CS 240 – Data Structures and Data Management

Module 11: External Memory

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Based on lecture notes by many previous cs240 instructors

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Outline

- External Memory
  - Motivation
  - Stream based algorithms
  - External sorting
  - External dictionaries
    - 2-4 Trees
    - (*a*, *b*)-Trees
    - B-Trees

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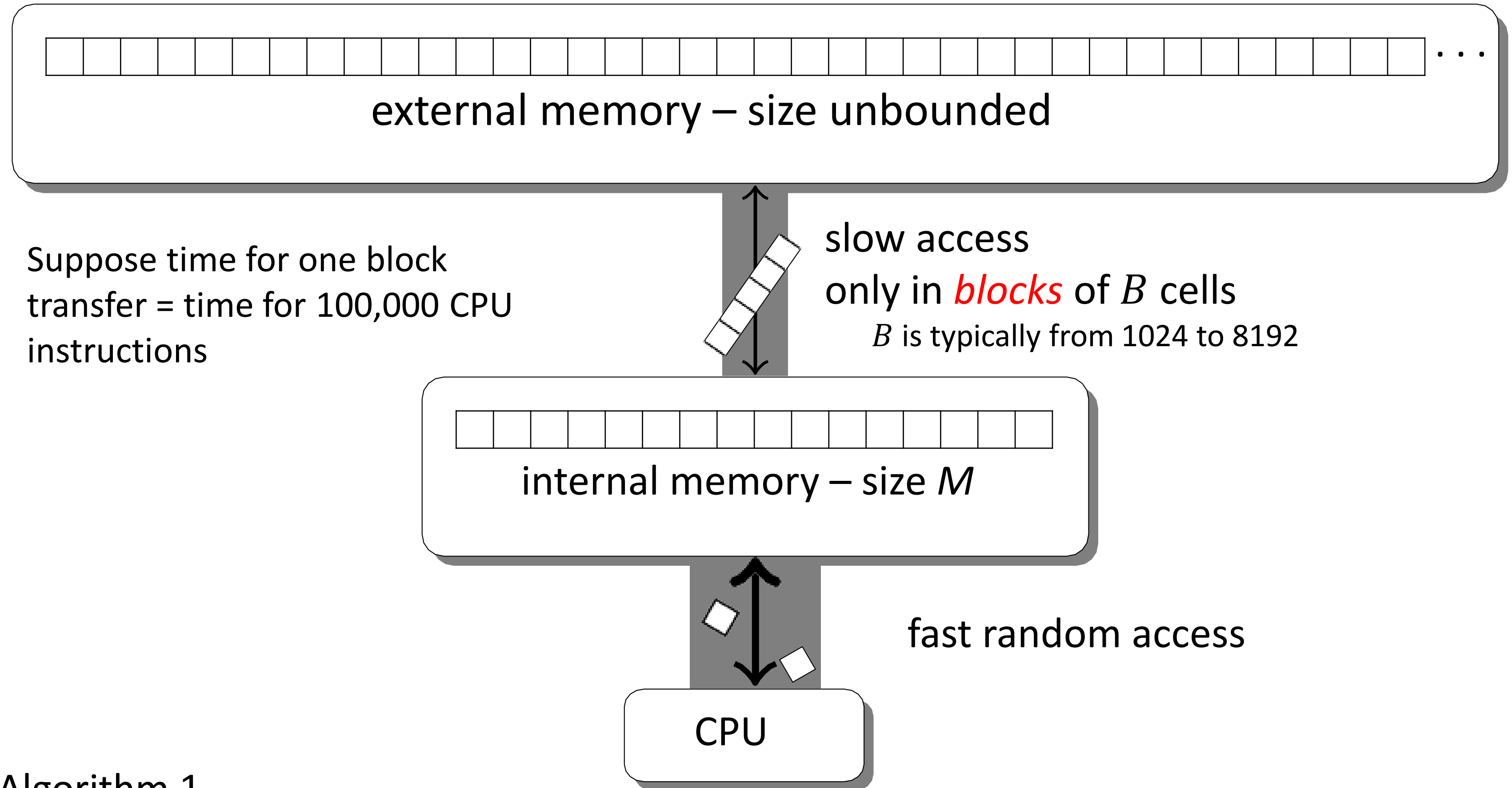
Different levels of memory

- Current architectures
    - registers: super fast, very small
    - cache L1, L2: very fast, less small
    - main memory: fast, large
    - disk or cloud: slow, very large
- How to adapt algorithms to take memory hierarchy into consideration?
    - desirable to minimize transfer between slow/fast memory
- To simplify, we focus on two levels of hierarchy
    - main (internal) memory and disk or cloud (external) memory
    - accessing a single location in external memory automatically loads a whole block (or “page”)
        - one block access can take as much time as executing 100,000 CPU instructions
        - need to care about the number of block accesses
    - new objective
        - revisit ADTs/problems with the objective of minimizing block transfers (“probes”, “disk transfers”, “page loads”)

Adding External-Memory Model (EMM)



Suppose time for one block transfer = time for 100,000 CPU instructions

- Algorithm 1
  ~~1,000 CPU instructions~~ + 1,000 block transfers $= \cancel{1{,}000} + 1{,}000 \cdot 100{,}000 = \cancel{10^3} + 10^8$
- Algorithm 2
  ~~10,000 CPU instructions~~ + 10 block transfers $= \cancel{10{,}000} + 10 \cdot 100{,}000 = \cancel{10^4} + 10^6$

dominating factors

- **New cost of computation:** number of blocks transferred between internal and external memory

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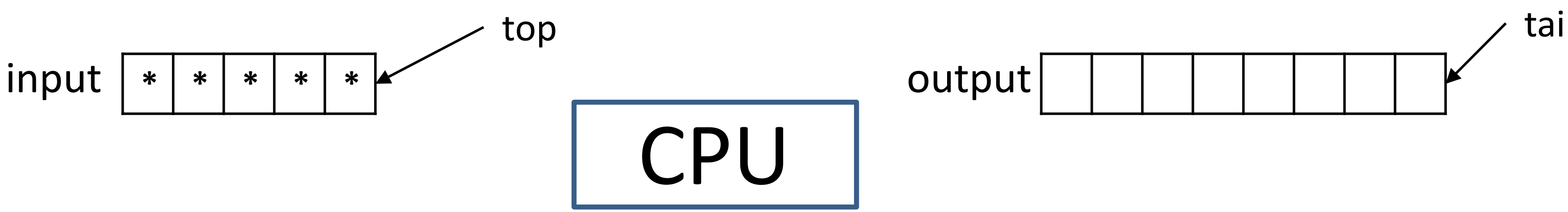
  - Extendible Hashing

Stream Based Algorithms in Internal Memory

- We studied some algorithms that handle input/output with streams
  - can access only the top item in input stream, can append only to tail of the output stream



- Repeat
  1. take item off top of the input
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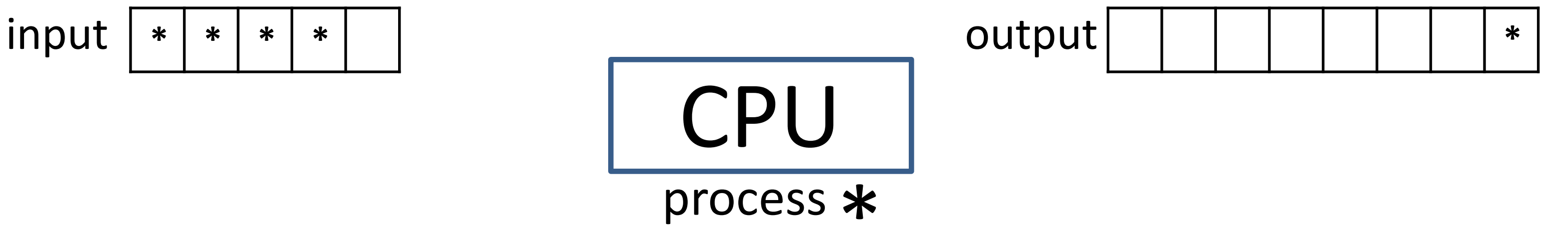

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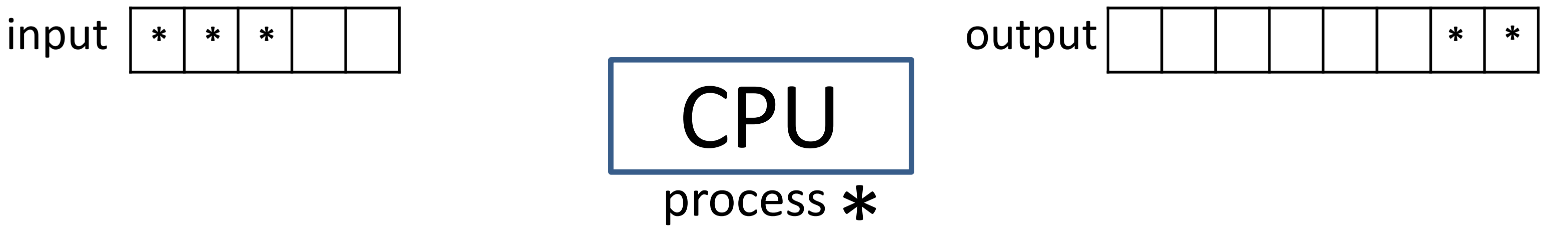

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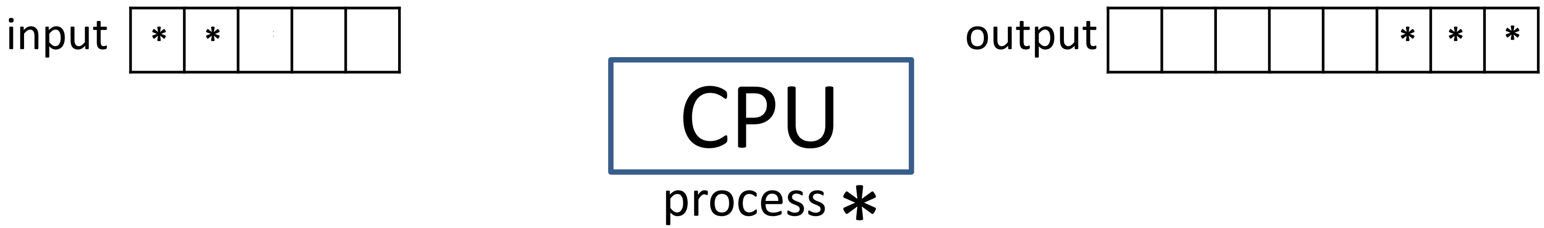

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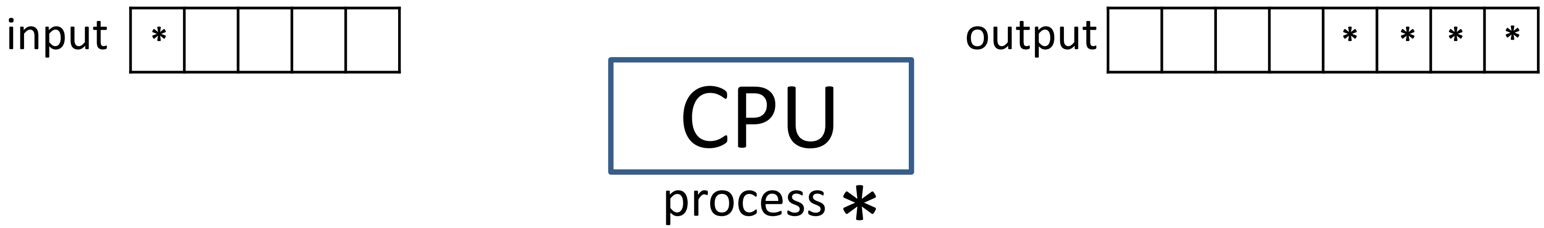

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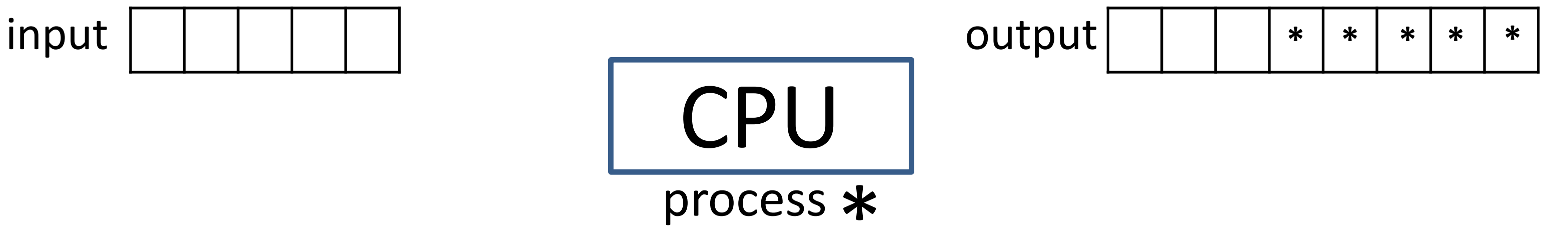

