

# CS 240 – Data Structures and Data Management

## Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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# Outline

## 1 Dictionaries for special keys

- Lower bound
- Interpolation Search
- Tries
  - Standard Tries
  - Variations of Tries
  - Compressed Tries

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## Lower bound for search

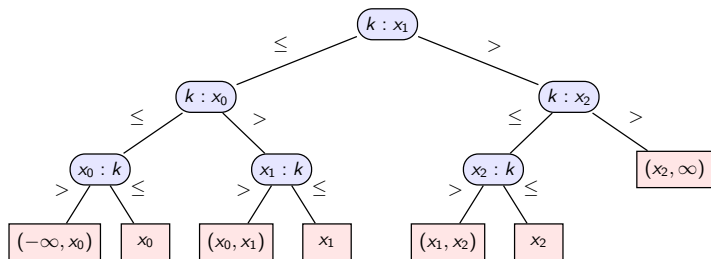
The fastest realizations of *ADT Dictionary* require  $\Theta(\log n)$  time to search among  $n$  items. Is this the best possible?

## Lower bound for search

The fastest realizations of *ADT Dictionary* require  $\Theta(\log n)$  time to search among  $n$  items. Is this the best possible?

**Theorem:** In the comparison model (on the keys),  $\Omega(\log n)$  comparisons are required to search a size- $n$  dictionary.

**Proof:** via decision tree



But can we beat the lower bound for special keys?

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- Lower bound
- **Interpolation Search**
- Tries
  - Standard Tries
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  - Compressed Tries

# Binary Search

Recall the run-times in a *sorted array*:

- *insert, delete*:  $\Theta(n)$
- *search*:  $\Theta(\log n)$

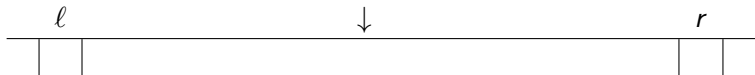
```
binary-search(A, n, k)
```

```
A: Sorted array of size n, k: key
```

1.  $\ell \leftarrow 0, r \leftarrow n - 1$
2. **while** ( $\ell \leq r$ )
3.      $m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
4.     **if** ( $A[m] < k$ ) **then**  $\ell = m + 1$
5.     **else if** ( $k < A[m]$ ) **then**  $r = m - 1$
6.     **else return** "found at  $A[m]$ "
7.     **return** "not found, but would be between  $A[\ell-1]$  and  $A[\ell]$ "

# Interpolation Search: Motivation

*binary-search*( $A[\ell, r], k$ ): Compare at index  $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$





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**Question:** If keys are *numbers*, where would you expect key  $k = 100$ ?

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$\ell$		$\downarrow$		$r$
40				120

**Question:** If keys are *numbers*, where would you expect key  $k = 100$ ?

*interpolation-search*( $A[\ell, r], k$ ): Compare at index  $\ell + \lfloor \frac{k-A[\ell]}{A[r]-A[\ell]}(r - \ell) \rfloor$

## Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

*interpolation-search*(A[0..10],449):

# Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500
$\ell$		$\uparrow$								$r$

*interpolation-search*(A[0..10],449):

- Initially  $\ell = 0$ ,  $r = n - 1 = 10$ ,  $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$



# Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

$\ell$        $\uparrow, r$

*interpolation-search*(A[0..10],449):

- Initially  $\ell = 0$ ,  $r = n - 1 = 10$ ,  $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$ ,  $r = 10$ ,  $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$ ,  $r = 4$ ,  $m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$ , found at A[4]

# Interpolation Search Example

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- $\ell = 3$ ,  $r = 4$ ,  $m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$ , found at A[4]

Works well if keys are *uniformly* distributed:

- Can show: Recurrence relation is  $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$ .
- This resolves to  $T^{(\text{avg})}(n) \in \Theta(\log \log n)$ .

But: Worst case performance  $\Theta(n)$

# Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash during computation of  $m$ .

