CS 240 – Data Structures and Data Management

Module 8: Range-Searching in Dictionaries for Points

M. Petrick V. Sakhnini O. Veksler
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Outline

1. Range-Searching in Dictionaries for Points
    - Range Searches
    - Multi-Dimensional Data
    - Quadtrees
    - kd-Trees
    - Range Trees
    - Conclusion

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Range searches

- So far: *search*(k) looks for *one* specific item.
- New operation **RangeSearch**: look for *all* items that fall within a given range.
  - Input: A **range**, i.e., an interval $I = (x, x')$
    It may be open or closed at the ends.
  - Want: Report all KVPs in the dictionary whose key k satisfies $k \in I$

Example:

5	10	11	17	19	33	45	51	55	59

RangeSearch((18,45]) should return $\{19, 33, 45\}$

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- Let s be the **output-size**, i.e., the number of items in the range.
- We need $\Omega(s)$ time simply to report the items.
- Note that sometimes $s = 0$ and sometimes $s = n$; we therefore keep it as a separate parameter when analyzing the run-time.

Range searches in existing dictionary realizations

Unsorted list/array/hash table: Range search requires $\Omega(n)$ time:
We have to check for each item explicitly whether it is in the range.

Sorted array: Range search in A can be done in $O(\log n + s)$ time:

RangeSearch((18,45])

5	10	11	17	19	33	45	51	55	59
			$\uparrow i$			$\uparrow i'$			

- Using binary search, find i such that x is at (or would be at) $A[i]$.
- Using binary search, find i' such that x' is at (or would be at) $A[i']$
- Report all items $A[i{+}1...i'{-}1]$
- Report $A[i]$ and $A[i']$ if they are in range

BST: Range searches can similarly be done in time $O(\text{height}+s)$ time.
We will see this in detail later.

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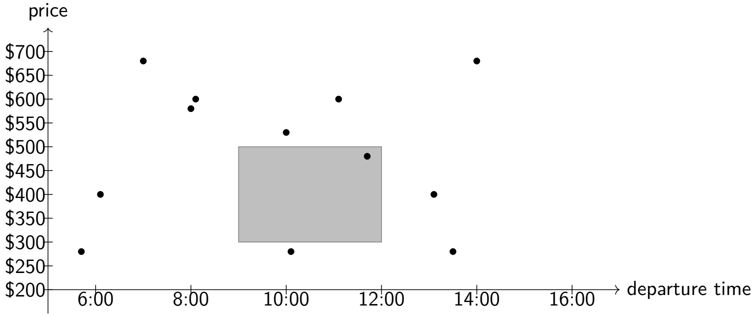
Multi-Dimensional Data

Range searches are of special interest for **multi-dimensional data**.

Example: flights that leave between 9am and noon, and cost \$300-\$500



- Each item has d **aspects** (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
- Aspect values (x_i) are numbers
- Each item corresponds to a point in d-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane

Multi-dimensional Range Search

(Orthogonal) ***d*-dimensional range search**: Given a **query rectangle** A, find all points that lie within A.

The time for range searches depends on how the points are stored.

- Could store a 1-dimensional dictionary (where the key is some combination of the aspects.)
  Problem: Range search on one aspect is not straightforward
- Could use one dictionary for each aspect
  Problem: inefficient, wastes space
- **Better idea**: Design new data structures specifically for points.
  - Quadtrees
  - kd-trees
  - range-trees
- **Assumption**: Point are in **general position**:
  No two x-coordinates or y-coordinates are the same.
  - Simplifies presentation; data structures can be generalized.

Outline

Quadtrees

We have n points $S = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.

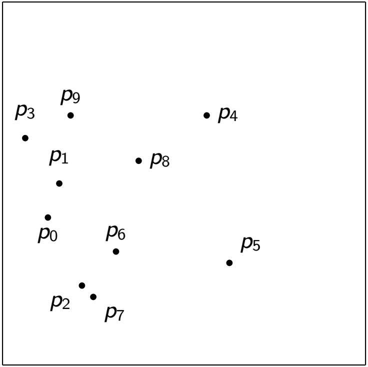
We need a **bounding box** R: a square containing all points.

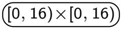
- Can find R by computing minimum and maximum x and y values in S
- The width/height of R should be a power of 2

Structure (and also how to *build* the quadtree that stores S):

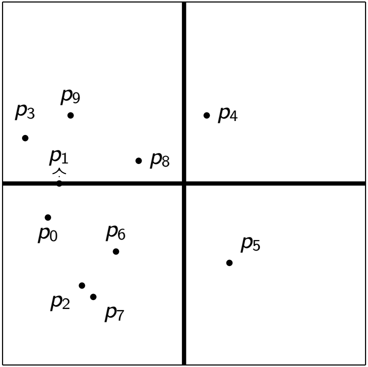
- Root r of the quadtree is associated with region R
- If R contains 0 or 1 points, then root r is a leaf that stores point.
- Else *split*: Partition R into four equal subsquares (**quadrants**) $R_{NE}, R_{NW}, R_{SW}, R_{SE}$
- Partition S into sets $S_{NE}, S_{NW}, S_{SW}, S_{SE}$ of points in these regions.
  - **Convention:** Points on split lines belong to right/top side
- Recursively build tree T_i for points S_i in region R_i and make them children of the root.