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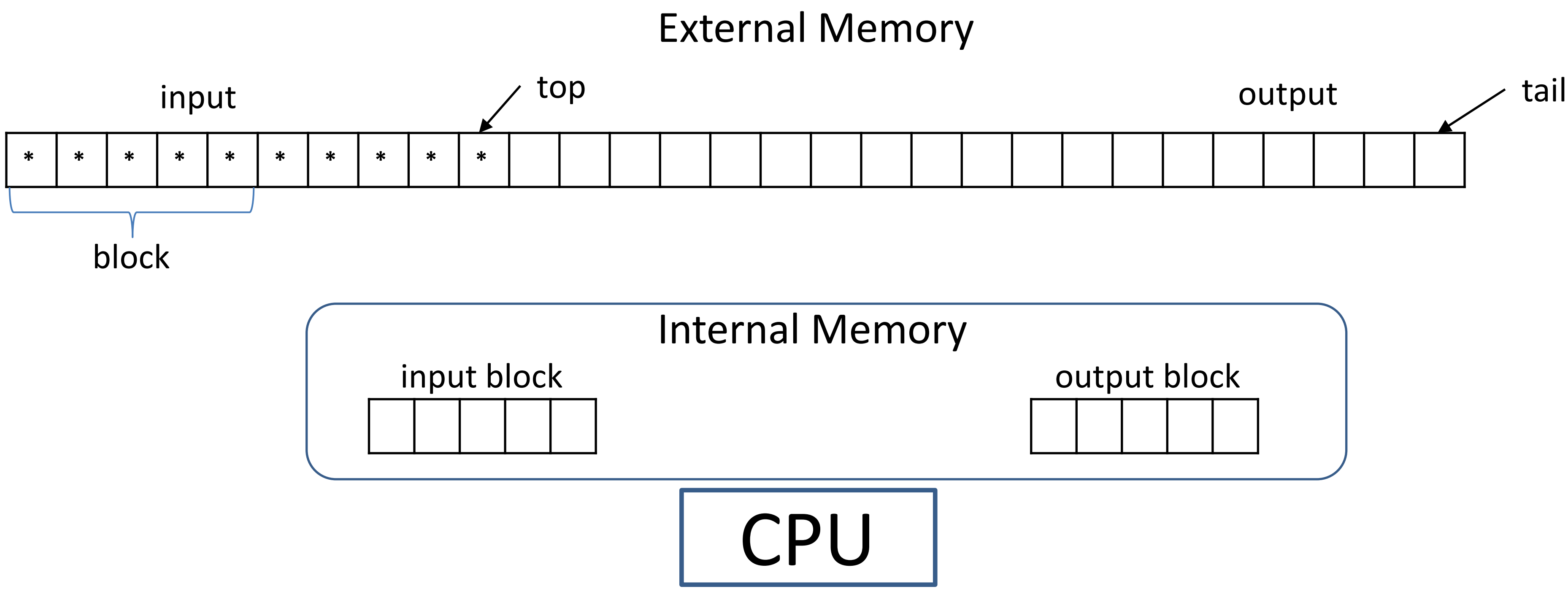
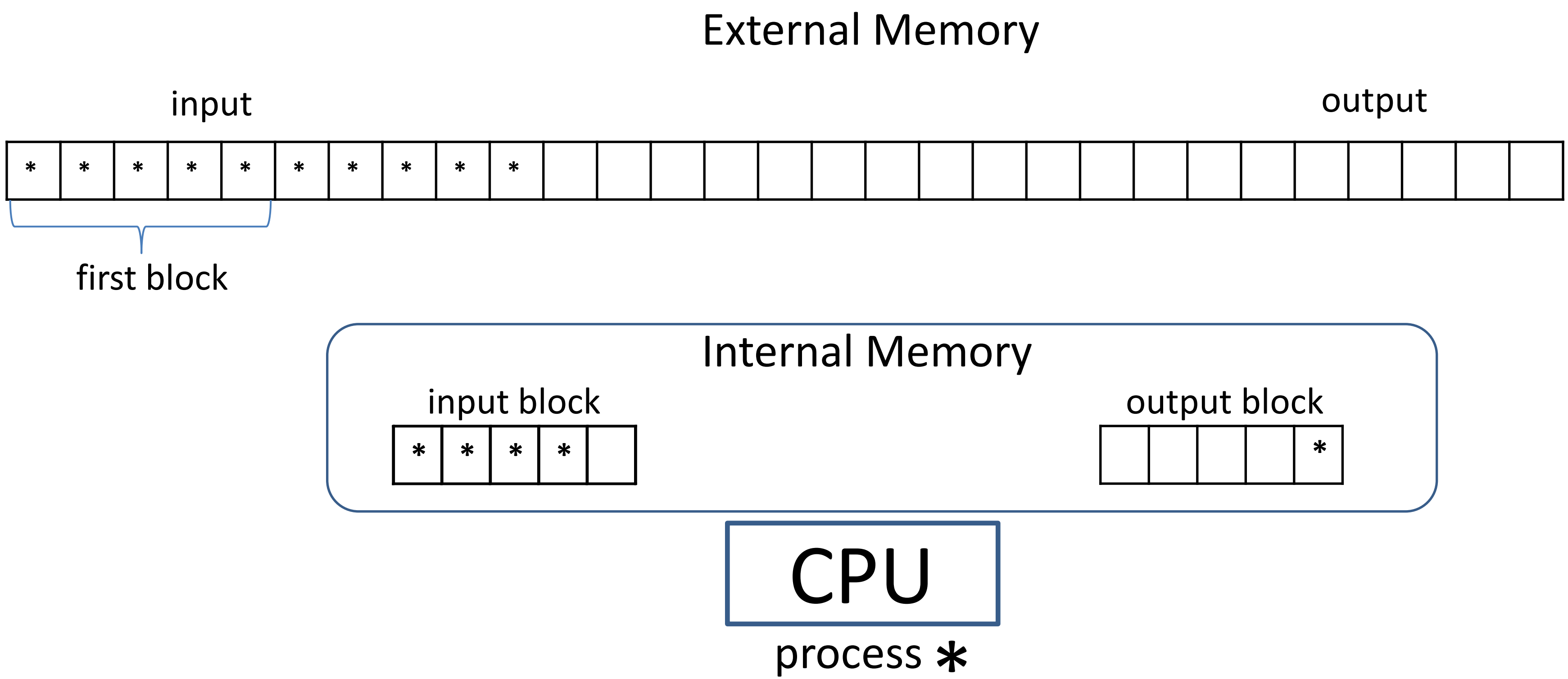


- Data in external memory has to be placed in internal memory before it can be processed
- Idea: perform the same algorithm as before, but in “block-wise” manner
  - have one block for input, one block for output in internal memory
  - transfer a block (size B) to internal memory, process it as before, store result in output block
  - when output stream is of size B (full block), transfer it to external memory
  - when current block is in internal memory is fully processed, transfer next unprocessed block from external memory

Stream Based Algorithms in External Memory



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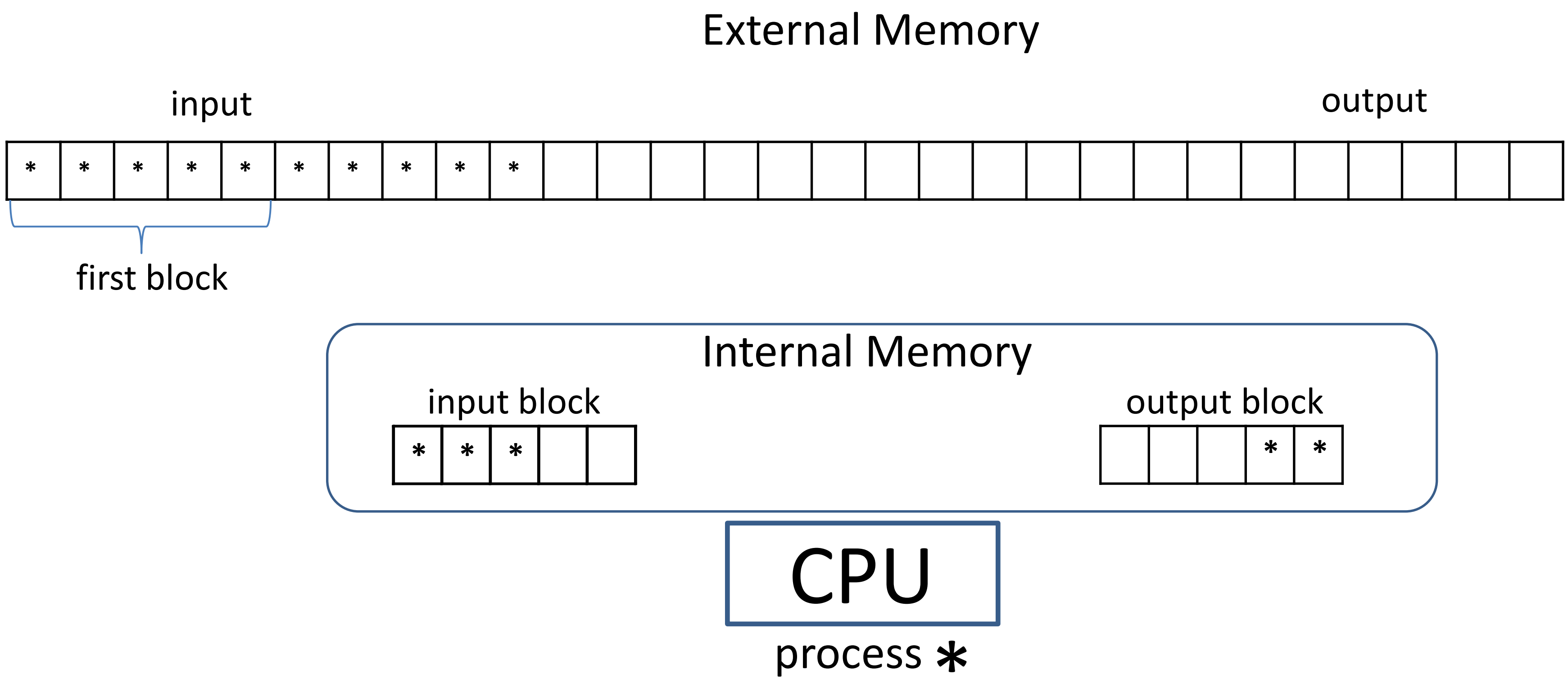

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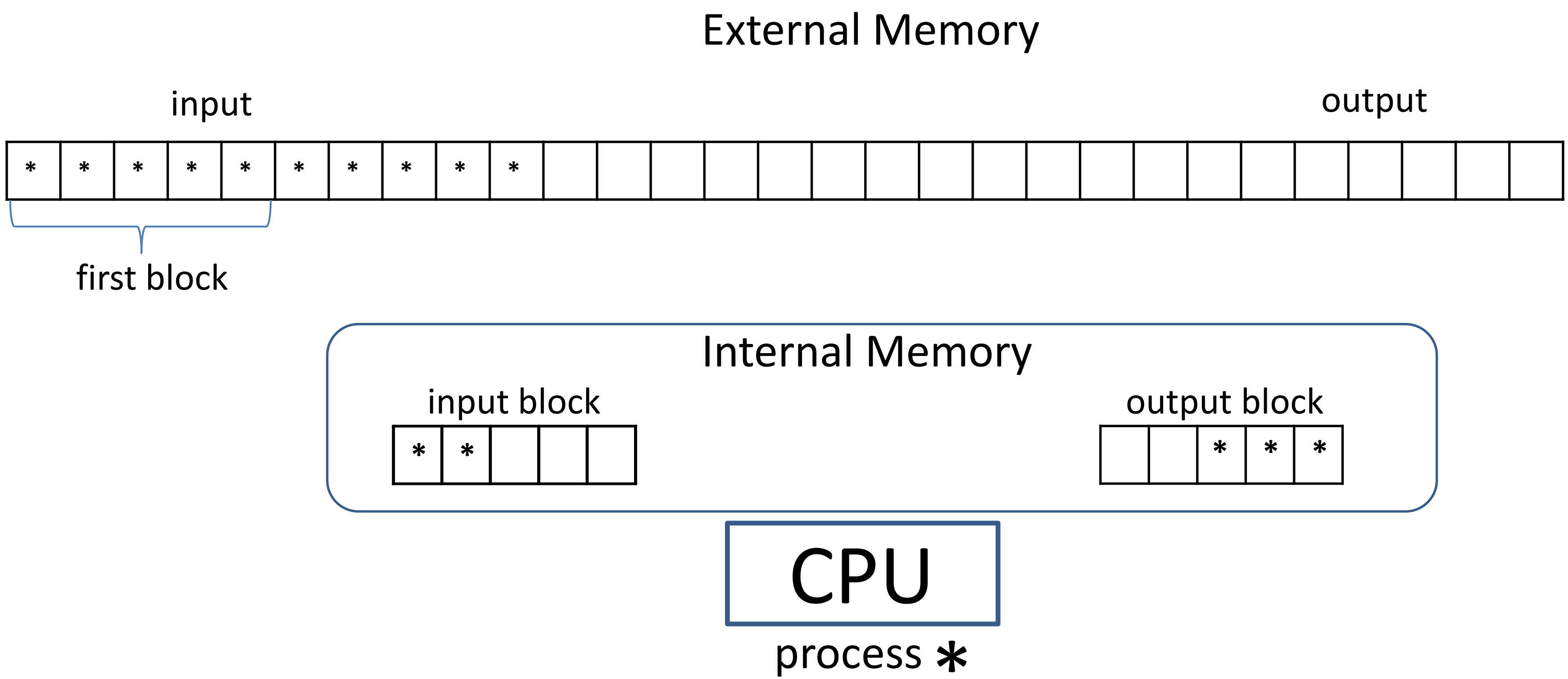

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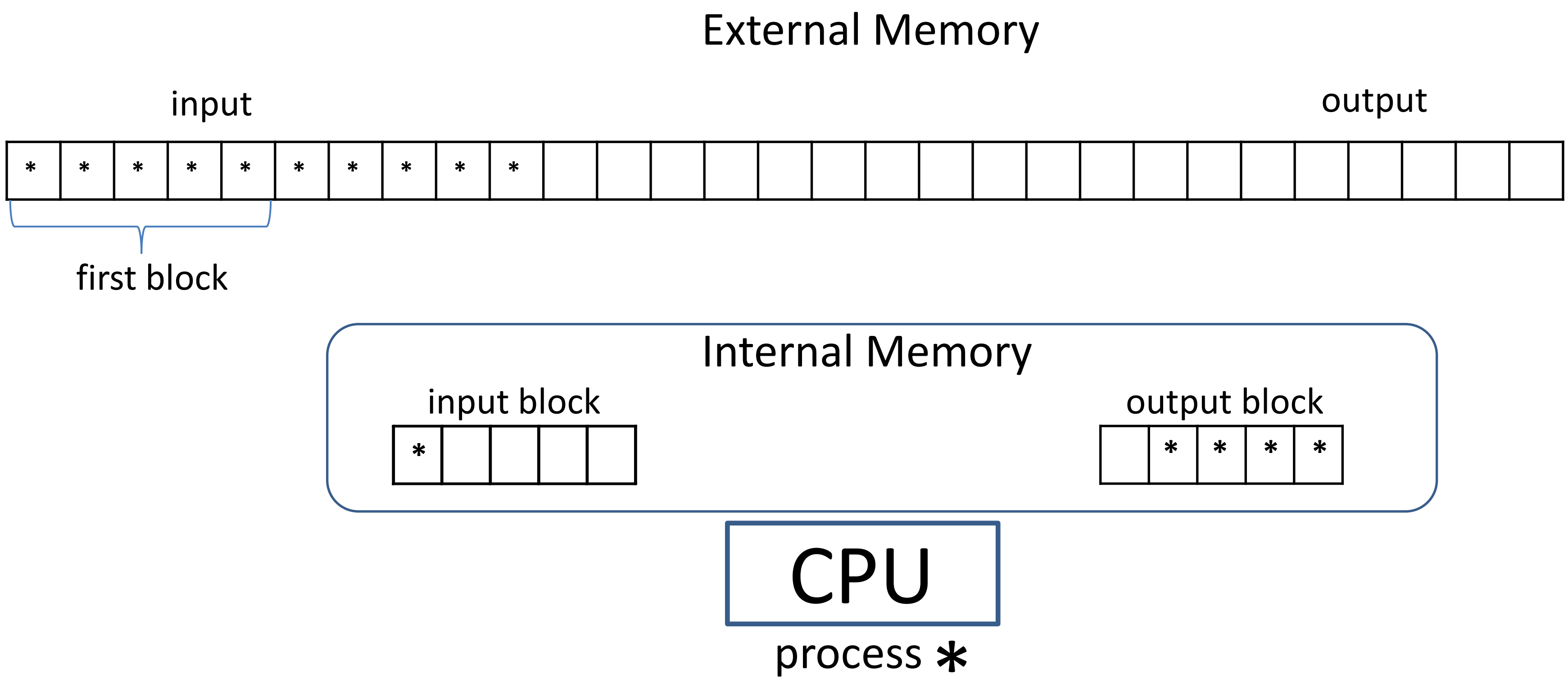