*interpolation-search*( $A, n, k$ )

$A$ : Sorted array of size  $n$ ,  $k$ : key

1.  $\ell \leftarrow 0, r \leftarrow n - 1$
2. **while** ( $\ell \leq r$ )
3.     **if** ( $k < A[\ell]$  or  $k > A[r]$ ) **return** “not found”
4.     **if** ( $A[\ell] = A[r]$ ) **then return** “found at  $A[\ell]$ ”
5.      $m \leftarrow \ell + \lfloor \frac{k - A[\ell]}{A[r] - A[\ell]} \cdot (r - \ell) \rfloor$
6.     **if** ( $A[m] < k$ ) **then**  $\ell = m + 1$
7.     **else if** ( $k < A[m]$ ) **then**  $r = m - 1$
8.     **else return** “found at  $A[m]$ ”
9.     // We always return from somewhere within while-loop



# Outline

## 1 Dictionaries for special keys

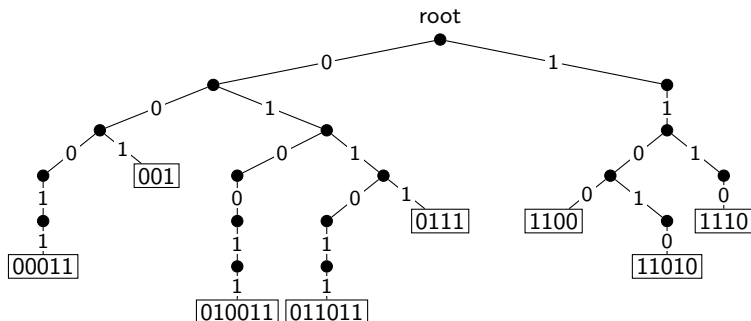
- Lower bound
- Interpolation Search
- **Tries**
  - Standard Tries
  - Variations of Tries
  - Compressed Tries

# Tries: Introduction

**Trie** (also known as **radix tree**): A dictionary for bitstrings.

(Should know: string, word,  $|w|$ , alphabet, prefix, suffix, comparing words,...)

- Comes from retrieval, but pronounced “try”
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key

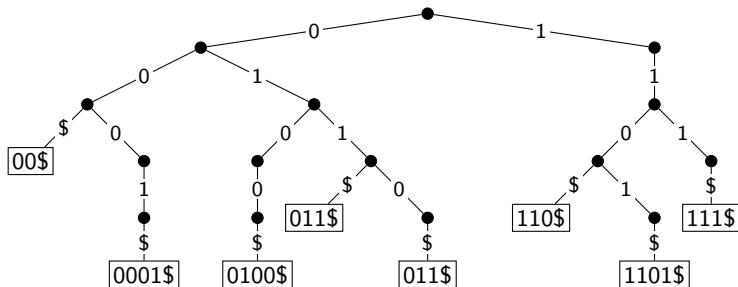


## More on tries

**Assumption:** Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with 'end-of-word' character \$.

**Example:** A trie for {00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$}

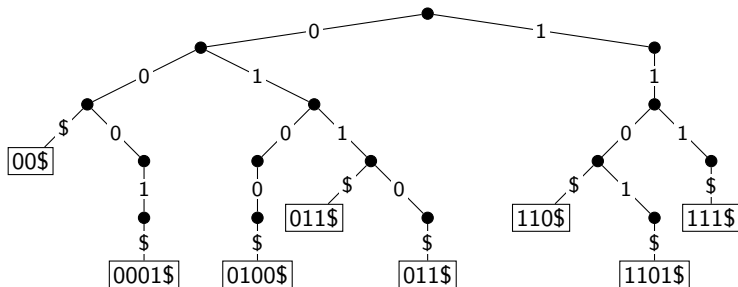


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**Example:** A trie for {00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$}