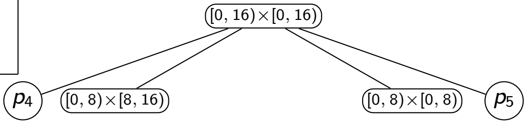
Quadtrees example



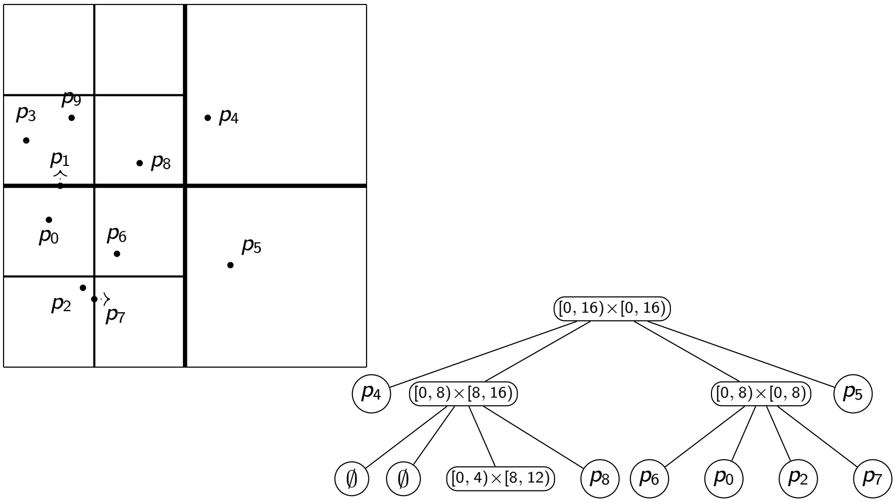


Quadtrees example

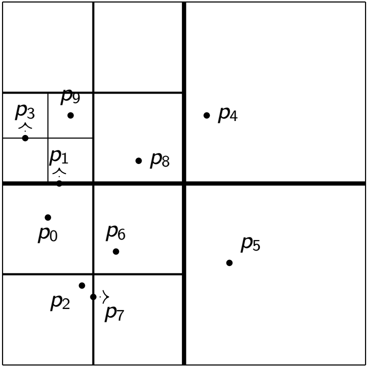


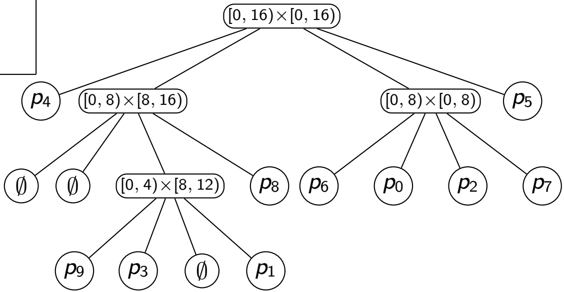


Quadtrees example

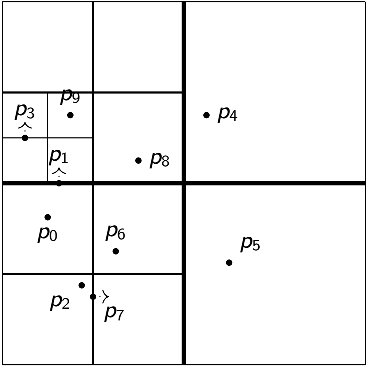


Quadtrees example

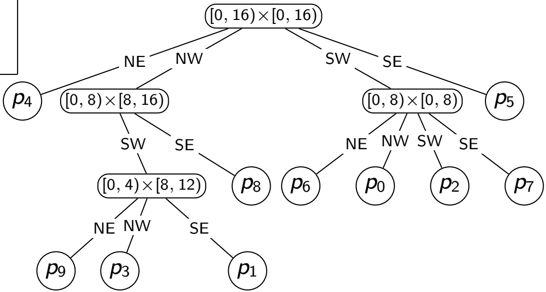




Quadtrees example



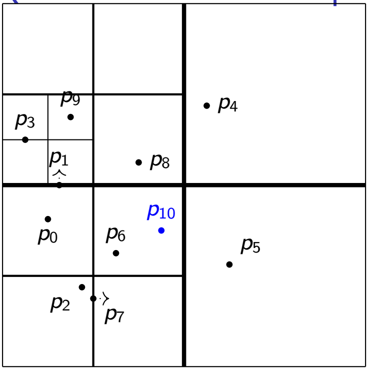
Easier for humans: omit empty sub-trees, label edges



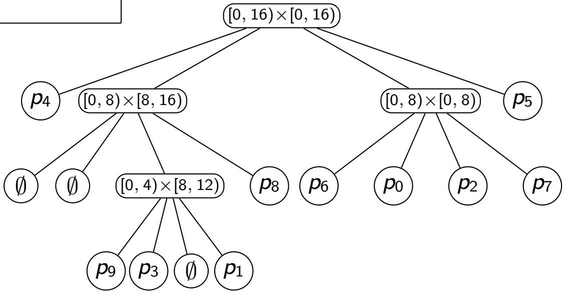
Quadtree Dictionary Operations

- *search*: Analogous to binary search trees and tries
- *insert*:
  - Search for the point
  - Split the leaf while there are two points in one region
- *delete*:
  - Search for the point
  - Remove the point
  - If its parent has only one point left: delete parent
    (and recursively all ancestors that have only one point left)

Quadtree Insert example



insert(p_{10})



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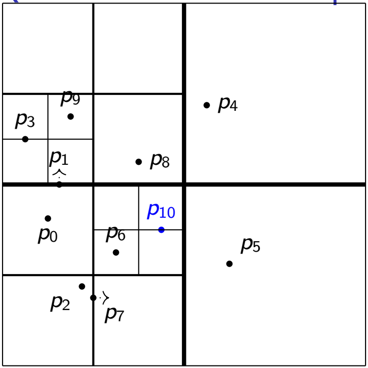

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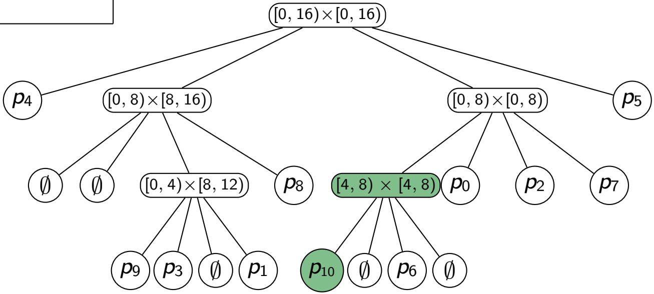


Quadtree Range Search

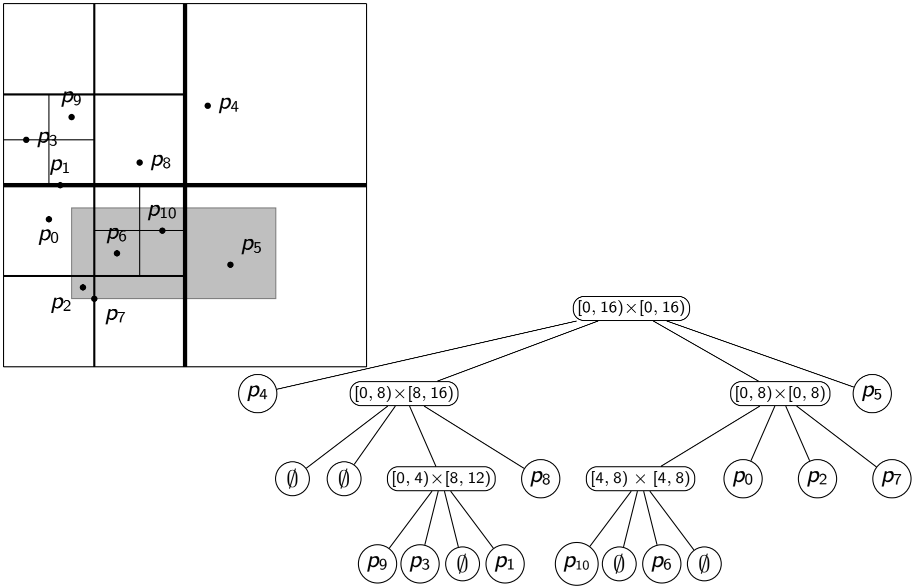
QTree::RangeSearch$(r \leftarrow root, A)$
r: The root of a quadtree, A: Query-rectangle

$R \leftarrow$ region associated with node r
if $(R \subseteq A)$ **then** // inside node
    report all points below r; **return**
if $(R \cap A$ is empty$)$ **then** // outside node
    return

    // The node is a boundary node, recurse
if $(r$ is a leaf$)$ **then**
    $p \leftarrow$ point stored at r
    if p is in A **return** p
    else return
for each child v of r **do**
    QTree::RangeSearch(v, A)

Note: We assume here that each node of the quadtree stores the associated square. Alternatively, these could be re-computed during the search (space-time tradeoff).