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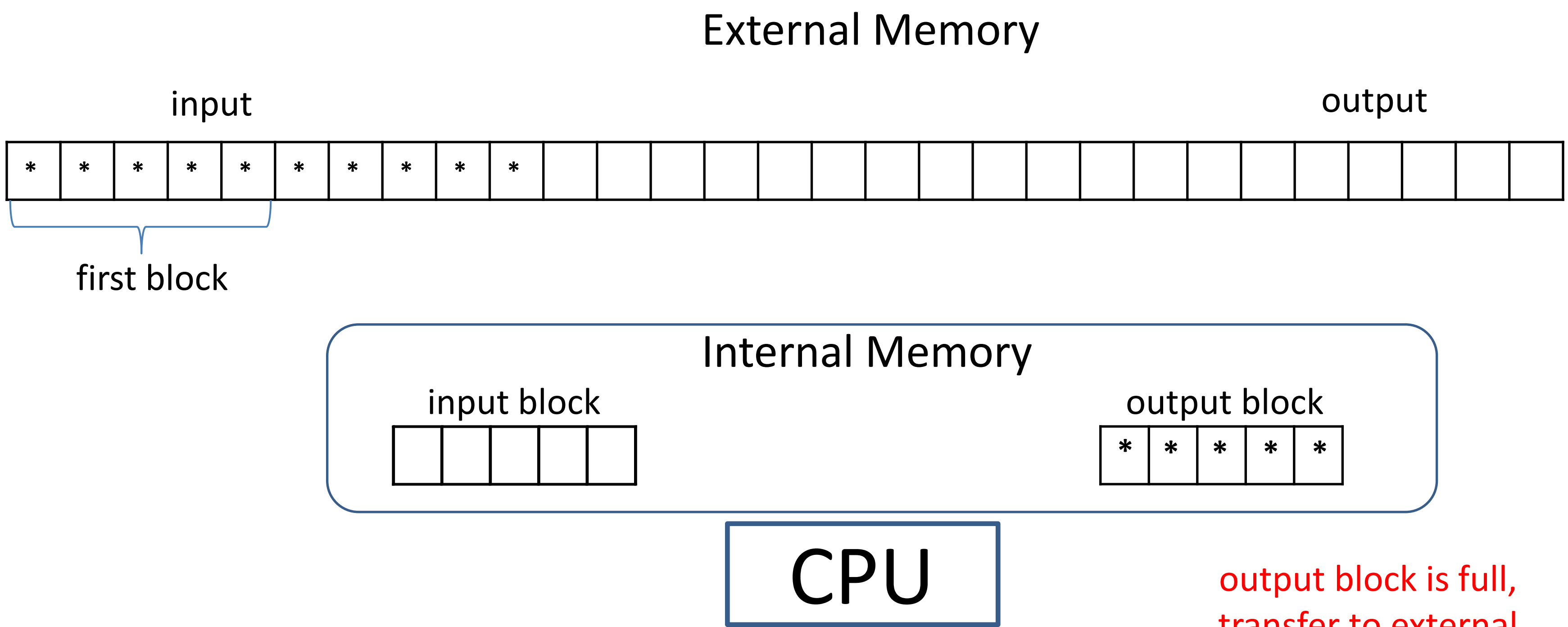
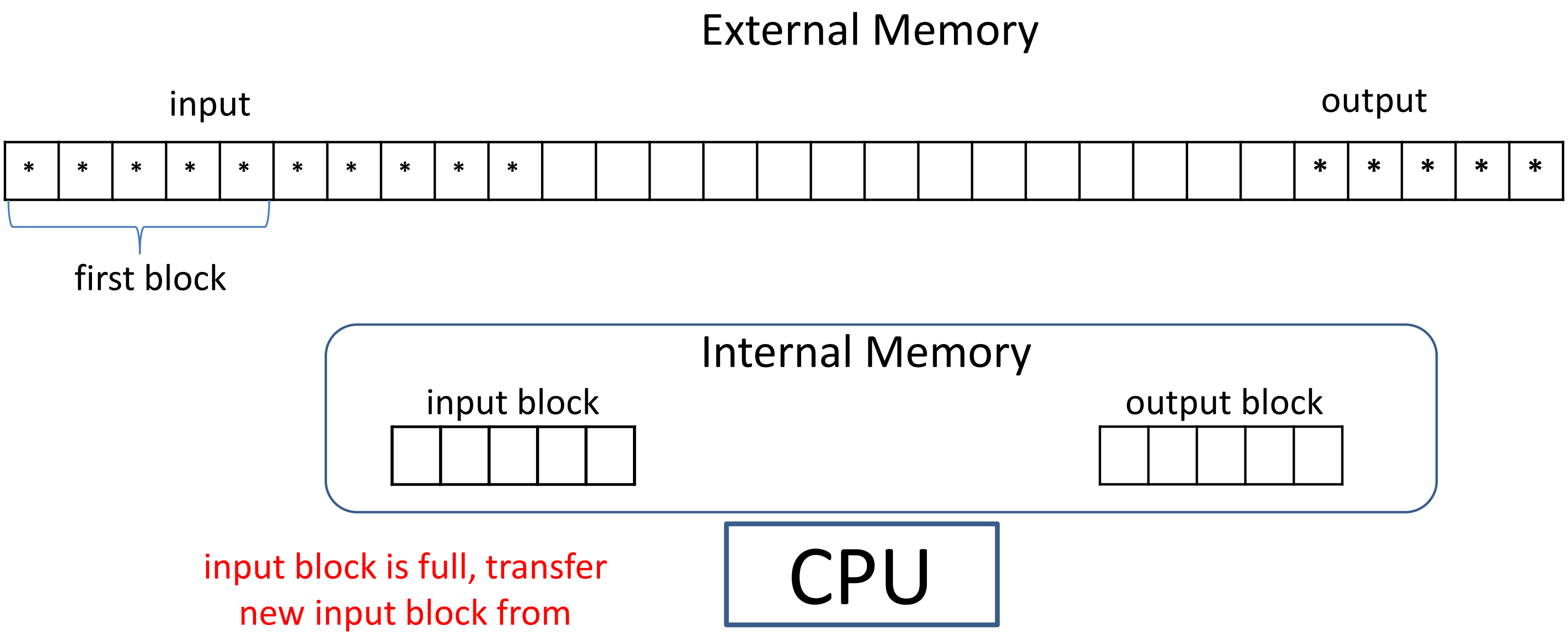


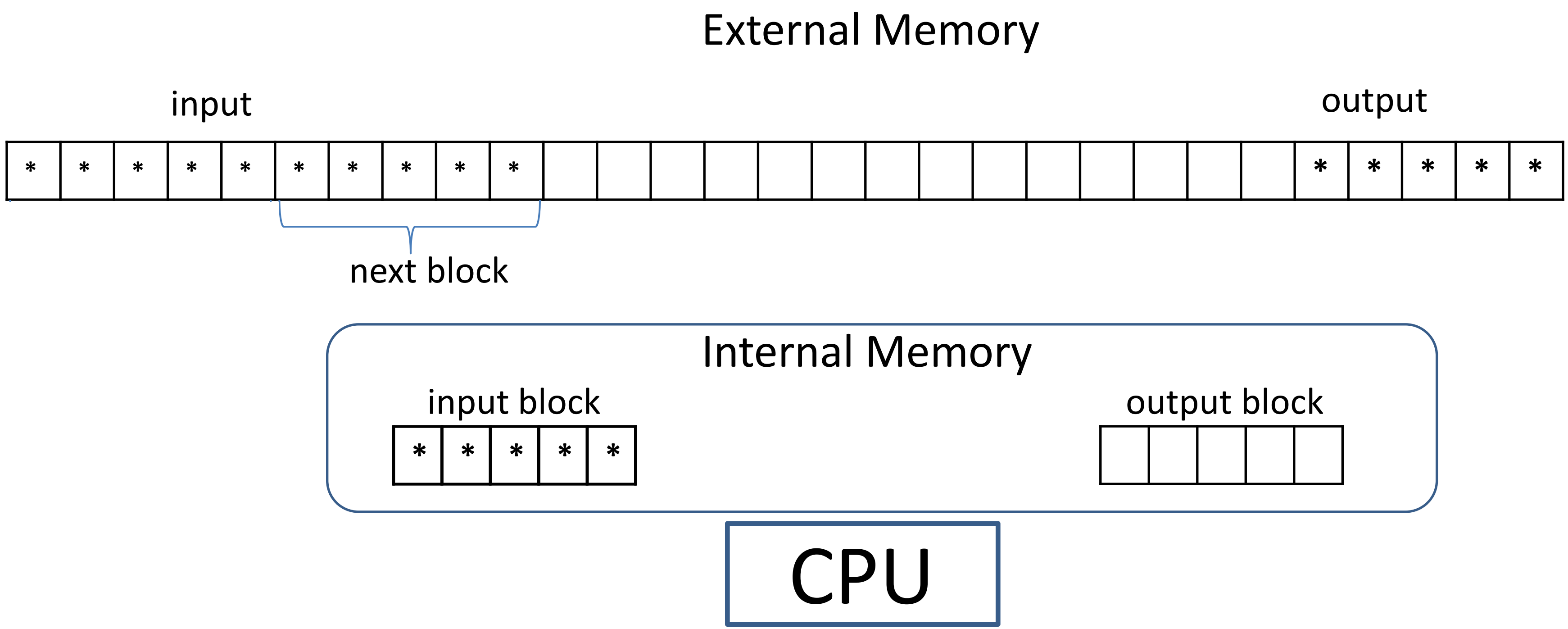
output block is full, transfer to external memory

Stream Based Algorithms in External Memory



input block is full, transfer
new input block from
external memory

Stream Based Algorithms in External Memory



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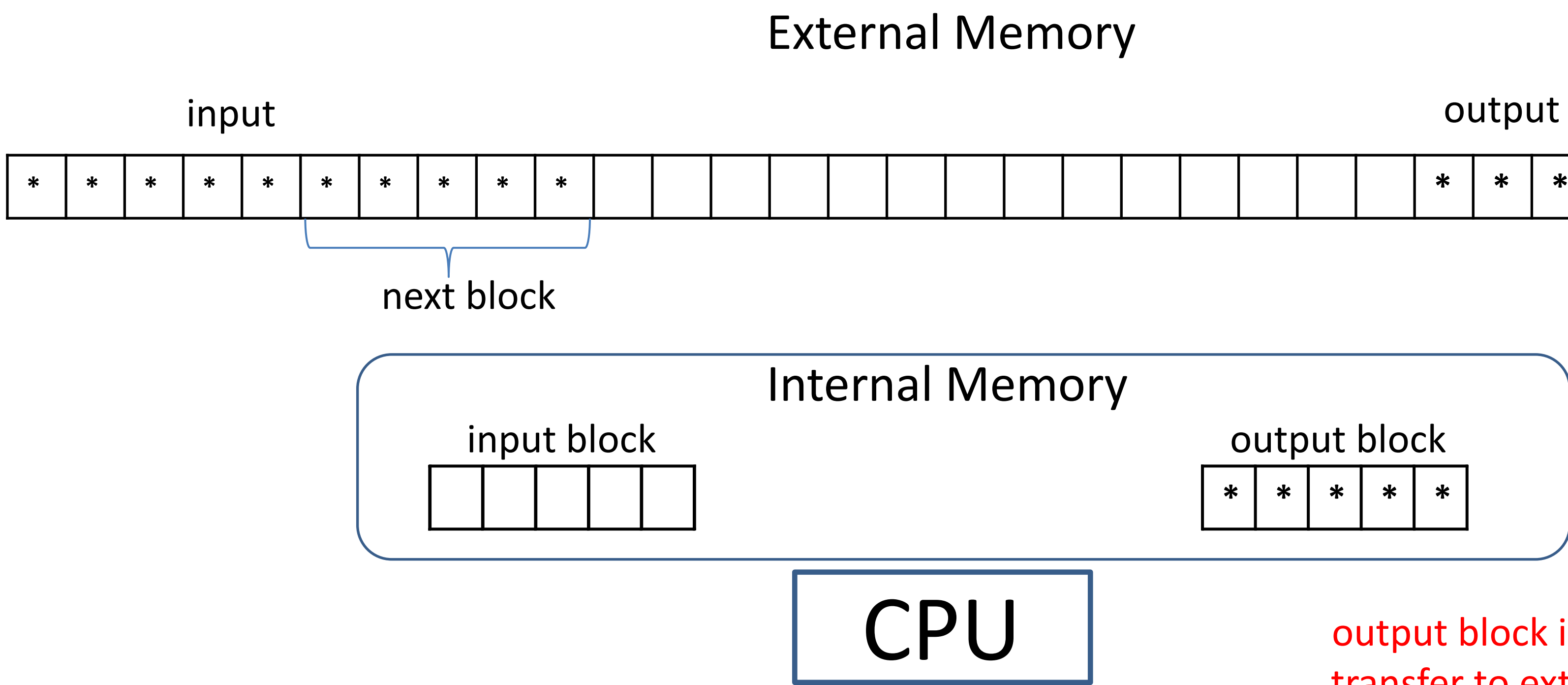