Then items (keys) are stored *only* in the leaf nodes

## Tries: Search

- start from the root and the most significant bit of  $x$
- follow the link that corresponds to the current bit in  $x$ ;  
return failure if the link is missing
- return success if we reach a leaf (it must store  $x$ )
- else recurse on the new node and the next bit of  $x$

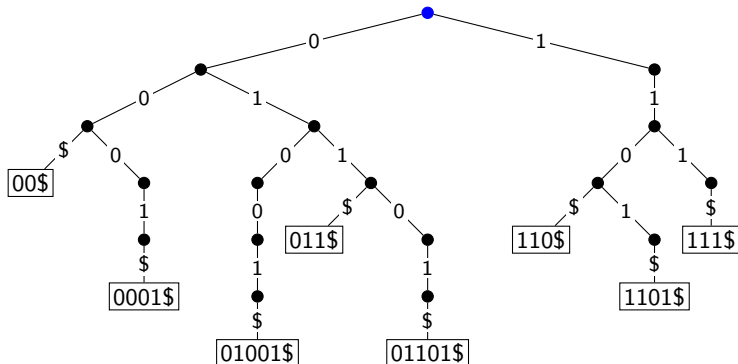
```
Trie::search( $v \leftarrow \text{root}, d \leftarrow 0, x$ )
```

$v$ : node of trie;  $d$ : level of  $v$ ,  $x$ : word stored as array of chars

1. **if**  $v$  is a leaf
2.     **return**  $v$
3. **else**
4.     let  $v'$  be child of  $v$  labelled with  $x[d]$
5.     **if** there is no such child
6.         **return** "not found"
7.     **else** *Trie::search*( $v', d + 1, x$ )

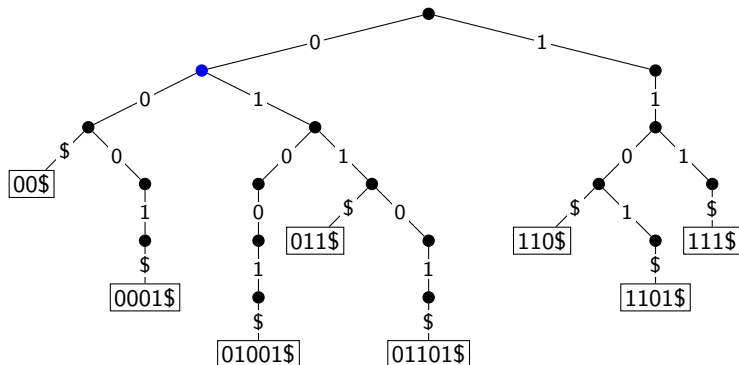
# Tries: Search Example

Example: Trie::search(011\$)



