary heap*.

Heaps – Definition

A **heap** is a binary tree with the following two properties:

- 1 **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
- 2 **Heap-order Property:** For any node i , the key of the parent of i is larger than or equal to key of i .

The full name for this is *max-oriented binary heap*.

Lemma: The height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

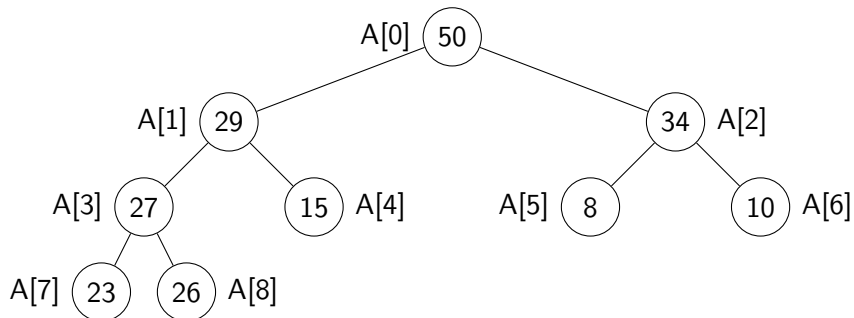
Heaps should *not* be stored as binary trees!

Let H be a heap of n items and let A be an array of size n . Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

Storing Heaps in Arrays

Heaps should *not* be stored as binary trees!

Let H be a heap of n items and let A be an array of size n . Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.



Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the *root* node is at index 0
(We use “node” and “index” interchangeably in this implementation.)
- the *last* node is $n - 1$ (where n is the size)
- the *left child* of node i (if it exists) is node $2i + 1$
- the *right child* of node i (if it exists) is node $2i + 2$
- the *parent* of node i (if it exists) is node $\lfloor \frac{i-1}{2} \rfloor$
- these nodes exist if the index falls in the range $\{0, \dots, n-1\}$

Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the *root* node is at index 0
(We use “node” and “index” interchangeably in this implementation.)
- the *last* node is $n - 1$ (where n is the size)
- the *left child* of node i (if it exists) is node $2i + 1$
- the *right child* of node i (if it exists) is node $2i + 2$
- the *parent* of node i (if it exists) is node $\lfloor \frac{i-1}{2} \rfloor$
- these nodes exist if the index falls in the range $\{0, \dots, n-1\}$

We should hide implementation details using helper-functions!

- functions *root()*, *last()*, *parent(i)*, etc.

Some of these helper-functions need to know n (but we omit this in the code for simplicity).

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- **Operations in Binary Heaps**
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

Insert in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *fix-up*:

Insert in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *fix-up*:

fix-up(A, i)

i : an index corresponding to a node of the heap

1. **while** $\text{parent}(i)$ exists **and** $A[\text{parent}(i)].\text{key} < A[i].\text{key}$ **do**
2. swap $A[i]$ and $A[\text{parent}(i)]$
3. $i \leftarrow \text{parent}(i)$

The new item “bubbles up” until it reaches its correct place in the heap.

Insert in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *fix-up*:

fix-up(A, i)

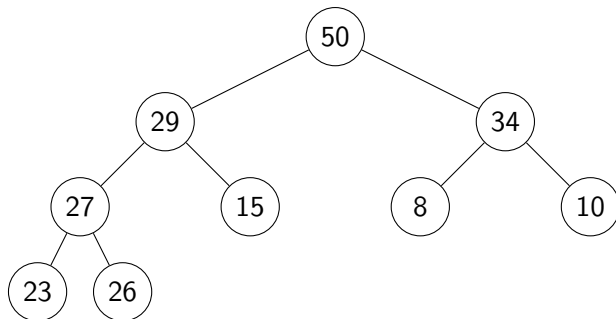
i : an index corresponding to a node of the heap

1. **while** $\text{parent}(i)$ exists **and** $A[\text{parent}(i)].\text{key} < A[i].\text{key}$ **do**
2. swap $A[i]$ and $A[\text{parent}(i)]$
3. $i \leftarrow \text{parent}(i)$