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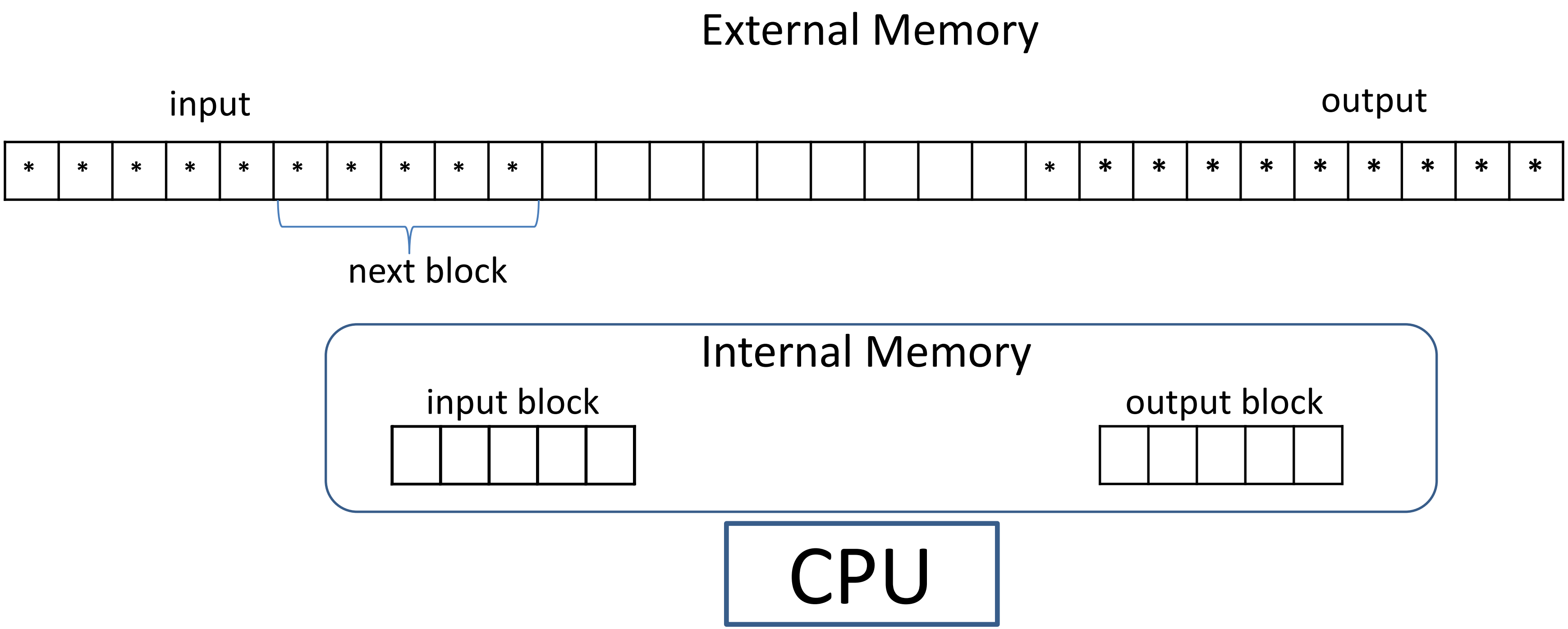

- Running time is (recall that we only count the block transfers now)
  - input stream: $\frac{n}{B}$ block transfers to read input of size n
  - output stream: $\frac{s}{B}$ block transfers to write output of size s
- Running time is *automatically* as efficient as possible for external memory
  - any algorithm needs at least $\frac{n}{B}$ block transfers to read input of size n and $\frac{s}{B}$ block transfers to write output of size s

Stream Based Algorithms in External Memory

- Methods below use stream input/output model, therefore need $\Theta\left(\frac{n}{B}\right)$ block transfers, assuming output size is $O(n)$
  - Pattern matching: Karp-Rabin, Knuth-Morris-Pratt, Boyer-Moore
    - assuming pattern P fits into internal memory
  - Text compression: Huffman, run-length encoding, Lempel-Ziv-Welch

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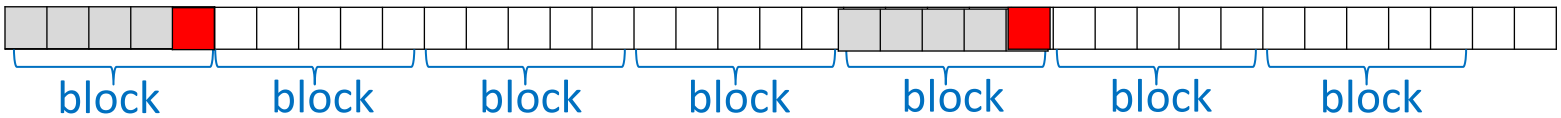
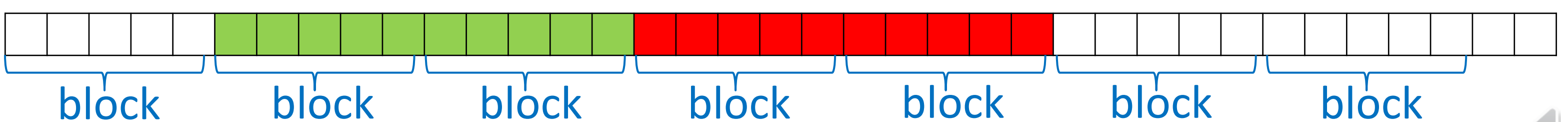
Sorting in external memory

- Sort array A of n numbers
  - n is huge so that A is stored in blocks in external memory
- Heapsort was optimal in time and space in RAM model
  - poor memory locality: accesses indices of A that are far apart

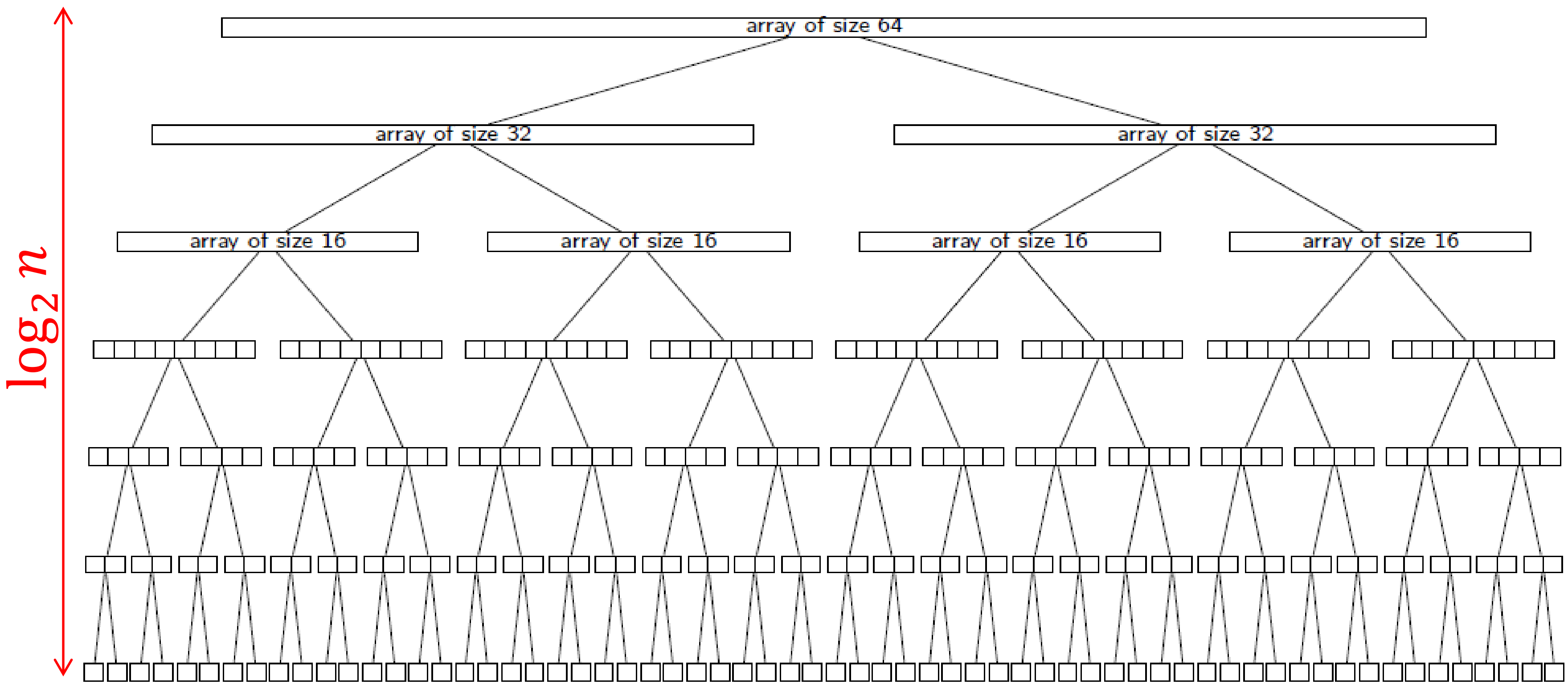


  - typically one block transfer per array access
    - access 2 blocks, but need only 2 elements in these blocks
    - all other data read in these 2 blocks is not used
  - heapsort does not adapt well to data stored in external memory
- Mergesort adapts well to array stored in external memory
  - based on merging already sorted parts of the array
  - access consecutive locations of A, ideal for reading in blocks



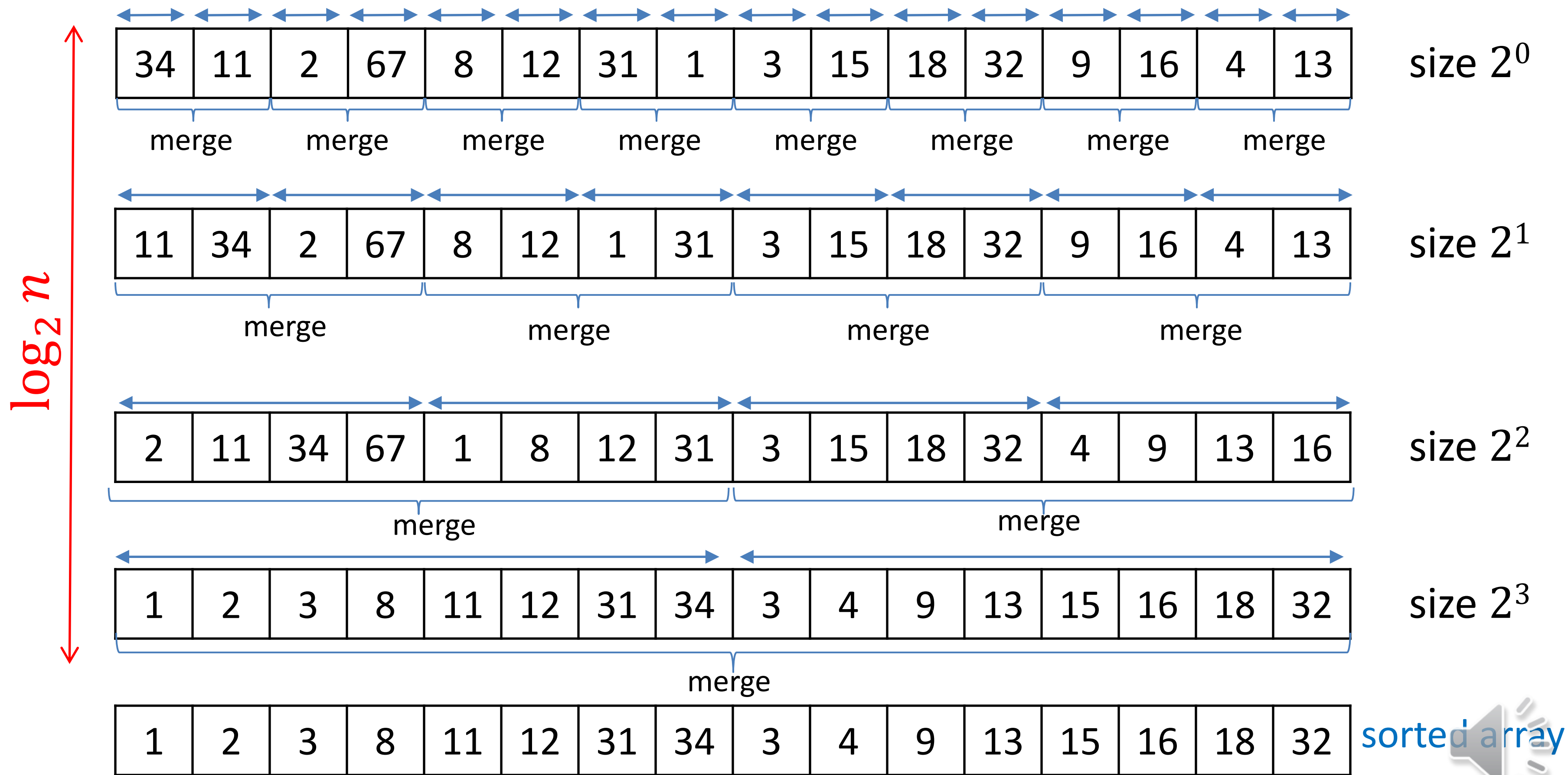
  - **key idea**: merge can be done with streams

Recall Mergesort



Recall Mergesort: non-recusive view

- Several rounds of merging adjacent pairs of sorted *runs* (run = subarray)
  - in round i, merge sorted runs of size 2^i
- Graphical notation ⟵ sorted run ⟶



Merging with Streams in External Memory

Merge(S_1, S_2, S)

S_1, S_2 are input streams in sorted order, S is output stream

while S_1 or S_2 is not empty **do**

- **if** S_1 is empty S.append$(S_2.pop(\,))$
- **else if** S_2 is empty S.append$(S_1.pop(\,))$
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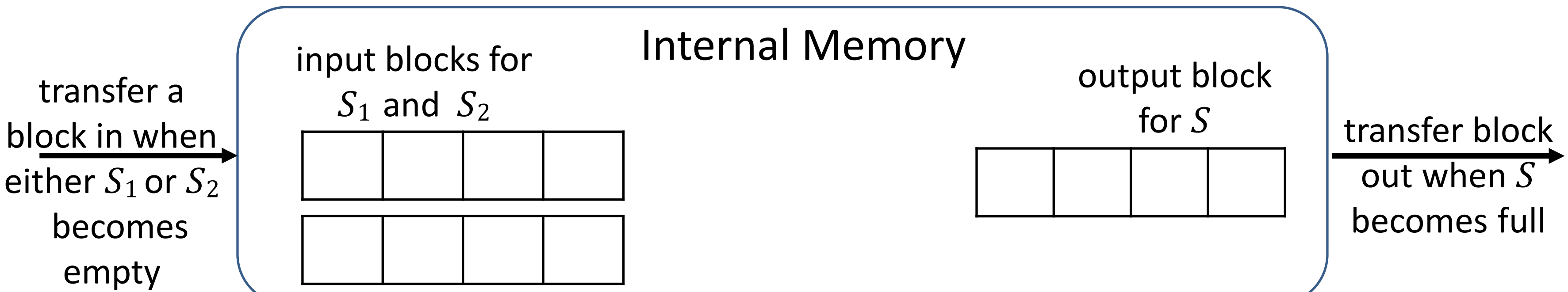