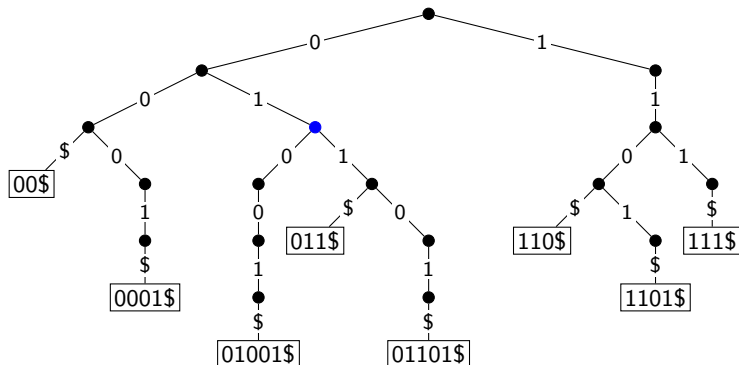
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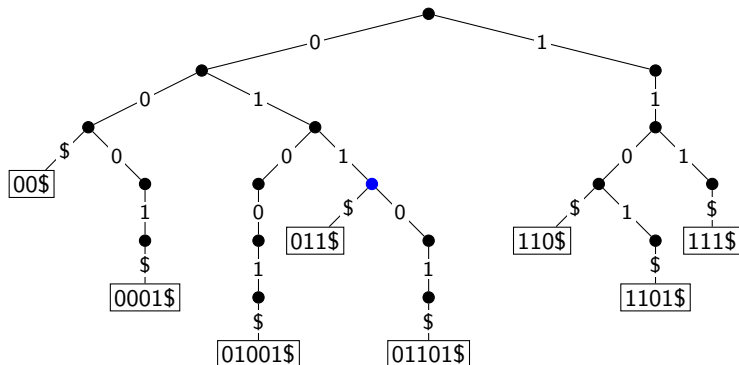
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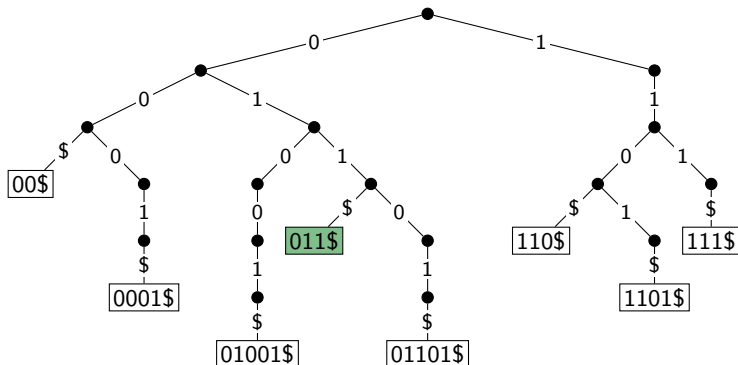
# Tries: Search Example

Example: Trie::search(011\$)



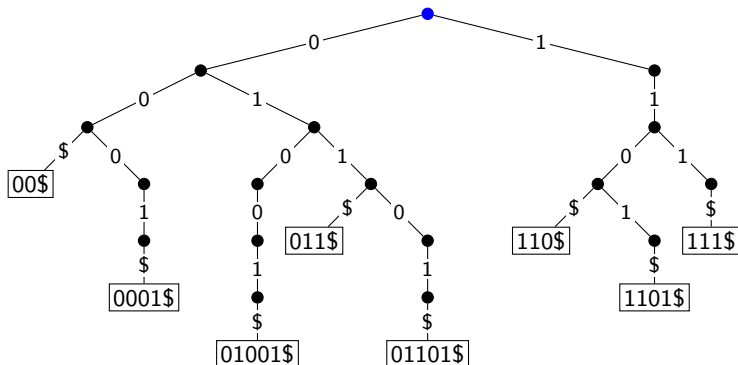
# Tries: Search Example

Example: Trie::search(011\$) **successful**



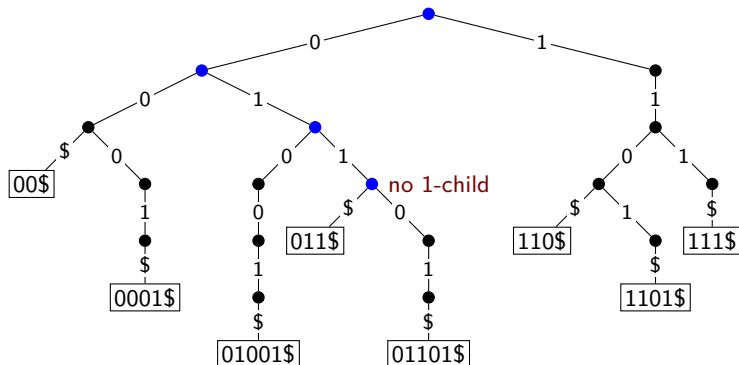
# Tries: Search Example

Example: Trie::search(0111\$)