Quadtree range search example



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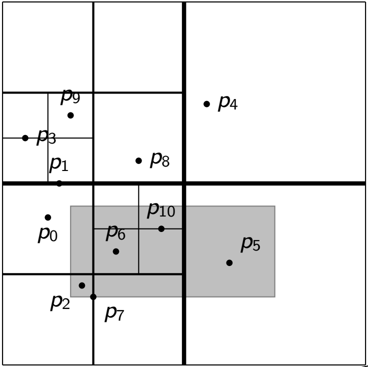
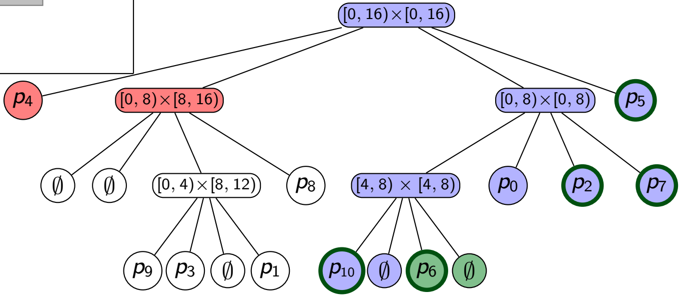
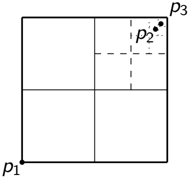


- Red: Search stopped due to $R \cap A = \emptyset$.
- Green: Search stopped due to $R \subseteq A$.
- Blue: Must continue search in children / evaluate.



Quadtree Analysis

- Crucial for analysis: what is the height of a quadtree?
  - Can have very large height for bad distributions of points



  - **spread factor** of points S:

$$\beta(S) = \frac{\text{sidelength of } R}{\text{minimum distance between points in } S}$$

  - Can show: height h of quadtree is in $\Theta(\log \beta(S))$
- Complexity to build initial tree: $\Theta(nh)$ worst-case
- Complexity of range search: $\Theta(nh)$ worst-case even if the answer is $\emptyset$
- But in practice much faster.

Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

"Points:"	0	9	12	14	24	26	28

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[0,32)

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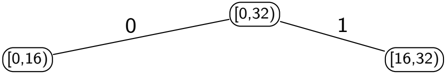

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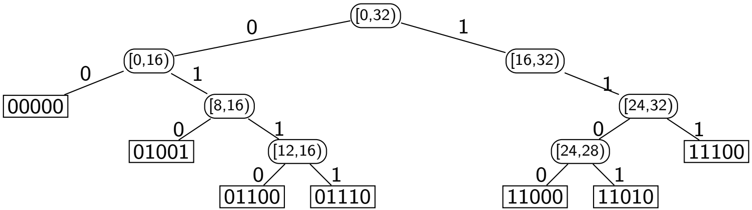

Same as a trie (with splitting stopped once key is unique)

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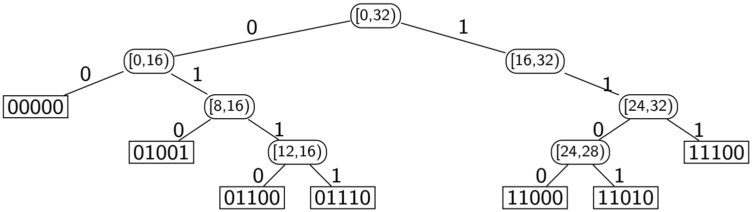

Same as a trie (with splitting stopped once key is unique)

- Quadtrees also easily generalize to higher dimensions (octrees, *etc.*) but are rarely used beyond dimension 3.

Quadtree summary

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (bit-shift!) if the width/height of R is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation: We could stop splitting earlier and allow up to S points in a leaf (for some fixed bound S).
- Variation: Store pixelated images by splitting until each region has the same color.

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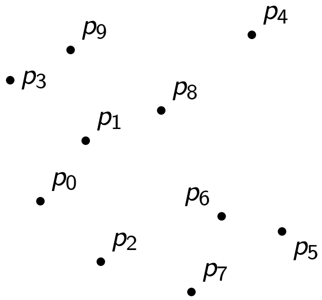
kd-trees

- We have n points $S = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$
- Quadtrees split square into quadrants regardless of where points are
- (Point-based) kd-tree idea: Split the region such that (roughly) half the point are in each subtree
- Each node of the kd-tree keeps track of a **splitting line** in one dimension (2D: either vertical or horizontal)
- **Convention:** Points on split lines belong to right/top side
- Continue splitting, switching between vertical and horizontal lines, until every point is in a separate region

  (There are alternatives, e.g., split by the dimension that has better aspect ratios for the resulting regions. No details.)

kd-tree example



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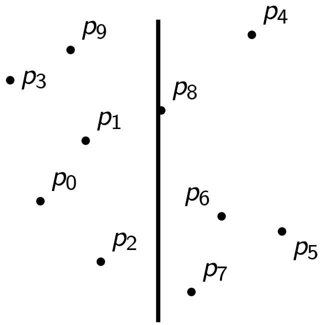

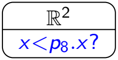

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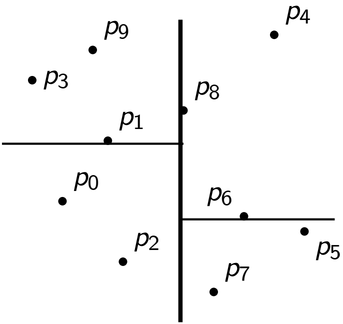
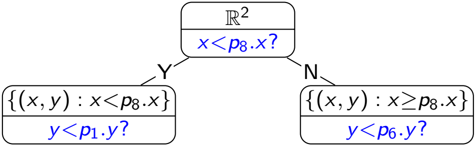
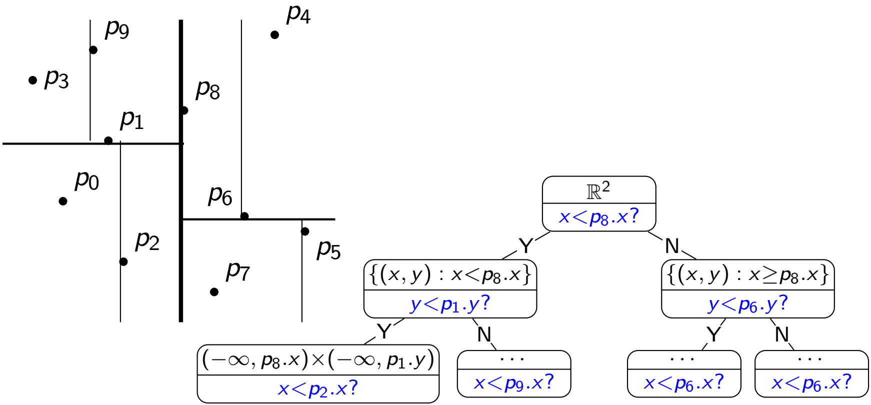