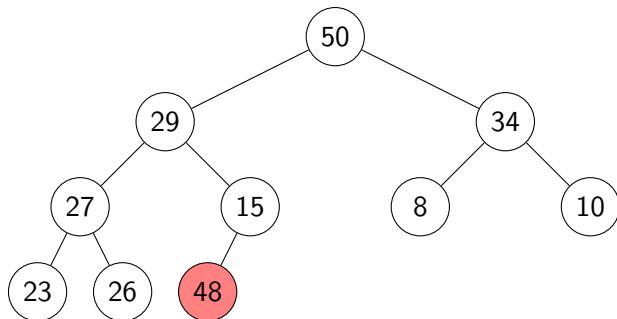
The new item “bubbles up” until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$.

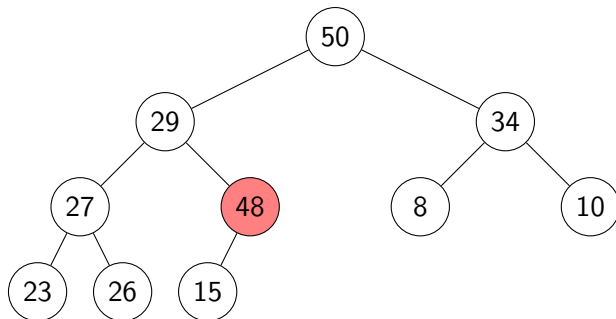
fix-up example



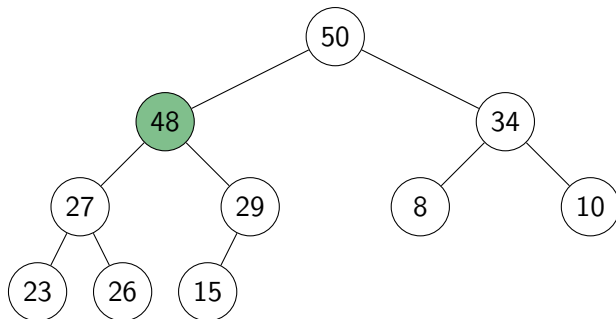
fix-up example



fix-up example



fix-up example



deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *fix-down*:

deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *fix-down*:

fix-down($A, i, n \leftarrow A.size$)

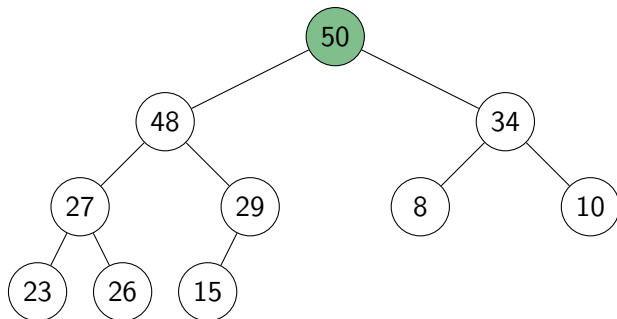
A : an array that stores a heap of size n

i : an index corresponding to a node of the heap

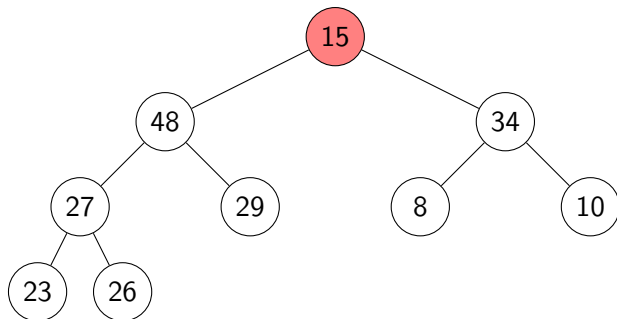
1. **while** i is not a leaf **do**
2. $j \leftarrow$ left child of i // Find the child with the larger key
3. if (i has right child and $A[\text{right child of } i].key > A[j].key$)
4. $j \leftarrow$ right child of i
5. **if** $A[i].key \geq A[j].key$ **break**
6. swap $A[j]$ and $A[i]$
7. $i \leftarrow j$

Time: $O(\text{height of heap}) = O(\log n)$.