# Tries: Search Example

Example: Trie::search(0111\$) **unsuccessful**

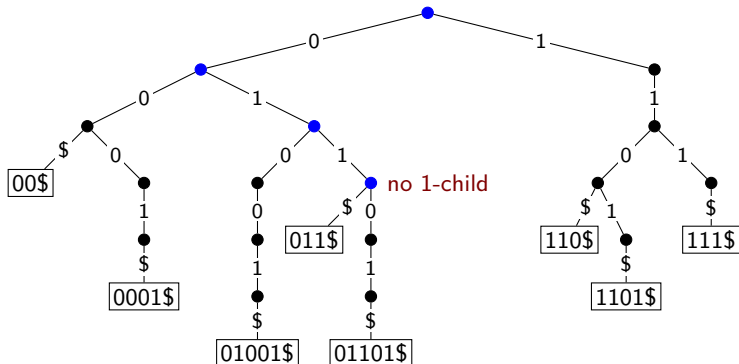


# Tries: Insert & Delete

- *Trie::insert(x)*
  - ▶ Search for  $x$ , this should be unsuccessful
  - ▶ Suppose we finish at a node  $v$  that is missing a suitable child.  
Note:  $x$  has extra bits left.
  - ▶ Expand the trie from the node  $v$  by adding necessary nodes that correspond to extra bits of  $x$ .
- *Trie::delete(x)*
  - ▶ Search for  $x$
  - ▶ let  $v$  be the leaf where  $x$  is found
  - ▶ delete  $v$  and all ancestors of  $v$  until we reach an ancestor that has two children.
- **Time Complexity** of all operations:  $\Theta(|x|)$   
 $|x|$ : length of binary string  $x$ , i.e., the number of bits in  $x$

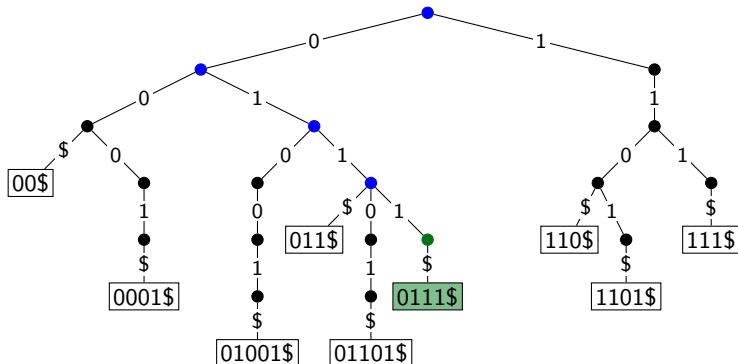
# Tries: Insert Example

Example: *Trie::insert*(0111\$)



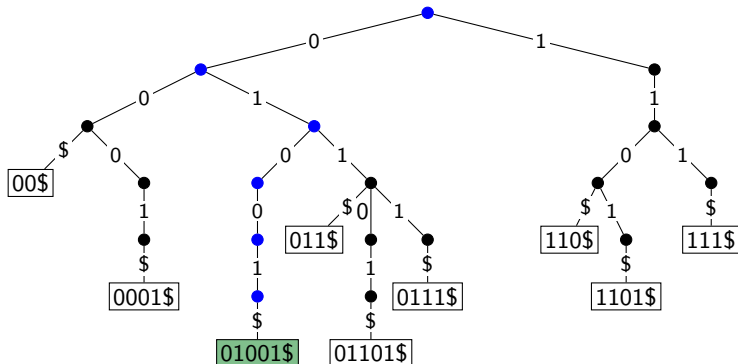
# Tries: Insert Example

Example: *Trie::insert*(0111\$)