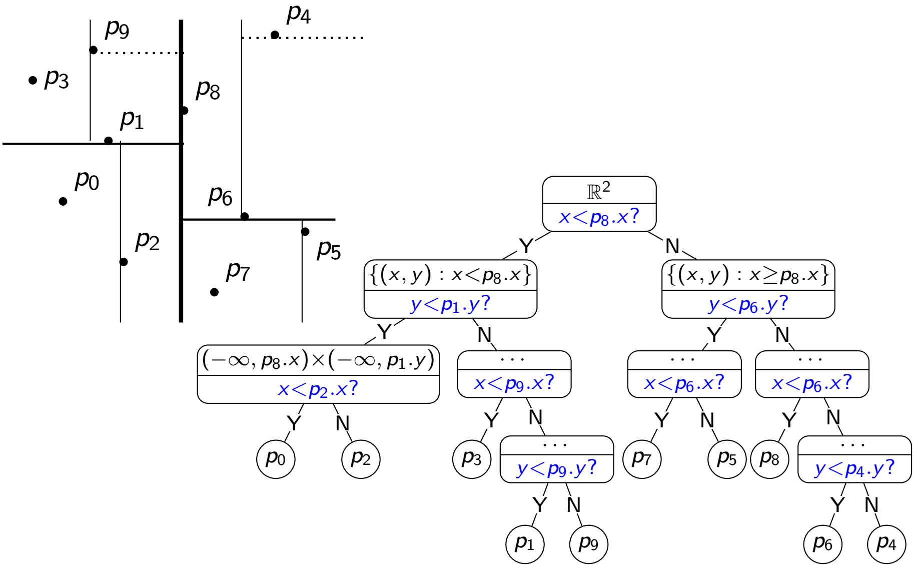


kd-tree example



For ease of drawing, we will usually not show the associated regions.

kd-tree example



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Constructing kd-trees

Build kd-tree with initial split by x on points S:

- If $|S| \leq 1$ create a leaf and return.
- Else $X :=$ *quick-select*$(S, \lfloor \frac{n}{2} \rfloor)$ (select by x-coordinate)
- Partition S by x-coordinate into $S_{x<X}$ and $S_{x \geq X}$
  - $\lfloor \frac{n}{2} \rfloor$ points on one side and $\lceil \frac{n}{2} \rceil$ points on the other.
    (Recall: Points in general position.)
- Create left subtree recursively (splitting by y) for points $S_{x<X}$.
- Create right subtree recursively (splitting by y) for points $S_{x \geq X}$.

Building with initial y-split symmetric.

Constructing kd-trees

Run-time:

- Find X and partition S in $\Theta(n)$ expected time using *randomized-quick-select*.
- Both subtrees have $\approx n/2$ points.

$$T^{\exp}(n) = 2T^{\exp}(n/2) + O(n) \quad \text{(sloppy recurrence)}$$

  This resolves to $\Theta(n \log n)$ expected time.

- This can be reduced to $\Theta(n \log n)$ *worst-case* time by pre-sorting (no details).

Height: $h(1) = 0$, $h(n) \leq h(\lceil n/2 \rceil) + 1$.

- This resolves to $O(\log n)$ (specifically $\lceil \log n \rceil$).

kd-tree Dictionary Operations

- *search* (for single point): as in binary search tree using indicated coordinate
- *insert*: search, insert as new leaf.
- *delete*: search, remove leaf.

Problem: After insert or delete, the split might no longer be at exact median and the height is no longer guaranteed to be $\lceil \log_2 n \rceil$.

We can maintain $O(\log n)$ height by occasionally re-building entire subtrees. (No details.) But *rangeSearch* will be slower.

kd-trees do not handle insertion/deletion well.

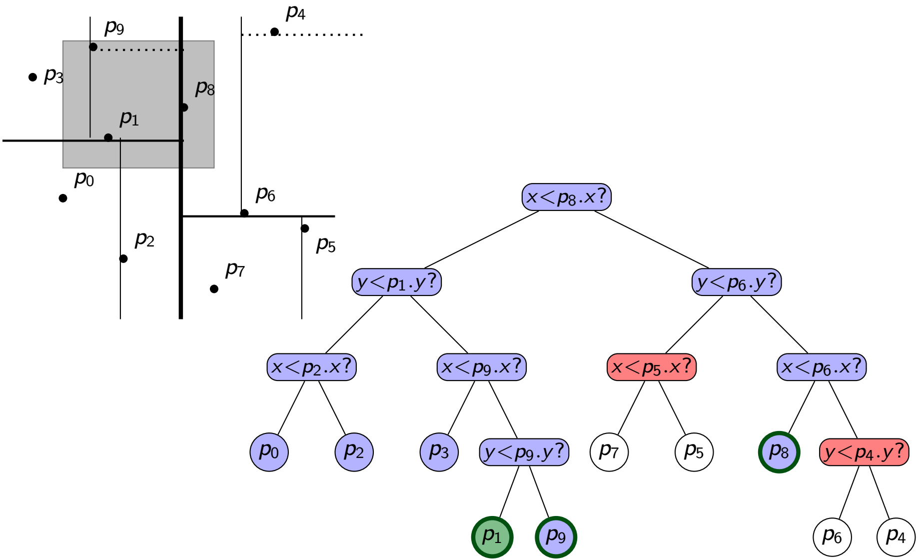
kd-tree Range Search

- Range search is *exactly* as for quad-trees, except that there are only two children.

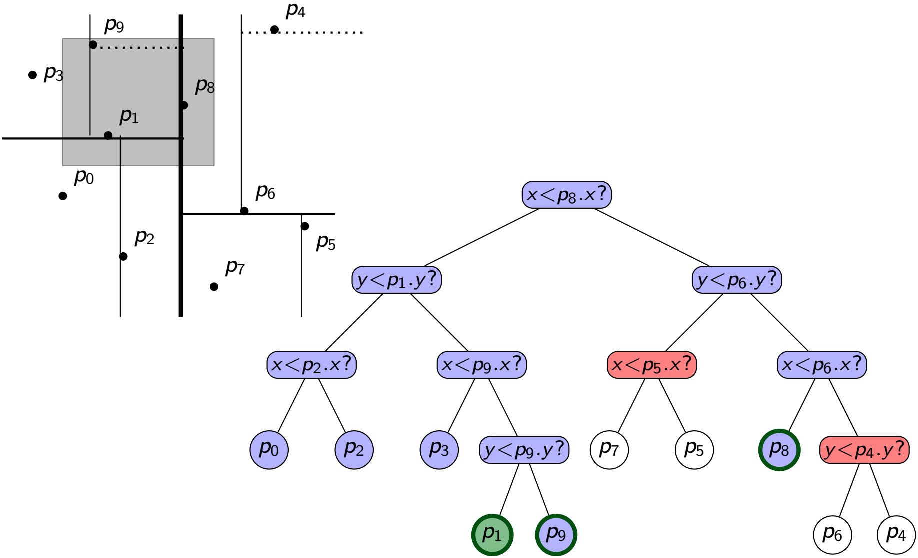
```
kdTree::RangeSearch(r ← root, A)
r: The root of a kd-tree, A: Query-rectangle
    R ← region associated with node r
    if (R ⊆ A) then report all points below r; return
    if (R ∩ A is empty) then return
    if (r is a leaf) then
        p ← point stored at r
        if p is in A return p
        else return
    for each child v of r do
        kdTree::RangeSearch(v, A)
```