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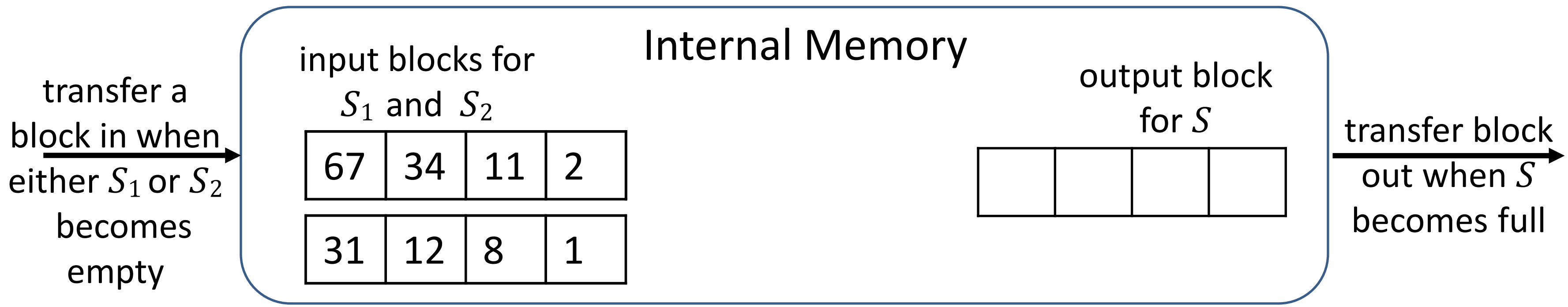

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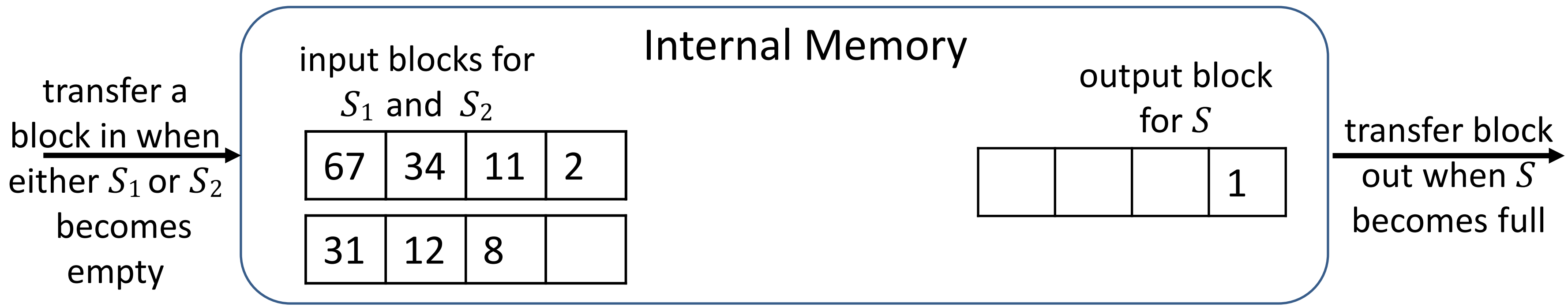

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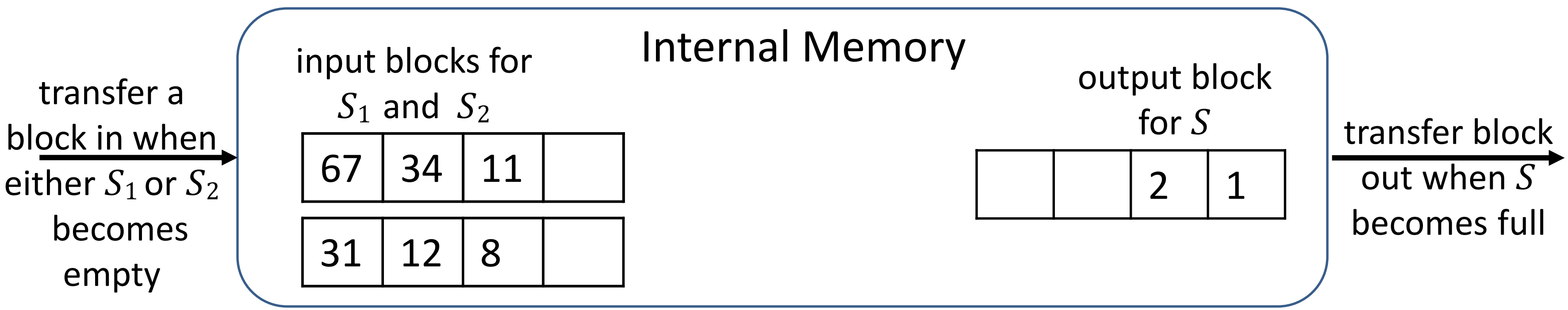

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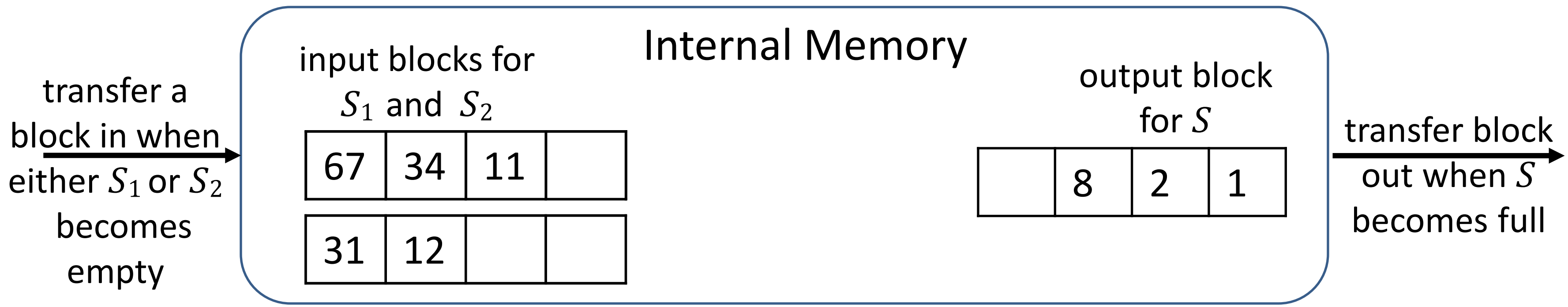

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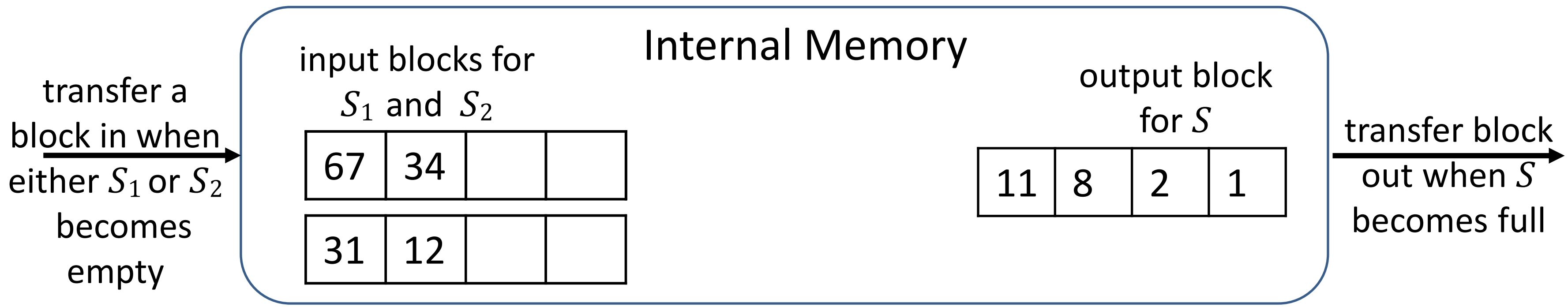

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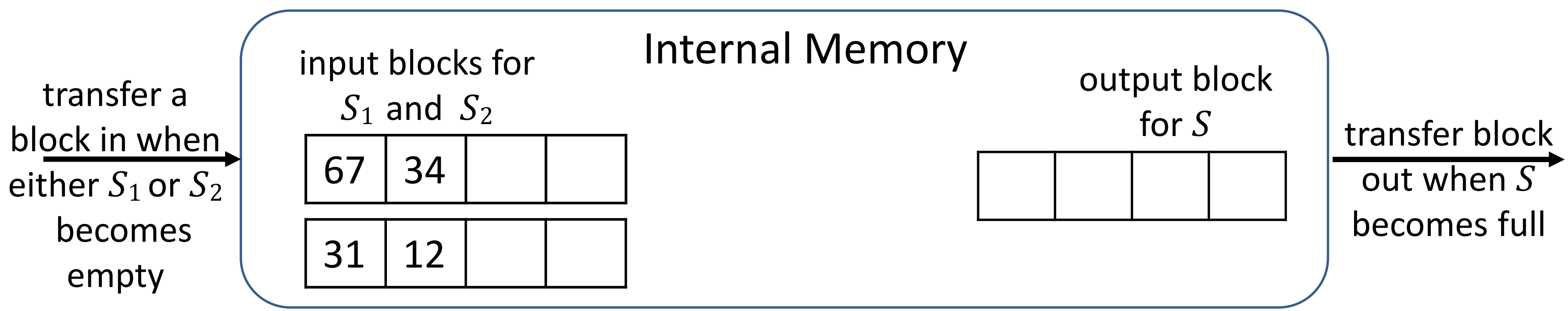

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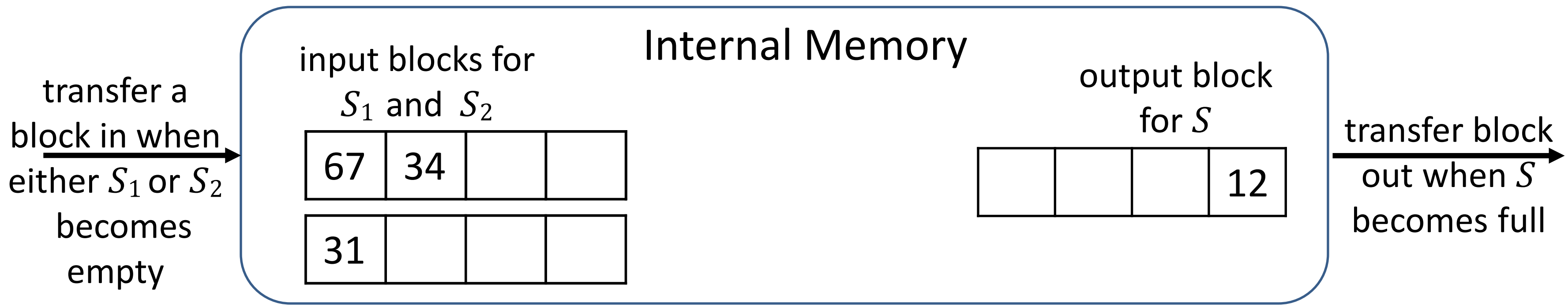

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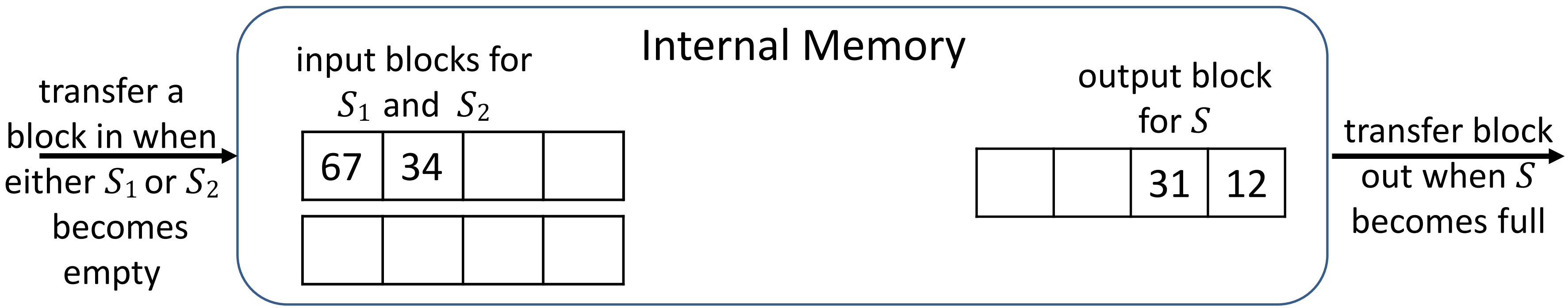

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input block for S_2 is empty, transfer next block for S_2 from external memory

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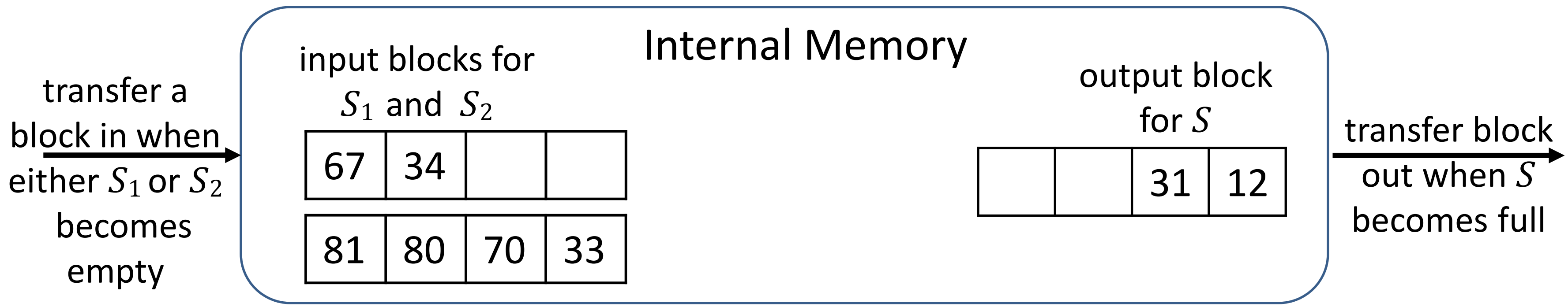