# Tries: Delete Example

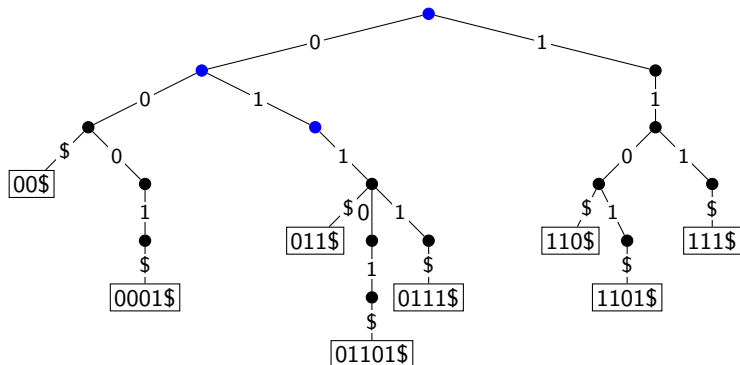
Example: *Trie::delete*(01001\$)





# Tries: Delete Example

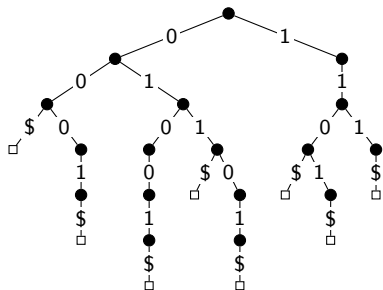
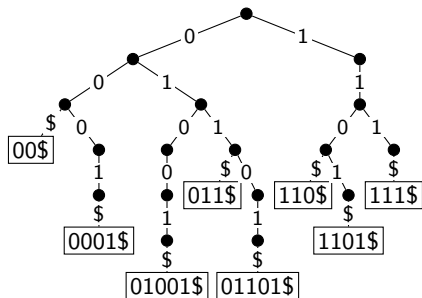
Example: *Trie::delete*(01001\$)



## Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

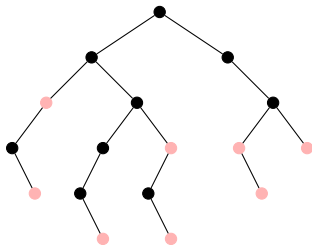
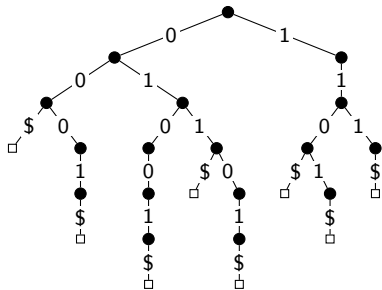
- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.



## Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

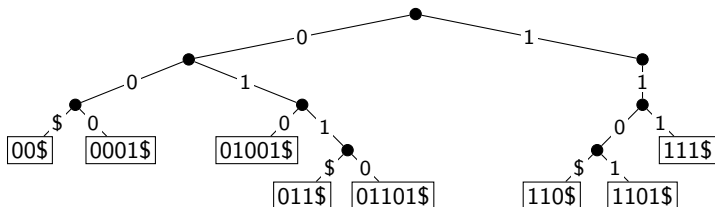
- Internal nodes may now also represent keys. Use a *flag* to indicate such nodes.
- No need for end-of-word character \$
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.



## Variations 3 of Tries

**Pruned Trie:** Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)

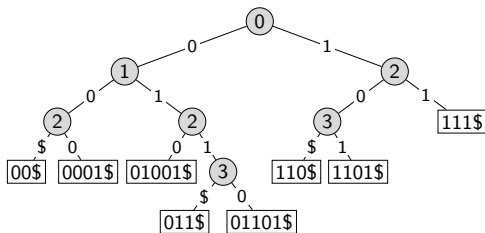
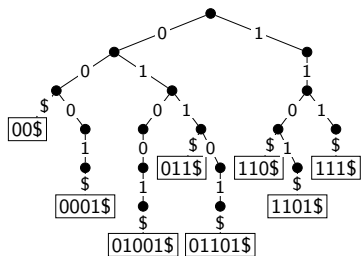


This is in practice the most efficient version of tries, but the operations get a bit more complicated.