- We assume again that each node stores its associated region.
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line.

kd-tree: Range Search Example



kd-tree: Range Search Example



Red: Search stopped due to $R \cap A = \emptyset$. Green: Search stopped due to $R \subseteq A$.

kd-tree: Range Search Complexity

- The complexity is $O(s + Q(n))$ where
  - s is the output-size
  - $Q(n)$ is the number of "boundary" nodes (blue):
    - *kdTree::RangeSearch* was called.
    - Neither $R \subseteq A$ nor $R \cap A = \emptyset$
- **Can show:** $Q(n)$ satisfies the following recurrence relation (no details):

$$Q(n) \leq 2Q(n/4) + O(1)$$

- This solves to $Q(n) \in O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(s + \sqrt{n})$

kd-tree: Higher Dimensions

- kd-trees for d-dimensional space:
  - At the root the point set is partitioned based on the first coordinate
  - At the subtrees of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth d we start all over again, partitioning on first coordinate
- **Storage**: $O(n)$
- **Height**: $O(\log n)$
- **Construction time**: $O(n \log n)$
- **Range search time**: $O(s + n^{1-1/d})$

This assumes that d is a constant.

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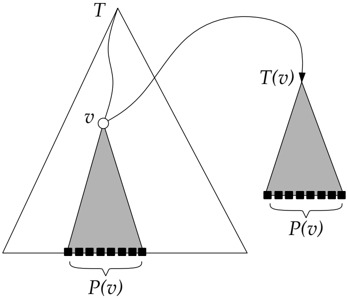
Towards Range Trees

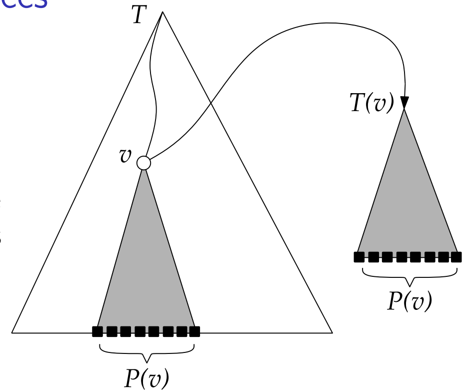
- Both Quadtrees and kd-trees are intuitive and simple.
- But: both may be very slow for range searches.
- Quadtrees are also potentially wasteful in space.

New idea: **Range trees**



- Somewhat wasteful in space, but much faster range search.
- **Tree of trees** (a *multi-level* data structure)

2-dimensional Range Trees

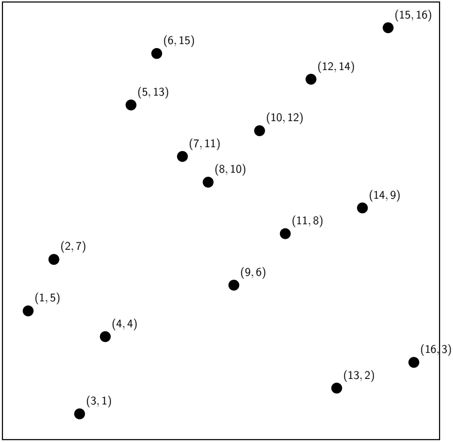


Primary structure:
Balanced binary search tree T that stores P and uses *x-coordinates* as keys.

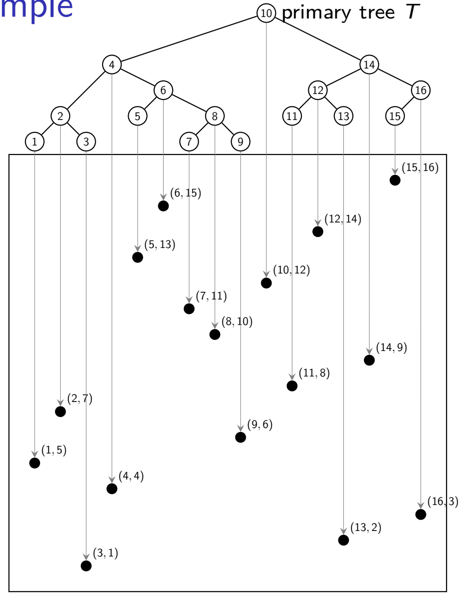
Each node v of T stores an **associate structure** $T(v)$:

- Let $P(v)$ be all points in subtree of v in T (including point at v)
- $T(v)$ stores $P(v)$ in a balanced binary search tree, using the *y-coordinates* as key
- Note: v is not necessarily the root of $T(v)$

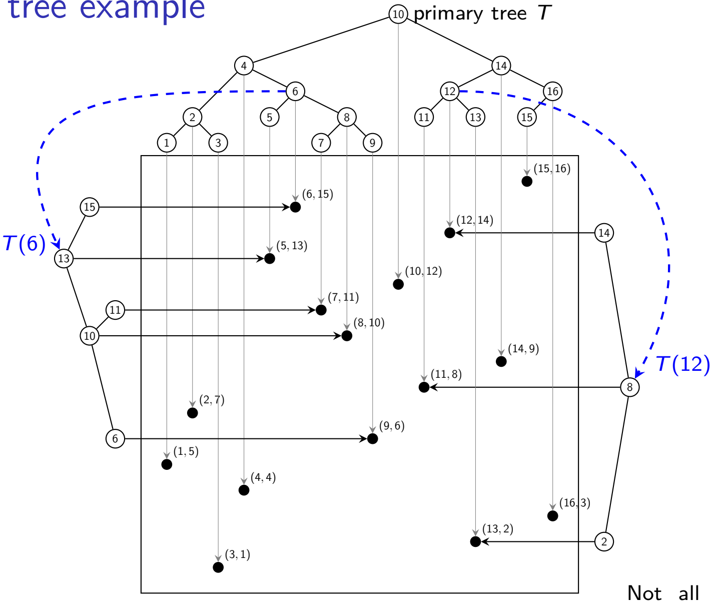
Range tree example



Range tree example



Range tree example



Not all associate trees are shown.

Range Tree Space Analysis

- Primary tree uses $O(n)$ space.
- Associate tree $T(v)$ uses $O(|P(v)|)$ space
  (where $P(v)$ are the points at descendants of v in T)
- **Key insight**: $w \in P(v)$ means that v is an ancestor of w in T
  - Every node w has $O(\log n)$ ancestors in T
    (Recall that we assume T to be balanced.)
  - Every node w belongs to $O(\log n)$ sets $P(v)$
  - So $\sum_v |P(v)| \leq \sum_w \#\{\text{ancestors of } w\} \in O(n \log n)$

Therefore: A range-tree with n points uses $O(n \log n)$ space.