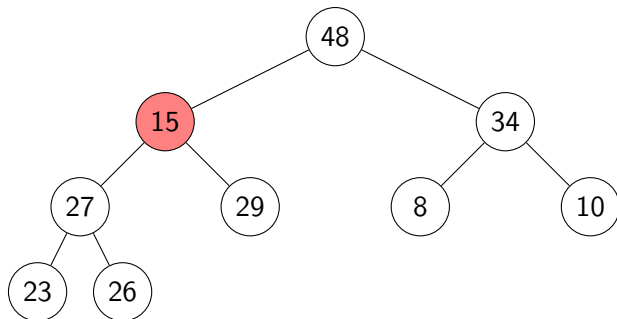
deleteMax example



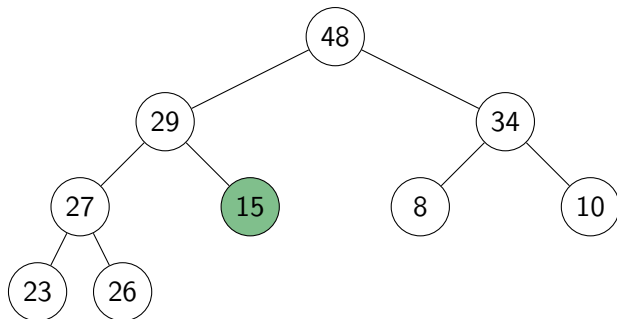
deleteMax example



deleteMax example



deleteMax example



Priority Queue Realization Using Heaps

- Store items in array A and globally keep track of $size$

insert(x)

1. increase $size$
2. $\ell \leftarrow last()$
3. $A[\ell] \leftarrow x$
4. *fix-up*(A, ℓ)

deleteMax()

1. $\ell \leftarrow last()$
2. swap $A[root()]$ and $A[\ell]$
3. decrease $size$
4. *fix-down*($A, root(), size$)
5. **return** $A[\ell]$

insert and *deleteMax*: $O(\log n)$ **time**

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$$

- Using the binary-heaps implementation of PQs, we obtain:

```
PQsortWithHeaps(A)
```

1. initialize H to an empty heap
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. $H.\textit{insert}(A[i])$
4. **for** $i \leftarrow n - 1$ **down to** 0 **do**
5. $A[i] \leftarrow H.\textit{deleteMax}()$

Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$$

- Using the binary-heaps implementation of PQs, we obtain:

```
PQsortWithHeaps(A)
1.  initialize H to an empty heap
2.  for i ← 0 to n - 1 do
3.      H.insert(A[i])
4.  for i ← n - 1 down to 0 do
5.      A[i] ← H.deleteMax()
```

- both operations run in $O(\log n)$ time for heaps

↪ *PQ-Sort* using heaps takes $O(n \log n)$ time.

- Can improve this with two simple tricks → **Heapsort**

- Heaps can be built faster if we know all input in advance.
- Can use the same array for input and heap. ↪ $O(1)$ auxiliary space!

Building Heaps with Fix-up

Problem: Given n items all at once (in $A[0 \dots n - 1]$) build a heap containing all of them.

Building Heaps with Fix-up

Problem: Given n items all at once (in $A[0 \dots n - 1]$) build a heap containing all of them.

Solution 1: Start with an empty heap and insert items one at a time:

```
simpleHeapBuilding( $A$ )
```

A : an array

1. initialize H as an empty heap
2. **for** $i \leftarrow 0$ **to** $A.size() - 1$ **do**
3. $H.insert(A[i])$

Building Heaps with Fix-up

Problem: Given n items all at once (in $A[0 \dots n - 1]$) build a heap containing all of them.

Solution 1: Start with an empty heap and insert items one at a time:

```
simpleHeapBuilding(A)
```

A: an array

1. initialize H as an empty heap
2. **for** $i \leftarrow 0$ **to** $A.size() - 1$ **do**
3. $H.insert(A[i])$

This corresponds to doing *fix-ups*

Worst-case running time: $\Theta(n \log n)$.