## Variation 4 of Tries

**Compressed Trie:** compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with  $n$  keys has at most  $n - 1$  internal nodes



Also known as **Patricia-Tries:**

*Practical Algorithm to Retrieve Information Coded in Alphanumeric*

## Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in  $x$ ;  
return failure if the link is missing
- if we reach a leaf, explicitly check whether word stored at leaf is  $x$
- else recurse on the new node and the next bit of  $x$

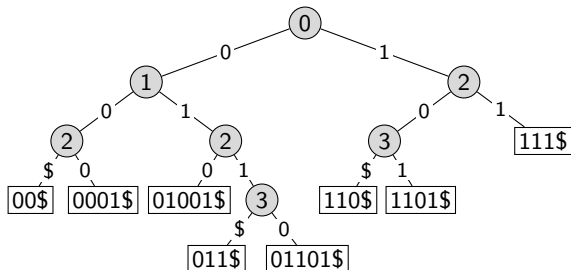
```
CompressedTrie::search( $v \leftarrow \text{root}, x$ )
```

$v$ : node of trie;  $x$ : word

1. **if**  $v$  is a leaf
2.     **return** *strcmp*( $x, v.\text{key}$ )
3.      $d \leftarrow$  index stored at  $v$
4.     **if**  $x$  has at most  $d$  bits
5.     **return** "not found"
6.      $v' \leftarrow$  child of  $v$  labelled with  $x[d]$
7.     **if** there is no such child
8.     **return** "not found"
9.     *CompressedTrie::search*( $v', x$ )

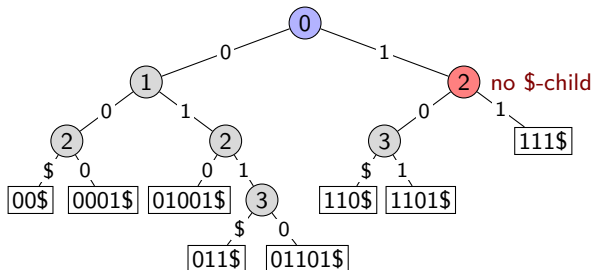
# Compressed Tries: Search Example

Example: CompressedTrie::search(10\$)



# Compressed Tries: Search Example

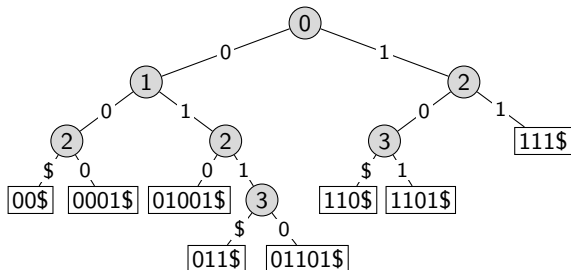
Example: CompressedTrie::search(10\$) **unsuccessful**





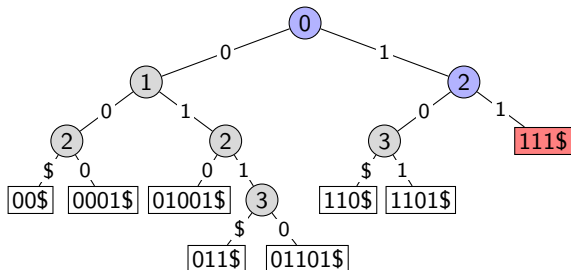
# Compressed Tries: Search Example

Example: CompressedTrie::search(101\$)