Range Trees Operations

- *search*: search by x-coordinate in T
- *insert*: First, insert point by x-coordinate into T.
  Then, walk back up to the root and insert the point by y-coordinate in *all* associate trees $T(v)$ of nodes v on path to the root.
- *delete*: analogous to insertion
- **Problem:** We want the binary search trees to be balanced.
  - This makes *insert*/*delete* very slow if we use AVL-trees.
    (A rotation at v changes $P(v)$ and hence requires a re-build of $T(v)$.)
  - **Solution**: Completely rebuild highly unbalanced subtrees (no details)

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    (A rotation at v changes $P(v)$ and hence requires a re-build of $T(v)$.)
  - **Solution**: Completely rebuild highly unbalanced subtrees (no details)
- *range-search*: search by x-range in T.
  Among found points, search by y-range in some associated trees.
- Must understand first: How to do (1-dimensional) range search in binary search tree?

BST Range Search

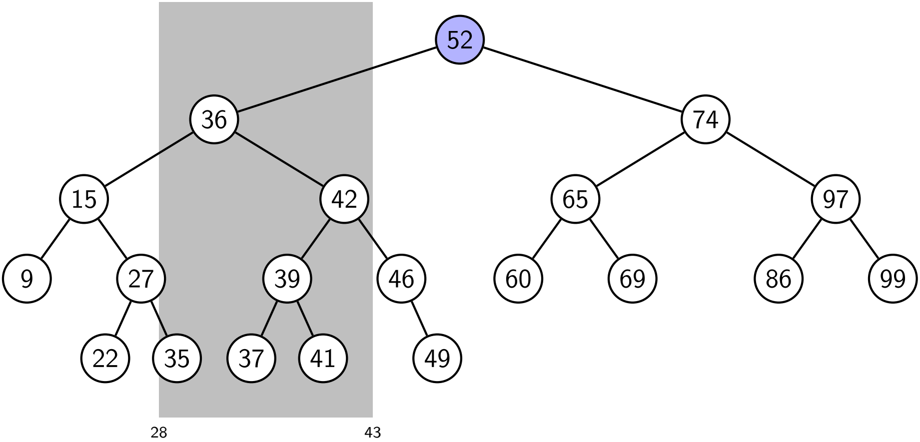
BST::RangeSearch$(r \leftarrow root, x_1, x_2)$
r: root of a binary search tree, x_1, x_2: search keys
Returns keys in subtree at r that are in range $[x_1, x_2]$

```
if r = NIL then return
if x1 ≤ r.key ≤ x2 then
    L ← BST::RangeSearch(r.left, x1, x2)
    R ← BST::RangeSearch(r.right, x1, x2)
    return L ∪ r.{key} ∪ R
if r.key < x1 then
    return BST::RangeSearch(r.right, x1, x2)
if r.key > x2 then
    return BST::RangeSearch(r.left, x1, x2)
```

Keys are reported in in-order, i. e., in sorted order.

BST Range Search example

BST::RangeSearch$(T, 28, 43)$



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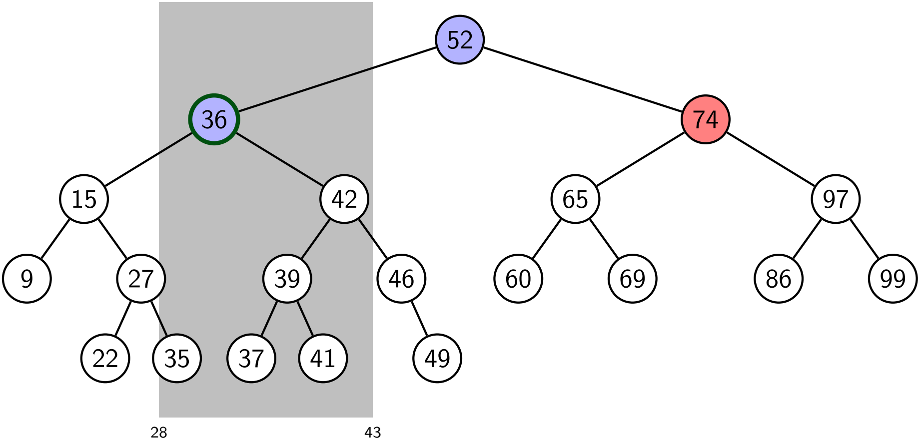

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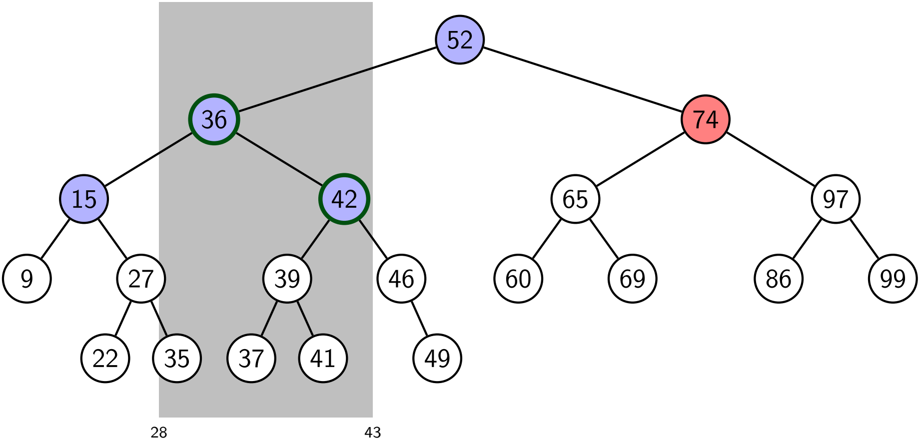

BST Range Search example

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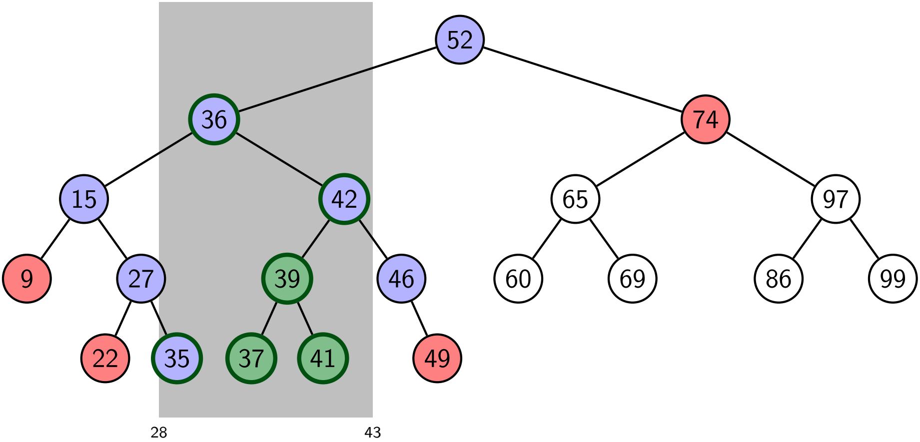