Building Heaps with Fix-down

Problem: Given n items all at once (in $A[0 \dots n - 1]$) build a heap containing all of them.

Building Heaps with Fix-down

Problem: Given n items all at once (in $A[0 \dots n - 1]$) build a heap containing all of them.

Solution 2: Using *fix-downs* instead:

```
heapify(A)
```

```
A: an array
```

1. $n \leftarrow A.size()$
2. **for** $i \leftarrow \textit{parent}(\textit{last}())$ **downto** $\textit{root}()$ **do**
3. $\textit{fix-down}(A, i, n)$

Building Heaps with Fix-down

Problem: Given n items all at once (in $A[0 \dots n - 1]$) build a heap containing all of them.

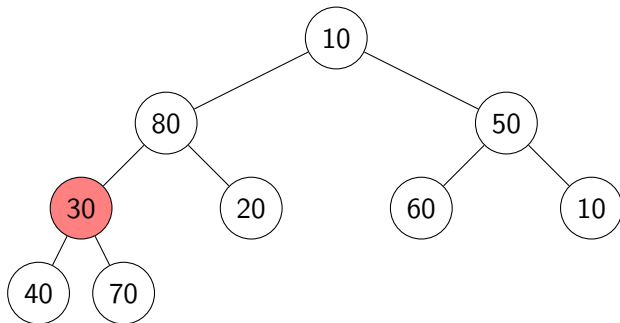
Solution 2: Using *fix-downs* instead:

```
heapify(A)
A: an array
1.    $n \leftarrow A.size()$ 
2.   for  $i \leftarrow \textit{parent}(\textit{last}())$  downto  $\textit{root}()$  do
3.       fix-down(A,  $i$ ,  $n$ )
```

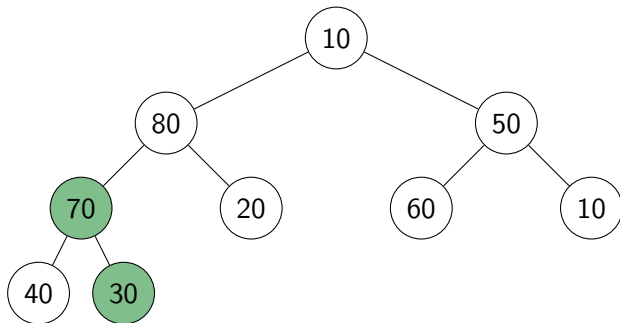
A careful analysis yields a worst-case complexity of $\Theta(n)$.

A heap can be built in linear time.