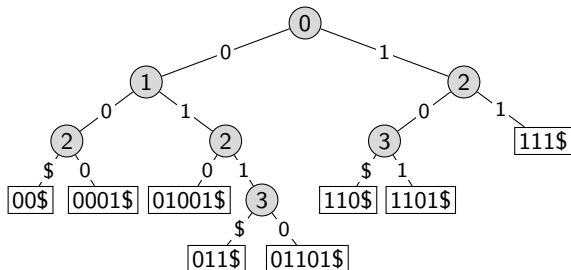
# Compressed Tries: Search Example

Example: CompressedTrie::search(101\$) **unsuccessful**



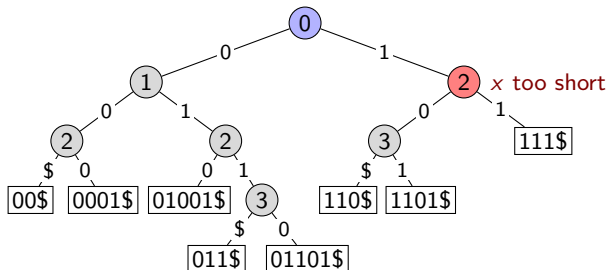
# Compressed Tries: Search Example

Example: CompressedTrie::search(1\$)



# Compressed Tries: Search Example

Example: CompressedTrie::search(1\$) **unsuccessful**



# Compressed Tries: Insert & Delete

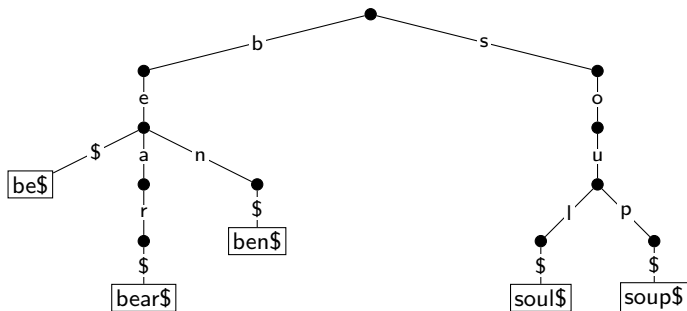
- *CompressedTrie::delete*( $x$ ):
  - ▶ Perform *search*( $x$ )
  - ▶ Remove the node  $v$  that stored  $x$
  - ▶ Compress along path to  $v$  whenever possible.
- *CompressedTrie::insert*( $x$ ):
  - ▶ Perform *search*( $x$ )
  - ▶ Let  $v$  be the node where the search ended.
  - ▶ Conceptually simplest approach:
    - ★ Uncompress path from root to  $v$ .
    - ★ Insert  $x$  as in an uncompressed trie.
    - ★ Compress paths from root to  $v$  and from root to  $x$ .

But it can also be done by only adding those nodes that are needed, see the textbook for details.

- All operations take  $O(|x|)$  time.

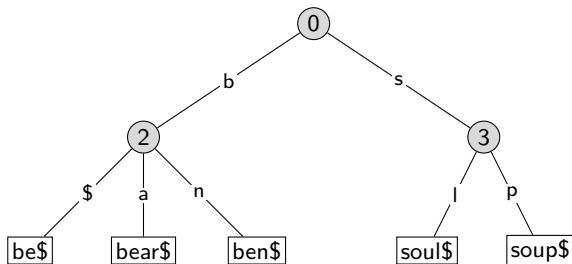
# Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet*  $\Sigma$
- Any node will have at most  $|\Sigma| + 1$  children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



# Compressed Multiway Tries

- **Variation:** Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



## Multiway Tries: Summary

- Operations *search*( $x$ ), *insert*( $x$ ) and *delete*( $x$ ) are exactly as for tries for bitstrings.
- Run-time  $O(|x| \cdot (\text{time to find the appropriate child}))$



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Each node now has up to  $|\Sigma| + 1$  children. How should they be stored?

**Solution 1:** Array of size  $|\Sigma| + 1$  for each node.

Complexity:  $O(1)$  time to find child,  $O(|\Sigma|n)$  space.

**Solution 2:** List of children for each node.

Complexity:  $O(|\Sigma|)$  time to find child,  $O(\#\text{children})$  space.

**Solution 3:** Dictionary (AVL-tree?) of children for each node.

Complexity:  $O(\log(\#\text{children}))$  time,  $O(\#\text{children})$  space.

Best in theory, but not worth it in practice unless  $|\Sigma|$  is huge.

In practice, use *hashing* (keys are in (typically small) range  $\Sigma$ ).