BST Range Search example

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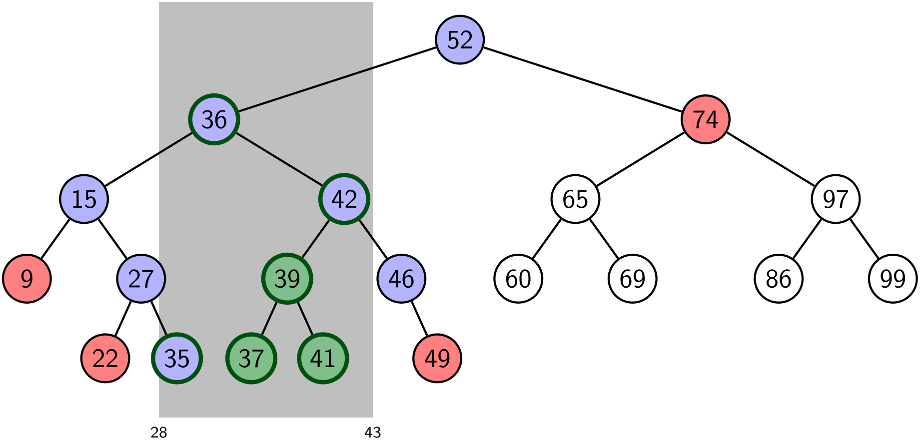
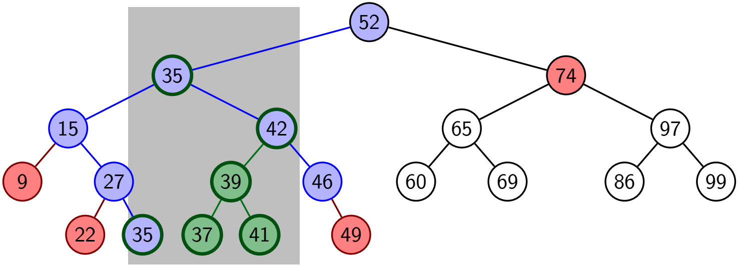


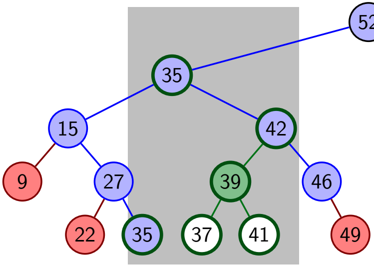
Note: Search from 39 was unnecessary: *all* its descendants are in range.

BST Range Search re-phrased



- Search for left boundary x_1: this gives path P_1
- Search for right boundary x_2: this gives path P_2
- This partitions T into three groups: outside, on, or between the paths.

BST Range Search re-phrased

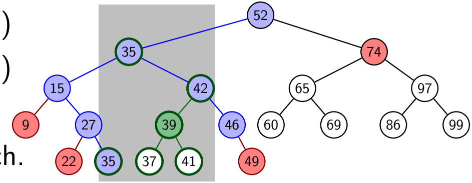


- boundary nodes: nodes in P_1 or P_2
  - For each boundary node, test whether it is in the range.
- outside nodes: nodes that are left of P_1 or right of P_2
  - These are *not* in the range, we stop the search at the topmost.
- inside nodes: nodes that are right of P_1 and left of P_2
  - We stop the search at the topmost inside node.
  - All descendants of such a node are *in* the range.
    For a 1d range search, report them.

BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path P_1: $O(\log n)$
- Search for path P_2: $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.



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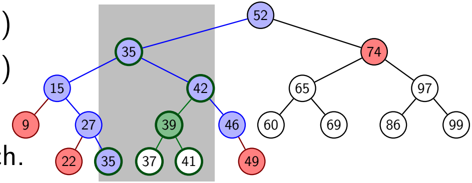


- We spend $O(1)$ time per topmost outside node.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- We spend $O(1)$ time per topmost inside node v.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- For 1d range search, also report the descendants of v.
  - We have $\sum_{v \text{ topmost inside}} \#\{\text{descendants of } v\} \leq s$ since subtrees of topmost inside nodes are disjoint. So this takes time $O(s)$ overall.

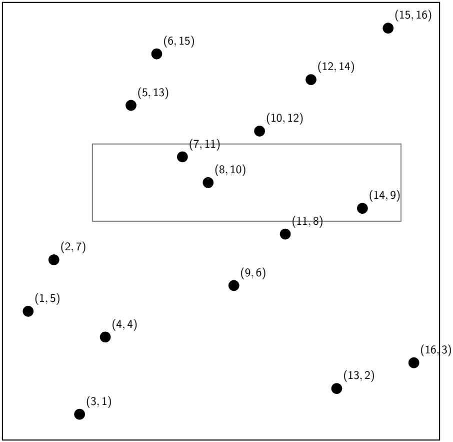
Run-time for 1d range search: $O(\log n + s)$. This is no faster overall, but topmost inside nodes will be important for 2d range search.

Range Trees: Range Search

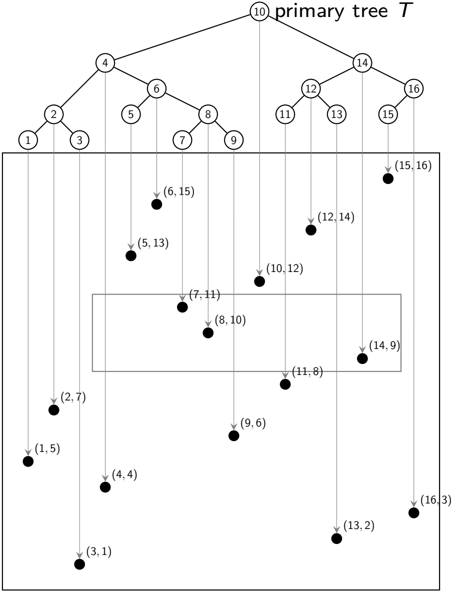
Range search for $A = [x_1, x_2] \times [y_1, y_2]$ is a two stage process:

- Perform a range search (on the x-coordinates) for the interval $[x_1, x_2]$ in primary tree T (*BST::RangeSearch*(T, x_1, x_2))
- Get boundary, topmost outside and topmost inside nodes as before.
- For every boundary node, test to see if the corresponding point is within the region A.
- For every topmost inside node v:
  - Let $P(v)$ be the points in the subtree of v in T.
  - We know that all x-coordinates of points in $P(v)$ are within range.
  - Recall: $P(v)$ is stored in $T(v)$.
  - To find points in $P(v)$ where the y-cordinates are within range as well, perform a range search in $T(v)$: *BST::RangeSearch*$(T(v), y_1, y_2)$