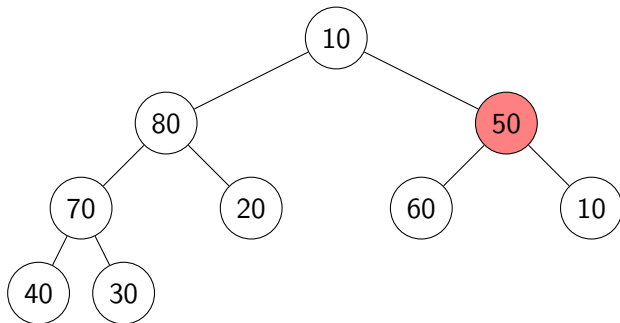
heapify example



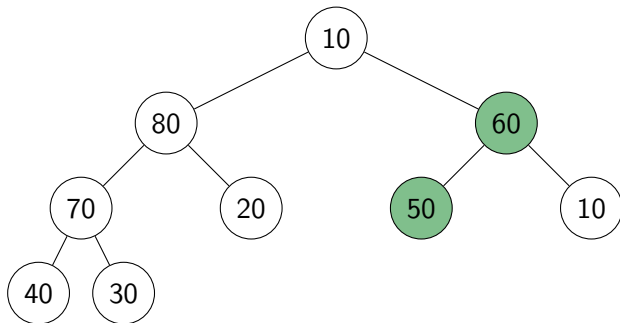
heapify example



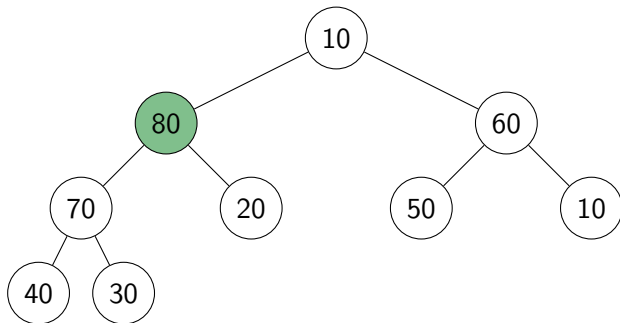
heapify example



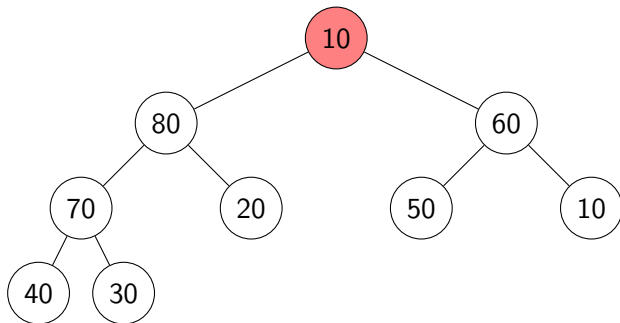
heapify example



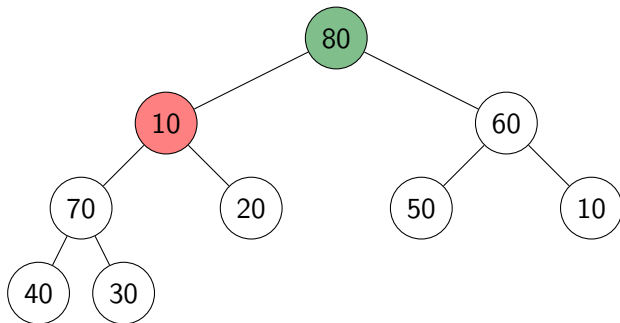
heapify example



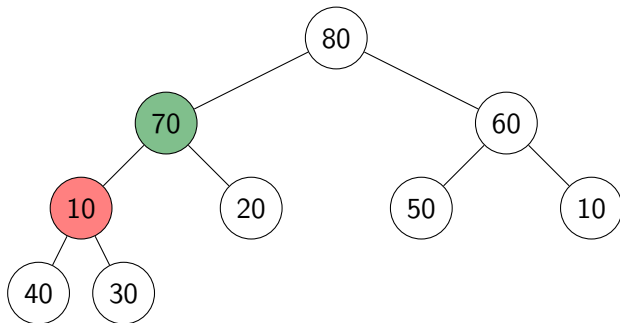
heapify example



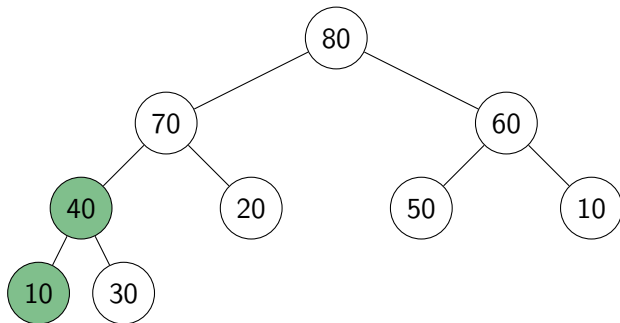
heapify example



heapify example



heapify example



HeapSort

- Idea: *PQ-sort* with heaps.
- $O(1)$ auxiliary space: Use same input-array A for storing heap.