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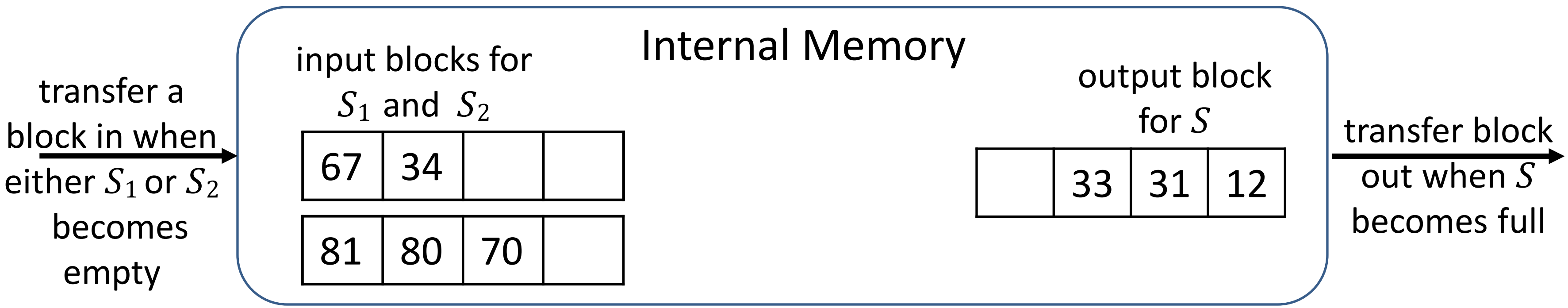

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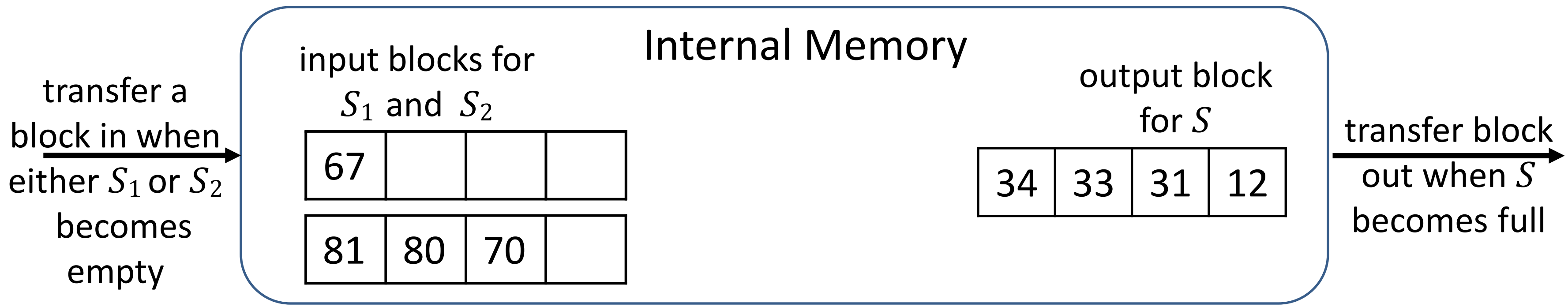

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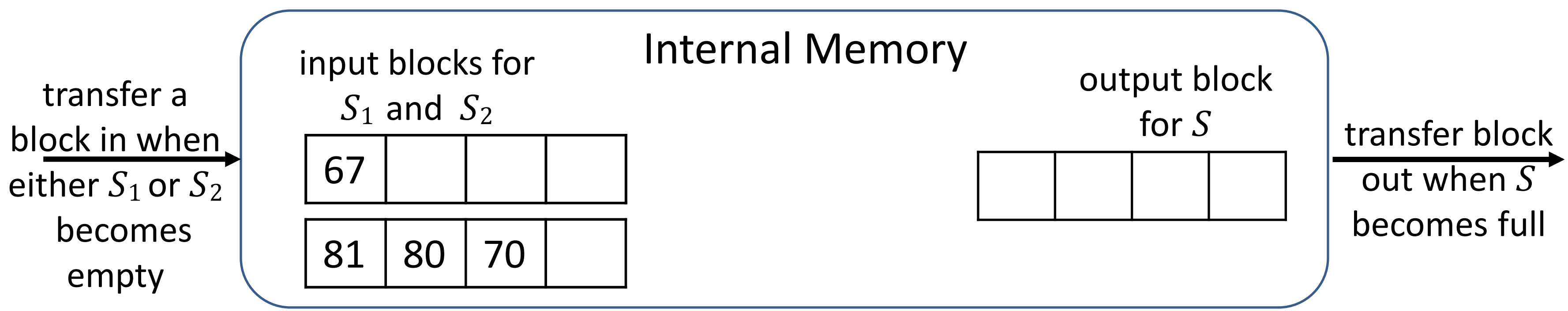

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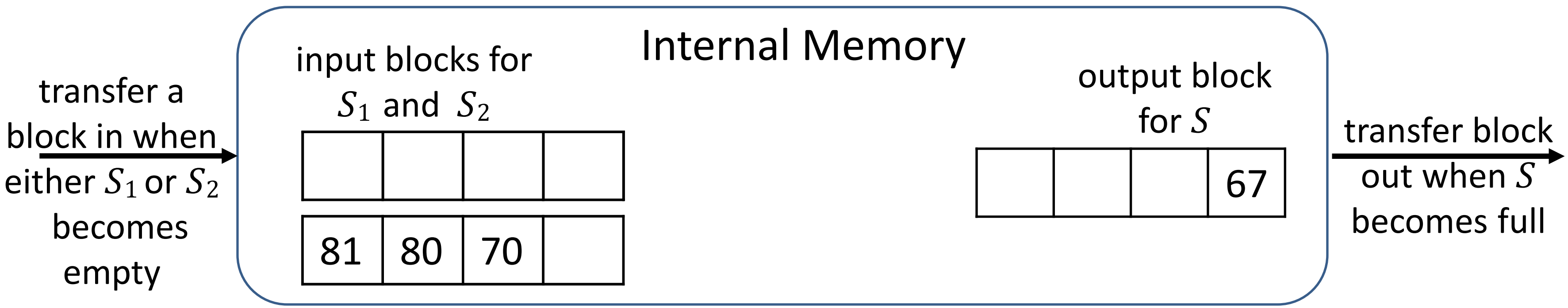

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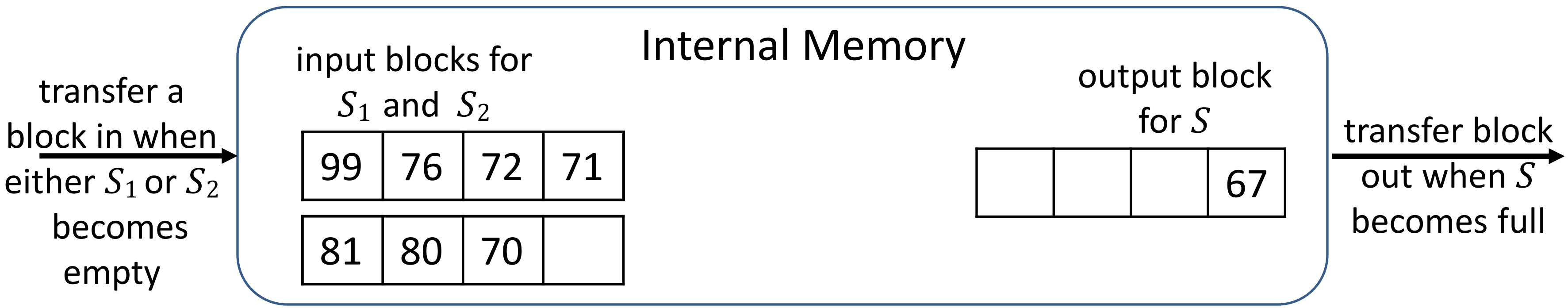

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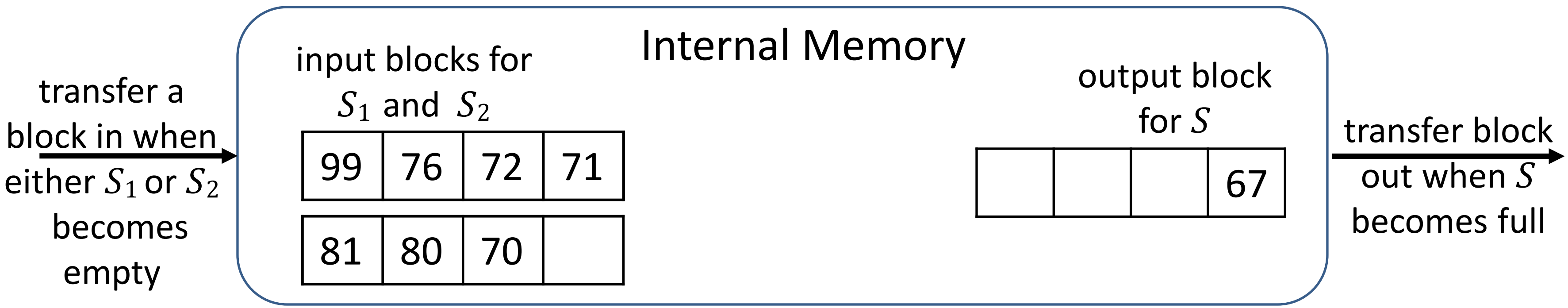
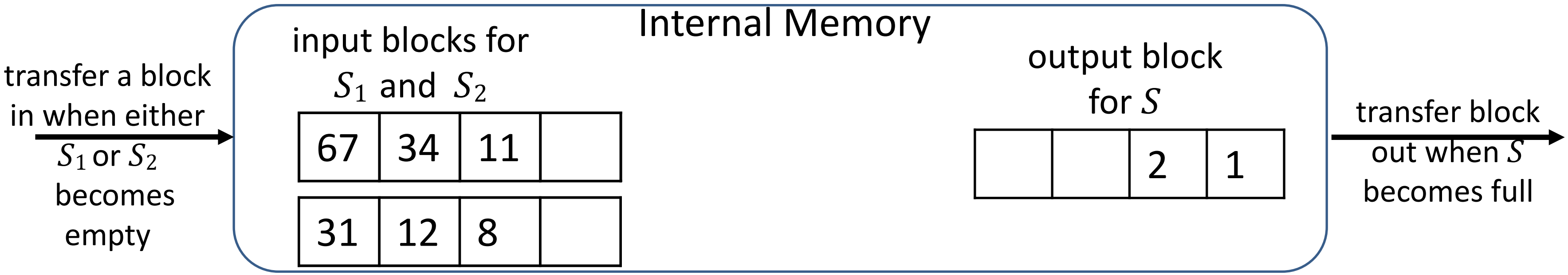
MergeSort Run Time in External Memory

- *Merge* uses streams S_1, S_2, S
  - each block in the stream is transferred exactly once
- *Merge* takes $\frac{n}{B}$ block transfers for input streams and $\frac{n}{B}$ for output stream, total $\frac{2n}{B}$
- Recall that MergeSort uses $\log_2 n$ rounds of merging
- *MergeSort* run-time to sort is $\frac{2n}{B} \cdot \log_2 n$ block transfers
  - not bad but we can do better

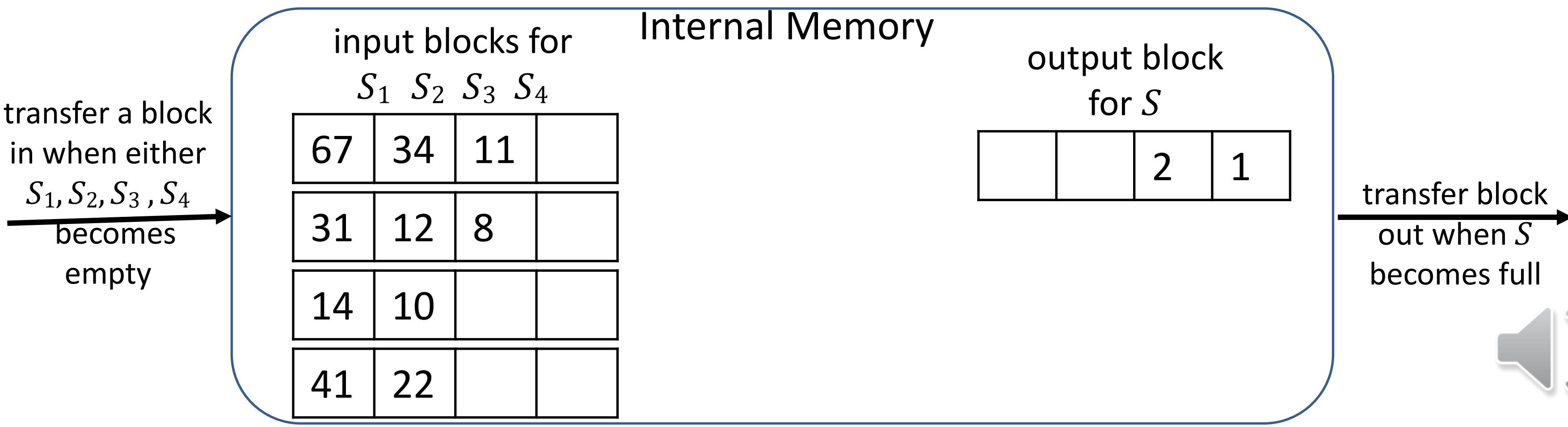


Towards d-way Mergesort

- Observe that we had space left in internal memory during *Merge*

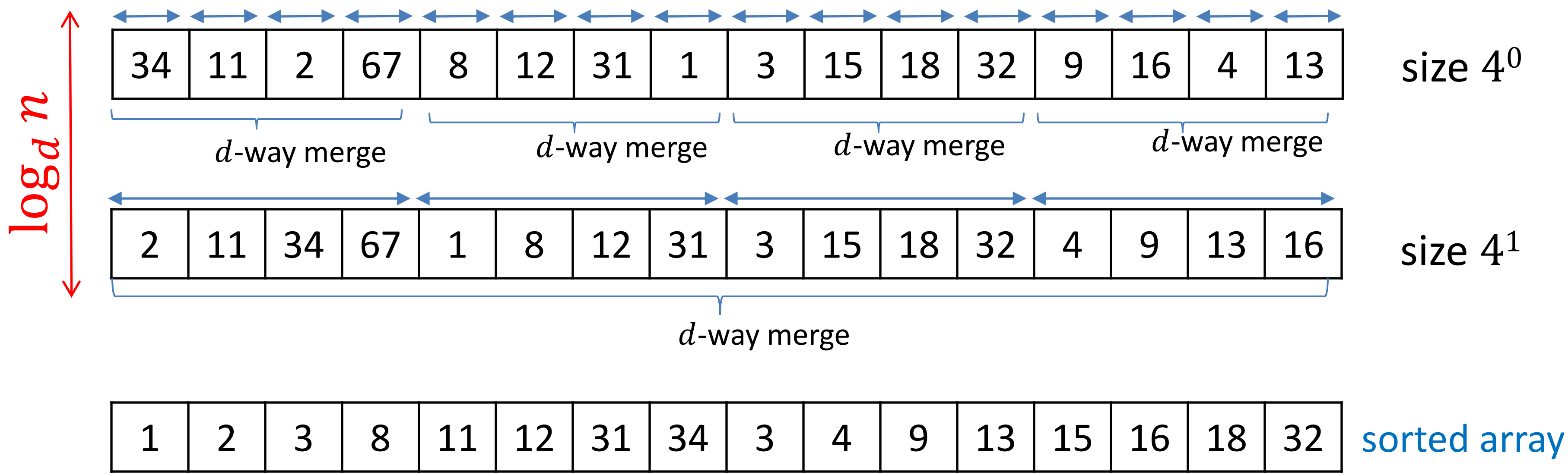


- We use only three blocks in internal memory, but typically $M \gg 3B$
  - M is the size of the internal memory
- Idea: can merge d parts at once, and it still takes $\frac{2n}{B}$ of block transfers
- Here $d \approx \frac{M}{B} - 1$ so that $d + 1$ blocks fit into internal memory



d-way Mergesort

- Merge d sorted runs at one time
  - $d = 2$ gives standard mergesort
- Example: $d = 4$