Range tree range search example



Range tree range search example



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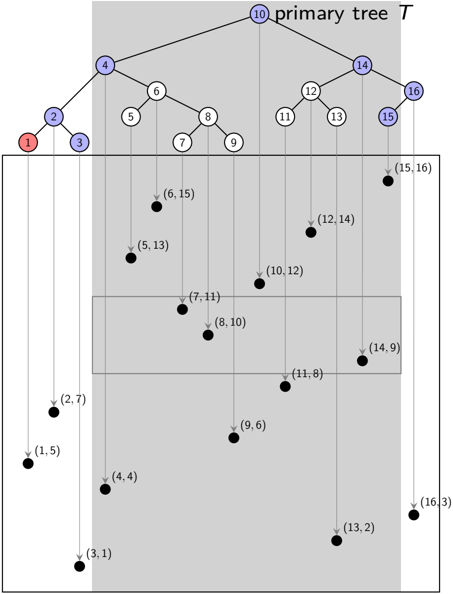
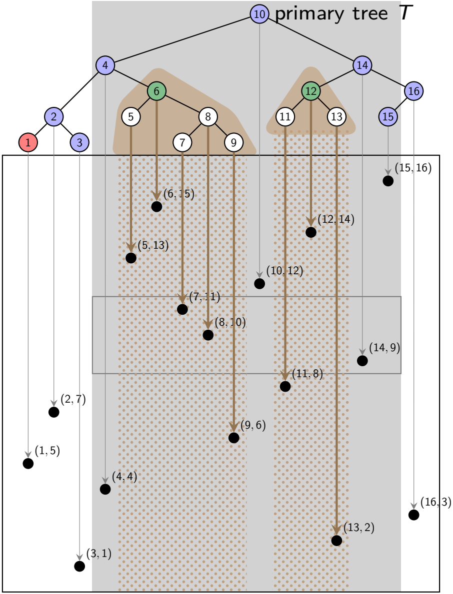


Range tree range search example



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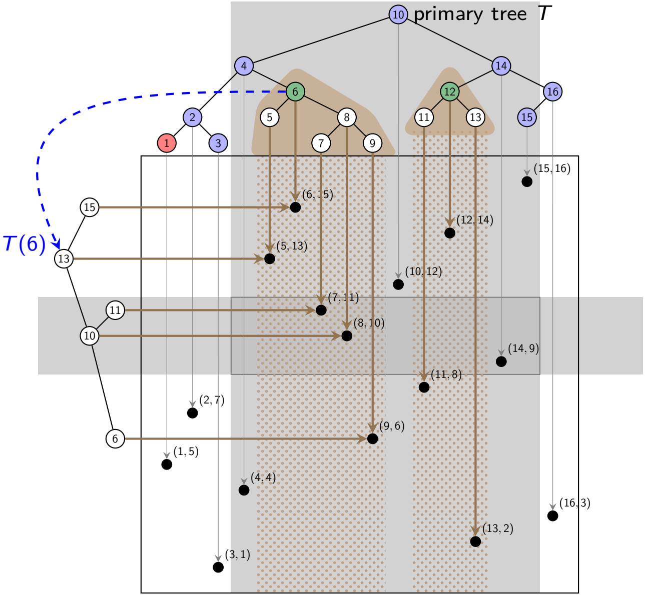
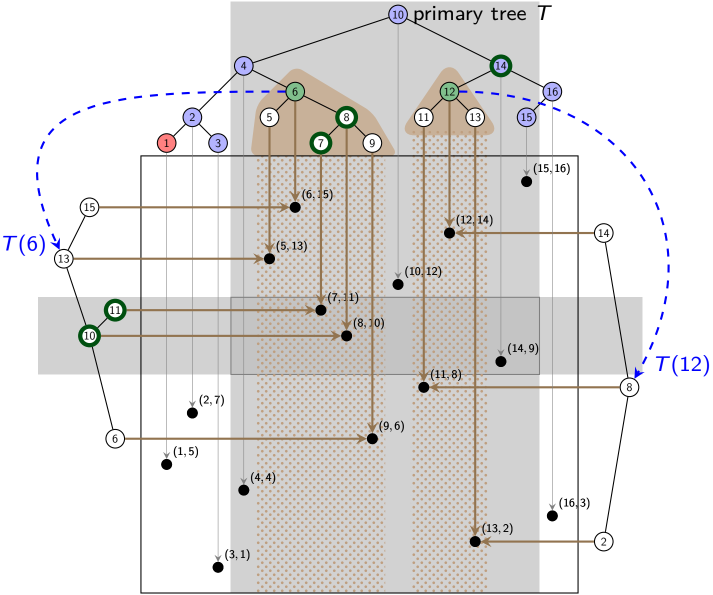


Range tree range search example



Range Trees: Range Search Run-time

- $O(\log n)$ time to find boundary and topmost inside nodes in primary tree.
- There are $O(\log n)$ such nodes.
- $O(\log n + s_v)$ time for each topmost inside node v, where s_v is the number of points in $T(v)$ that are reported
- Two topmost inside nodes have no common point in their trees
  $\Rightarrow$ every point is reported in at most one associate structure
  $\Rightarrow \sum_{v \text{ topmost inside}} s_v \leq s$

Time for range search in range-tree is proportional to

$$\sum_{v \text{ topmost inside}} (\log n + s_v) \in O(\log^2 n + s)$$

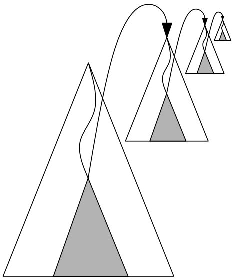
(There are ways to make this even faster. No details.)

Range Trees: Higher Dimensions

- Range trees can be generalized to d-dimensional space.

Space	$O(n\,(\log n)^{d-1})$
Construction time	$O(n\,(\log n)^{d})$
Range search time	$O(s + (\log n)^{d})$

(Note: d is considered to be a constant.)



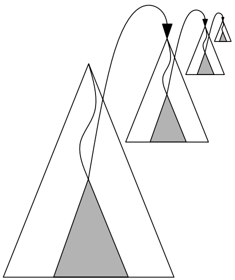
Range Trees: Higher Dimensions

- Range trees can be generalized to d-dimensional space.

Space	$O(n(\log n)^{d-1})$	kd-trees: $O(n)$
Construction time	$O(n(\log n)^{d})$	kd-trees: $O(n \log n)$
Range search time	$O(s + (\log n)^{d})$	kd-trees: $O(s + n^{1-1/d})$

(Note: d is considered to be a constant.)

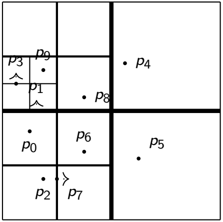
- Space/time trade-off compared to kd-trees.



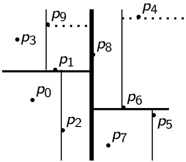
Outline

Range search data structures summary

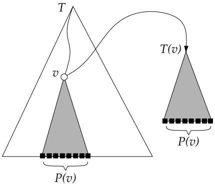
- Quadtrees
  - simple (also for dynamic set of points)
  - work well only if points evenly distributed
  - wastes space for higher dimensions



- kd-trees
  - linear space
  - range search time $O(\sqrt{n} + s)$
  - inserts/deletes destroy balance and range search time (no simple fix)



- range-trees
  - range search time $O(\log^2 n + s)$
  - wastes some space
  - inserts/deletes destroy balance (can fix this with occasional rebuilt)



Convention: Points on split lines belong to right/top side.