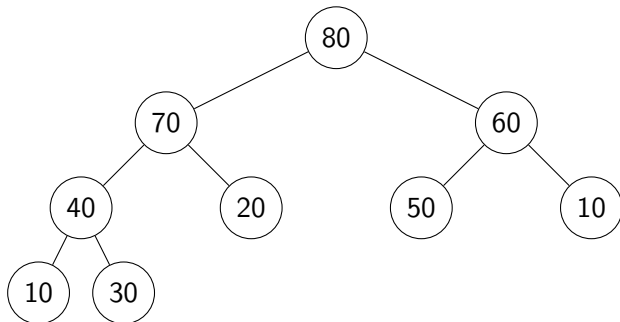
```
HeapSort( $A, n$ )
1. // heapify
2.  $n \leftarrow A.size()$ 
3. for  $i \leftarrow parent(last())$  downto 0 do
4.     fix-down( $A, i, n$ )

5. // repeatedly find maximum
6. while  $n > 1$ 
7.     // 'delete' maximum by moving to end and decreasing  $n$ 
8.     swap items at  $A[root()]$  and  $A[last()]$ 
9.     decrease  $n$ 
10.    fix-down( $A, root(), n$ )
```

The for-loop takes $\Theta(n)$ time and the while-loop takes $O(n \log n)$ time.

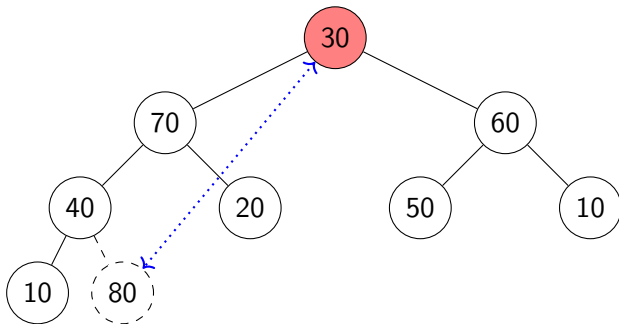
Heapsort example

Continue with the example from heapify:



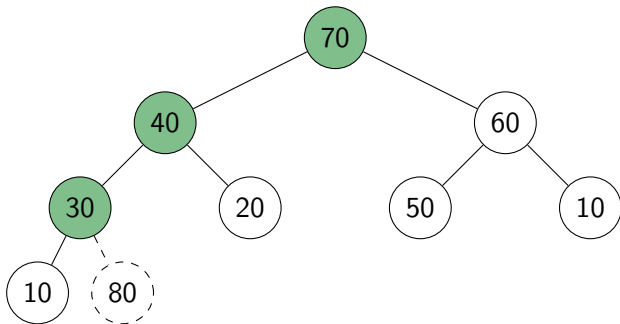
Heapsort example

Continue with the example from heapify:



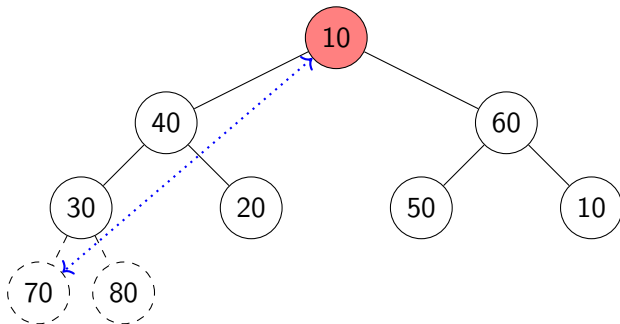
Heapsort example

Continue with the example from heapify:



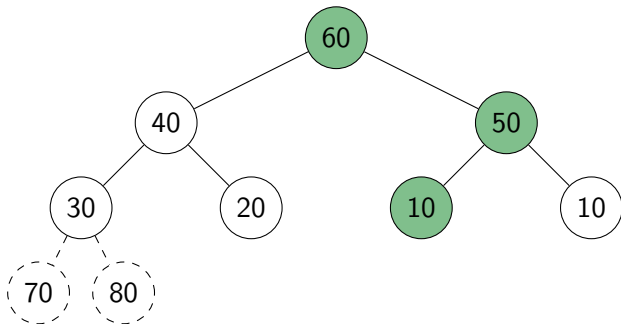
Heapsort example

Continue with the example from heapify:



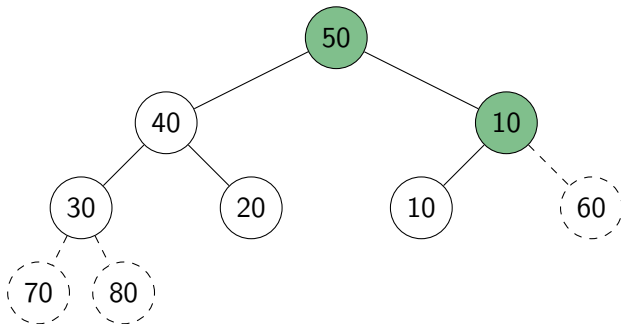
Heapsort example

Continue with the example from heapify:



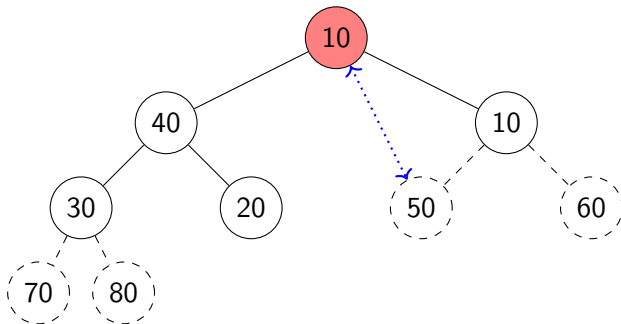
Heapsort example

Continue with the example from heapify:



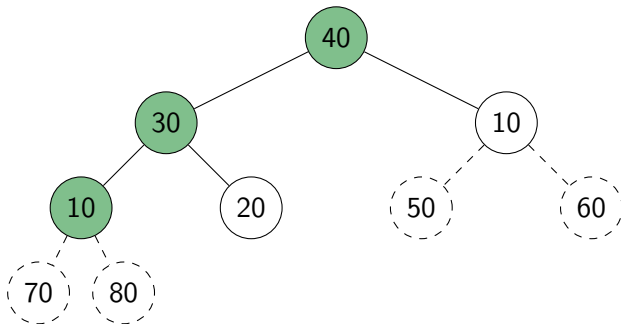
Heapsort example

Continue with the example from heapify:



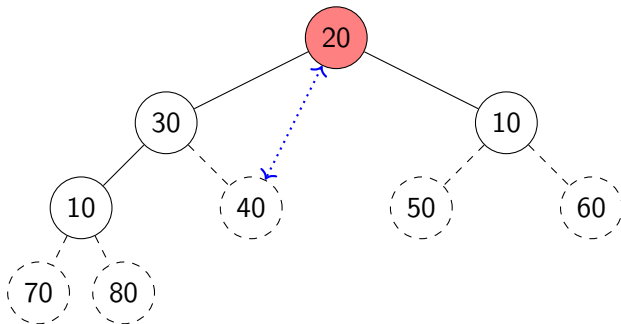
Heapsort example

Continue with the example from heapify:



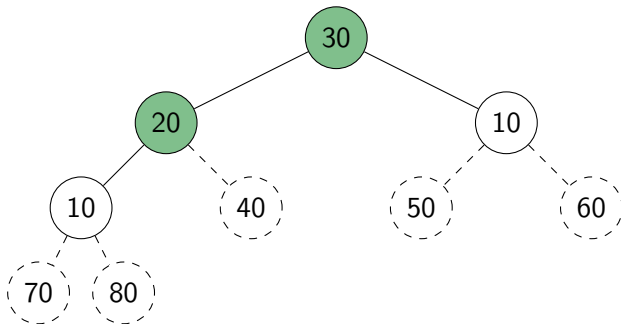
Heapsort example

Continue with the example from heapify:



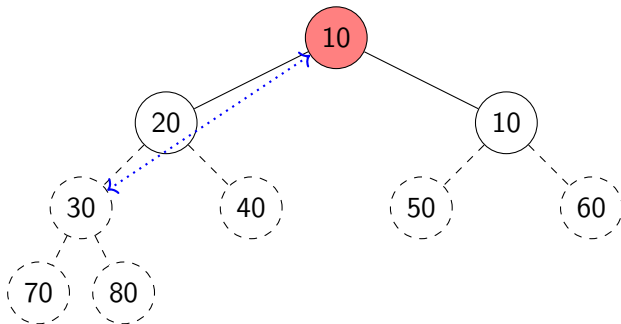
Heapsort example

Continue with the example from heapify:



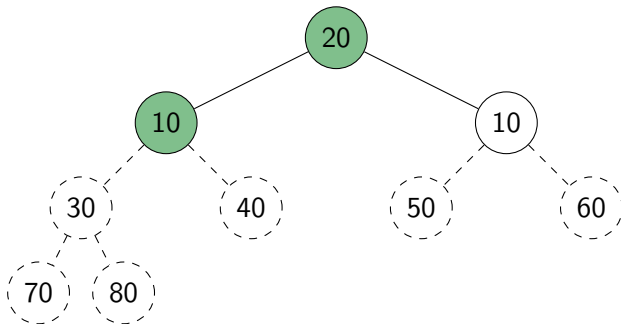
Heapsort example

Continue with the example from heapify:



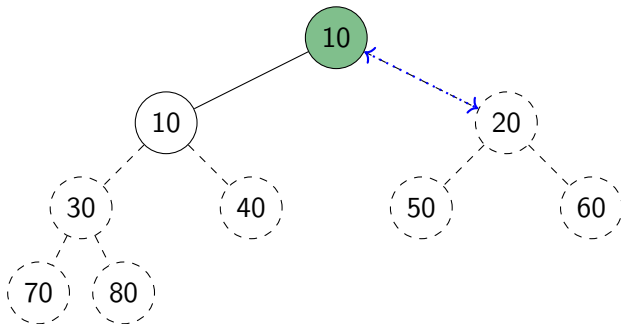
Heapsort example

Continue with the example from heapify:



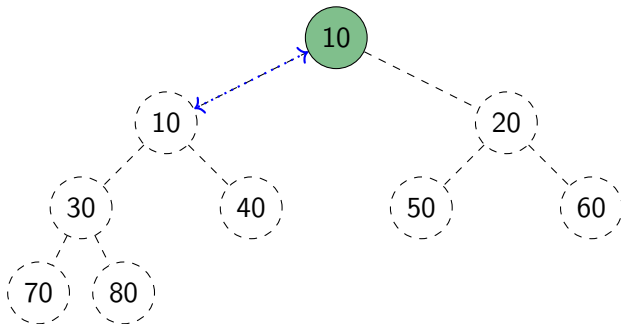
Heapsort example

Continue with the example from heapify:



Heapsort example

Continue with the example from heapify:



The array (i.e., the heap in level-by-level order) is now in sorted order.

Heap summary

- **Binary heap**: A binary tree that satisfies structural property and heap-order property.
- Heaps are one possible realization of ADT PriorityQueue:
 - ▶ *insert* takes time $O(\log n)$
 - ▶ *deleteMax* takes time $O(\log n)$
 - ▶ Also supports *findMax* in time $O(1)$
- A binary heap can be built in linear time.
- *PQ-sort* with binary heaps leads to a sorting algorithm with $O(n \log n)$ worst-case run-time (\rightsquigarrow *HeapSort*)
- We have seen here the *max-oriented version* of heaps (the maximum priority is at the root).
- There exists a symmetric *min-oriented version* that supports *insert* and *deleteMin* with the same run-times.

Outline

1 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

Finding the largest items

Problem: Find the *k*th largest item in an array A of n distinct numbers.

Solution 1: Make k passes through the array, deleting the maximum number each time.

Complexity: $\Theta(kn)$.

Solution 2: Sort A , then return $A[n-k]$.

Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the k largest numbers seen so far in a min-heap

Complexity: $\Theta(n \log k)$.

Solution 4: Create a max-heap with *heapify*(A). Call *deleteMax*(A) k times.

Complexity: $\Theta(n + k \log n)$.