- $\log_d n = \dfrac{\log_2 n}{\log_2 d}$ rounds
  - the larger is d the less rounds
- How to merge d sorted runs efficiently?
  - d-way merge

d-way Merge

- $d = 3$

2	11	34	8	9	12	1	11	31

- $d = 5$

2	11	34	8	9	12	1	11	31	15	18	32	9	12	13

- $d = 16$

34	11	2	67	8	12	31	1	3	15	18	32	9	16	4	13

- Need efficient data structure to find the minimum among d current tops
  - although it does not effect efficiency in terms of block transfers

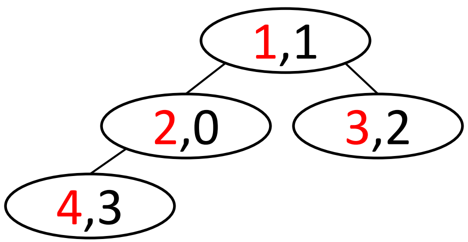
d-way Merge with Min-Heap

- Use min heap to find the smallest element among of d current tops
  - (key,value) = (element, sorted run)
- $d = 4$

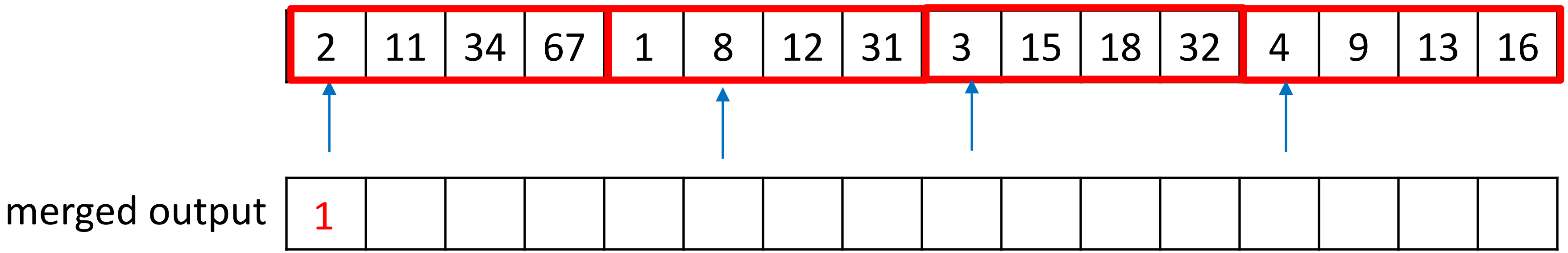
2	11	34	67	1	8	12	31	3	15	18	32	4	9	13	16

merged output

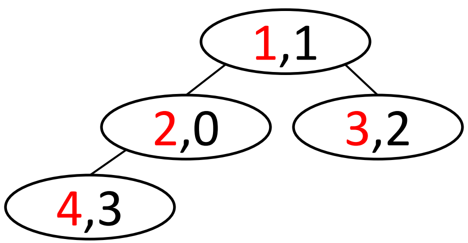
1) insert(2,0), insert(1,1), insert(3,2), insert(4,3)



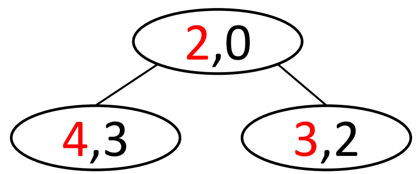
d-way Merge with Min-Heap



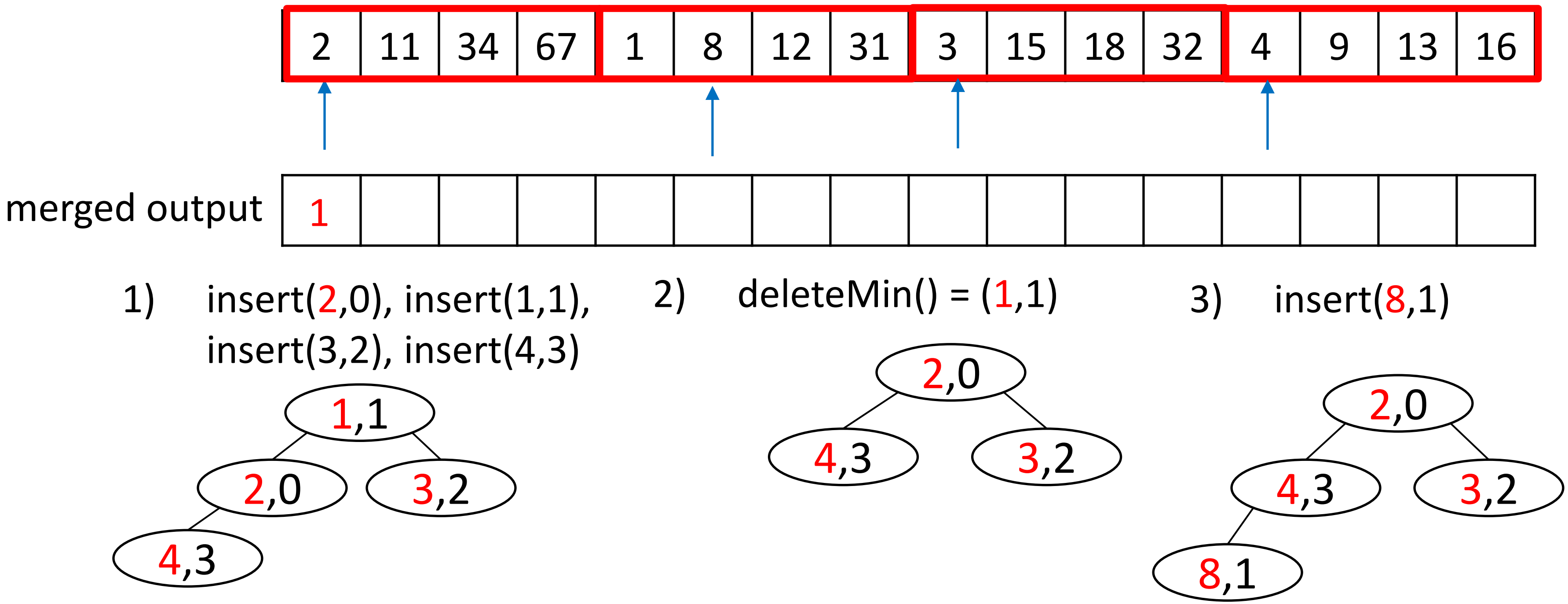
1) insert(2,0), insert(1,1), insert(3,2), insert(4,3)



2) deleteMin() = (1,1)

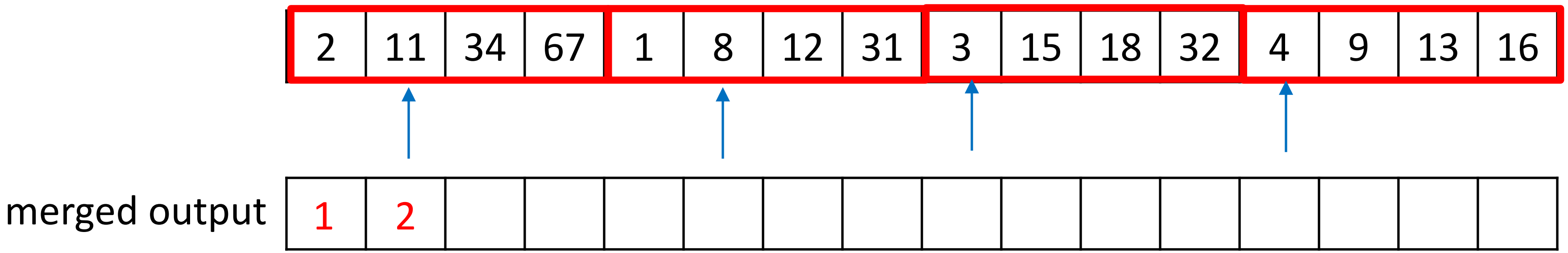


d-way Merge with Min-Heap

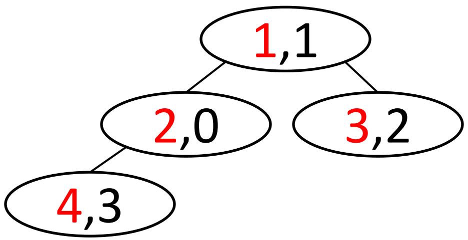


- Heap must have current fronts from all sorted runs
  - unless some sorted run ends

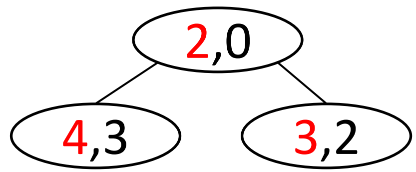
d-way Merge with Min-Heap



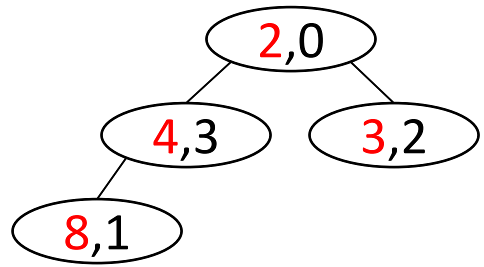
1) insert(2,0), insert(1,1), insert(3,2), insert(4,3)



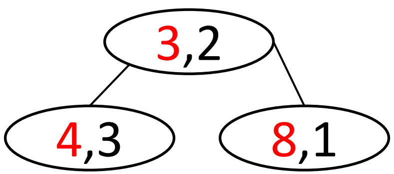
2) deleteMin() = (1,1)



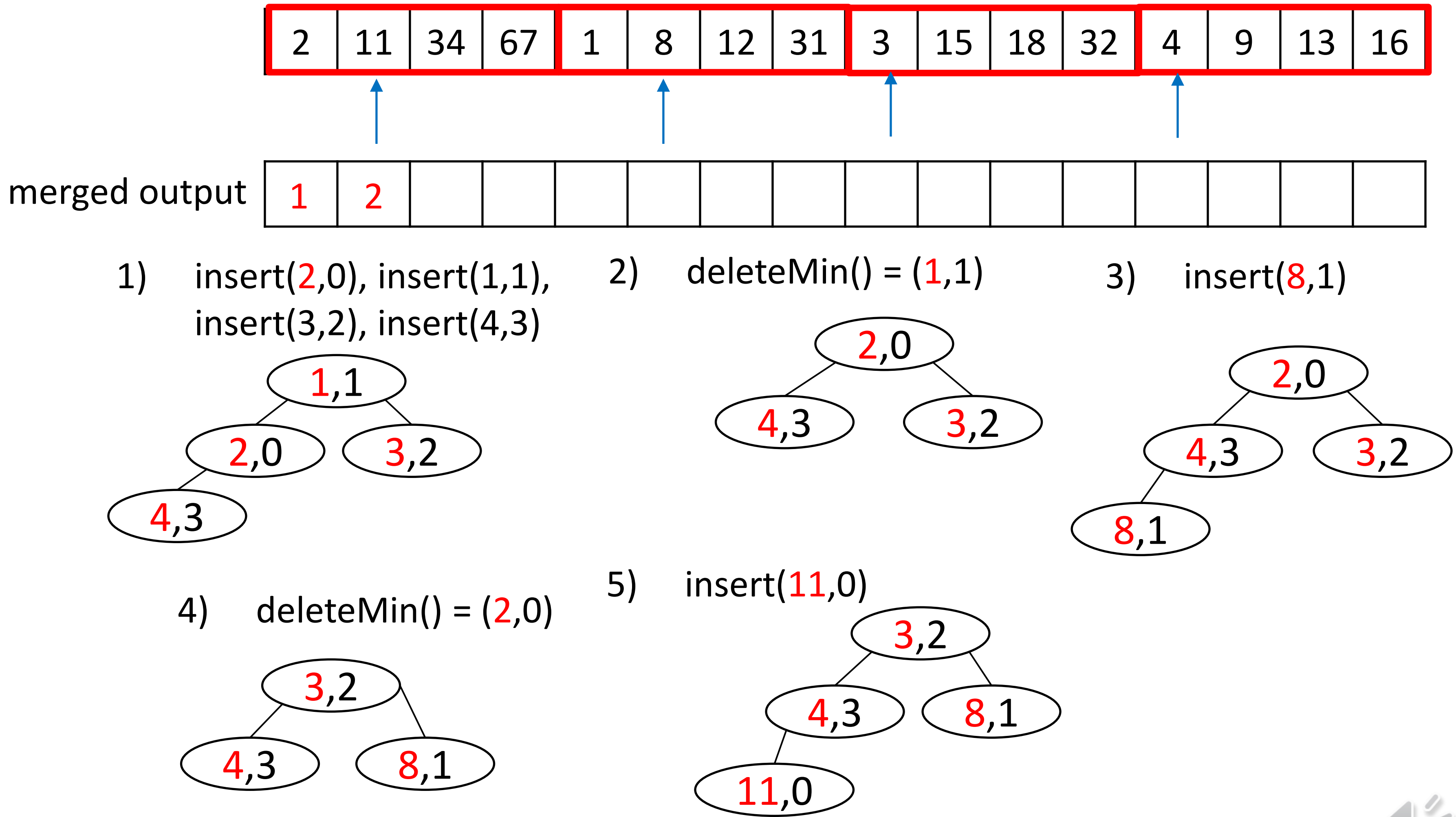
3) insert(8,1)



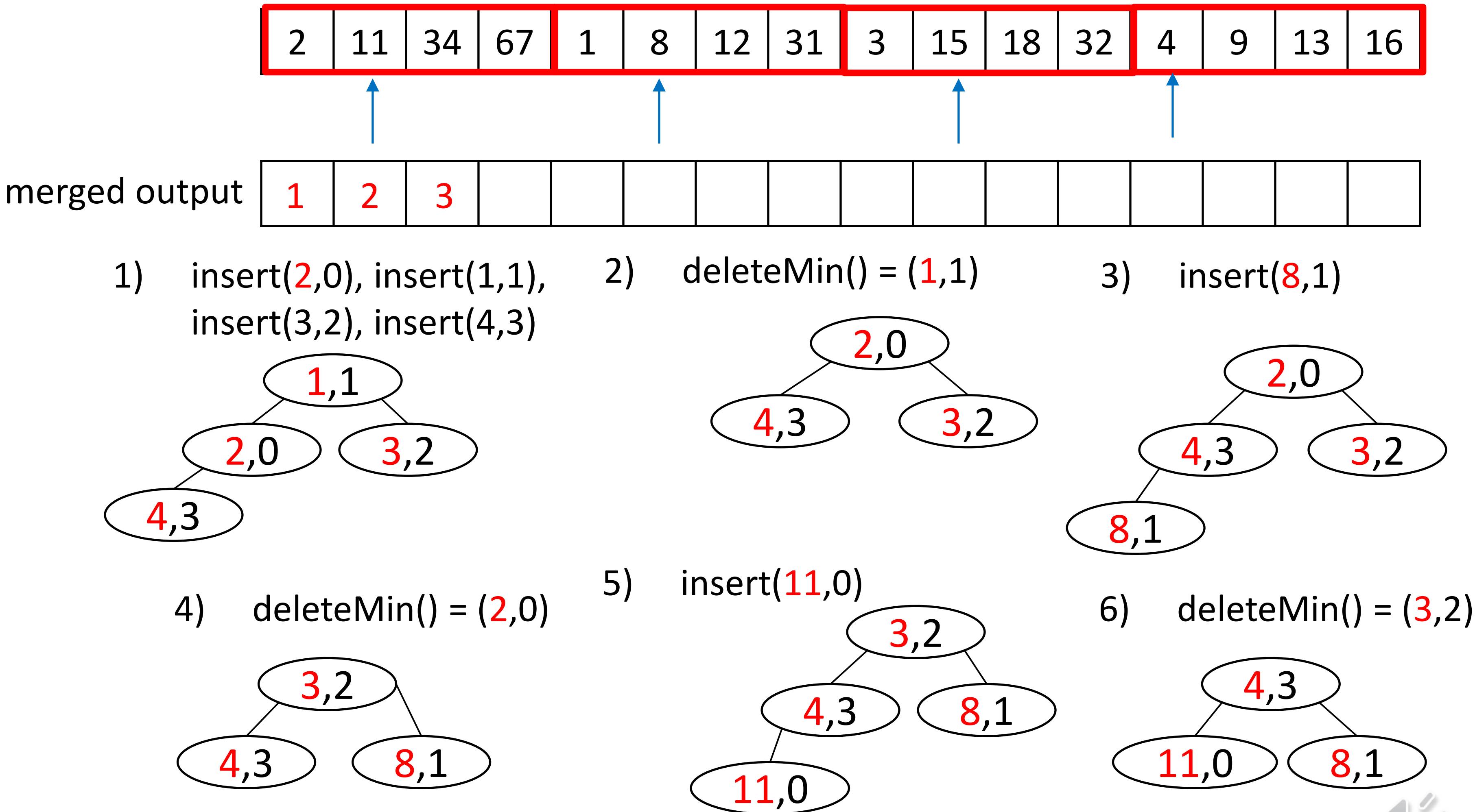
4) deleteMin() = (2,0)



d-way Merge with Min-Heap



d-way Merge with Min-Heap



d-way Merge with Min Heap Pseudo Code

d-Way-Merge$(S_1, \ldots, S_d, S)$
$S_1, \ldots, S_d$ are sorted input streams, S is output stream
$P \leftarrow$ empty min-priority queue
// P always holds current top elements of $S_1, \ldots, Sd$

$\Theta(d \log_2 d)$ $\left\{ \right.$
for $i \leftarrow 1$ to d **do**
P.insert$(S_i.top(\,),, i)$

$\Theta(n \log_2 d)$ $\left\{ \right.$
while P is not empty **do**
$(x, i) \leftarrow P$.*deleteMin*() // removes current top of S_i from P
S.append(x)
if S_i is not empty **do**
// current top of S_i is not represented in P, add it
P.insert$(S_i.top(\,), i)$

- Running time of operations in internal memory
  - priority queue P has size d at all times
  - while loop runs for $n - d$ iterations, where $n = |S_1| + \cdots + |S_d|$ at each iteration
    - one *deleteMin*() on heap of size d, time is $\Theta(\log_2 d)$
    - one *insert*() on heap of size d, time is $\Theta(\log_2 d)$
  - Total time is $\Theta(n \log_2 d)$

d-way Merge with Min Heap Pseudo Code

d-Way-Merge$(S_1, \dots, S_d, S)$

$S_1, \dots, S_d$ are sorted input streams, S is output stream

$P \leftarrow$ empty min-priority queue

// P always holds current top elements of $S_1, \dots, Sd$

$\Theta(d \log_2 d)$

for $i \leftarrow 1$ to d **do**

P.insert($S_i.top($ $),, i)$

$\Theta(n \log_2 d)$

while P is not empty **do**

$(x, i) \leftarrow P$.*deleteMin*() // removes current top of S_i from P

S.append(x)

if S_i is not empty **do**

// current top of S_i is not represented in P, add it

P.insert$(S_i.top($ $), i)$

- Running time of operations in internal memory
  - priority queue P has size d at all times
  - while loop runs for $n - d$ iterations, where $n = |S_1| + \cdots + |S_d|$ at each iteration
    - one *deleteMin*() on heap of size d, time is $\Theta(\log_2 d)$
    - one *insert*() on heap of size d, time is $\Theta(\log_2 d)$
  - Total time is $\Theta(n \log_2 d)$
- Number of block transfers is $\frac{2n}{B}$, assuming $d + 1$ blocks and P fit into main memory

One Round of d-way Mergesort Running time

- In internal memory, d-way merge is $\Theta(n \log_2 d)$
  - $n = |S_1| + \cdots + |S_d|$
- We need to d-way merge multiple number of times for one round of d-way Mergesort

34	11	2	67	8	12	31	1	3	15	18	32	9	16	4	13

d-way merge d-way merge d-way merge d-way merge

- let m_1 be the number of elements in the first set of d sequences we merge
  - time to merge is $\Theta(m_1 \log_2 d)$
- let m_2 be the number of elements in the second set of d sequences we merge
  - time to merge is $\Theta(m_2 \log_2 d)$
- ....
- let m_k be the number of elements in the last set of d sequences we merge
  - time to merge is $\Theta(m_k \log_2 d)$
- Total time to merge is $\Theta(m_1 \log_2 d + m_2 \log_2 d + \ldots + m_k \log_2 d) = \Theta(n \log_2 d)$
  - since $m_1 + m_2 + \cdots + m_k = n$
  - where n is the size of the whole sequence
- Similarly, for external memory analysis, the total number of block transfers is $\frac{2n}{B}$

d-way Mergesort Complexity In Internal Memory

- $\log_d n$ rounds
- Running time for one round is $\Theta(n \log_2 d)$
- Total time $\Theta(\log_d n \cdot n \log_2 d) = \Theta\left(\frac{\log_2 n}{\cancel{\log_2 d}} \cdot n \cancel{\log_2 d}\right) = \Theta(n \log_2 n)$

- In internal memory, d-way merge sort has the same running time theoretically
  - in practice , d-way merge is slower due to the overhead of maintaining a heap

d-way Mergesort Complexity In External Memory

- How do we gain advantage in external memory?
  - only block transfers count, each round is $\Theta\left(\frac{n}{B}\right)$ block transfers, no matter what d is
    - assuming d is such that $d+1$ blocks plus priority queue fit into internal memory
- $\log_d n$ rounds, time for each round is $\Theta\left(\frac{n}{B}\right)$ block transfers
- Total time $\Theta\left(\frac{n}{B} \cdot \log_d n\right)$
  - better than $\Theta\left(\frac{n}{B} \cdot \log_2 n\right)$ for large d

d-way Mergesort Complexity In External Memory

- Further improvements
  - proceed bottom-up with while loops, rather than top-down with recursion
  - reduce number of rounds by starting immediately with runs of length M

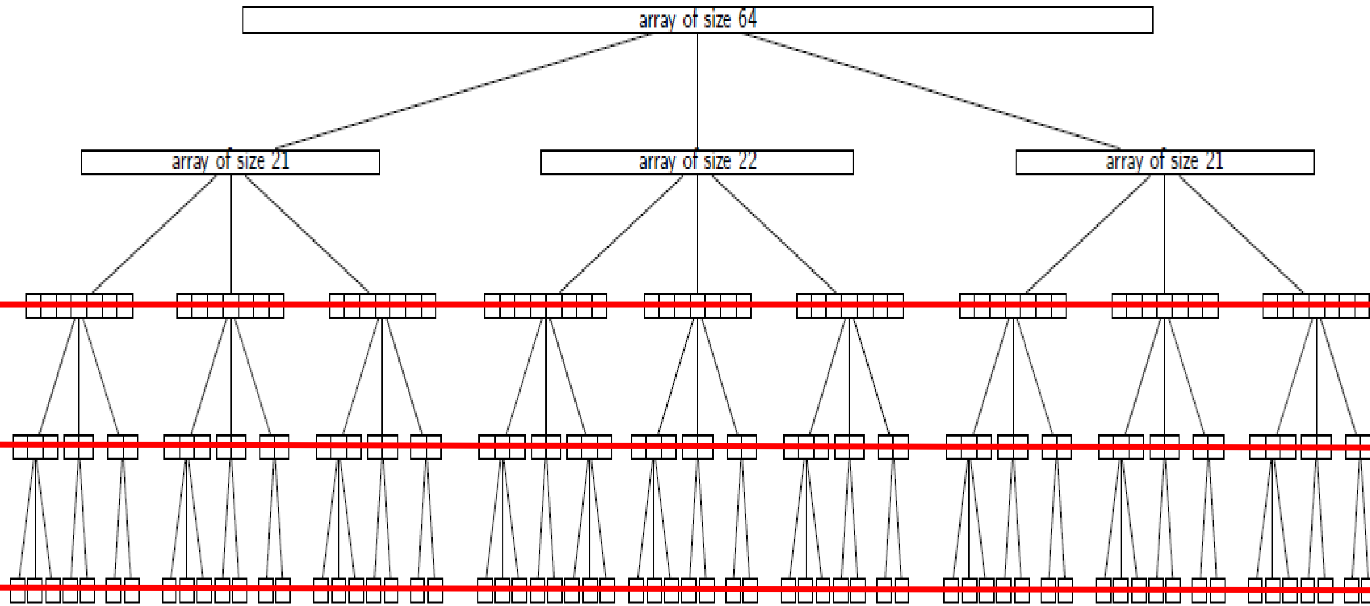
$\Theta\left(\frac{n}{B}\right)$ block transfers

$\Theta\left(\frac{n}{B}\right)$ block transfers

~~$\Theta\left(\frac{n}{B}\right)$ block transfers~~

~~$\Theta\left(\frac{n}{B}\right)$ block transfers~~

~~$\Theta\left(\frac{n}{B}\right)$ block transfers~~



- Suppose $M = 22$
  - start by sorting subarrays of size 22 in the main memory
  - avoids several rounds of merging

d-Way merge in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53

Internal memory $M = 8$

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

39	5	28	22	10	33	29	37	8	30	54	40	31	52	21	45	35	11	42	53	13	12	49	36	4	14	27	9	44	3	32	15

Internal (M = 8):

39	5	28	22	10	33	29	37

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

39	5	28	22	10	33	29	37	8	30	54	40	31	52	21	45	35	11	42	53	13	12	49	36	4	14	27	9	44	3	32	15

Internal (M = 8):

5	10	22	28	29	33	37	39

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	30	54	40	31	52	21	45	35	11	42	53	13	12	49	36	4	14	27	9	44	3	32	15

sorted run

Internal (M = 8):

5	10	22	28	29	33	37	39

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	30	54	40	31	52	21	45	35	11	42	53	13	12	49	36	4	14	27	9	44	3	32	15

sorted run

Internal (M = 8):

8	30	54	40	31	52	21	45

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	30	54	40	31	52	21	45	35	11	42	53	13	12	49	36	4	14	27	9	44	3	32	15

sorted run

Internal (M = 8):

8	21	30	31	40	45	52	54

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	35	11	42	53	13	12	49	36	4	14	27	9	44	3	32	15

sorted run sorted run

Internal (M = 8):

8	**21**	**30**	**31**	**40**	**45**	**52**	**54**

1. Create $\frac{n}{M}$ sorted runs of length M
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory: Initialization

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run | sorted run | sorted run | sorted run

Internal (M = 8):

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run — sorted run — sorted run — sorted run

Internal (M = 8):

S_1		S_2		S_3		S	
5	10	8	21	11	12		

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run | sorted run | sorted run | sorted run

Internal (M = 8):

S_1		S_2		S_3		S	
	10	8	21	11	12	5	

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run sorted run sorted run sorted run

Internal (M = 8):

S_1		S_2		S_3		S	
	10		**21**	**11**	**12**	**5**	**8**

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run | sorted run | sorted run | sorted run

5	8																														

Internal (M = 8):

S_1		S_2		S_3		S	
	10		21	11	12		

($d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run — sorted run — sorted run — sorted run

5	8																														

Internal $(M = 8)$:

S_1		S_2		S_3		S	
			21	11	12	10	

$(d = 3$, priority queue not shown$)$

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run | sorted run | sorted run | sorted run

5	8																														

Internal (M = 8):

S_1		S_2		S_3		S	
22	28		21	11	12	10	

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
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2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run | sorted run | sorted run | sorted run

5	8																														

Internal (M = 8):

S_1		S_2		S_3		S	
22	28		21		12	10	11

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
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2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run — sorted run — sorted run — sorted run

5	8	10	11																												

Internal (M = 8):

S_1		S_2		S_3		S	
22	**28**		**21**		**12**		

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run sorted run sorted run sorted run

5	8	10	11	12	13	21	22	28	29	30	31	33	35	36	37	39	40	42	45	49	52	53	54								

sorted run

S_1 S_2 S_3 S

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run — sorted run — sorted run — sorted run

5	8	10	11	12	13	21	22	28	29	30	31	33	35	36	37	39	40	42	45	49	52	53	54	3	4	9	14	15	27	32	44

sorted run — sorted run

S_1		S_2		S_3		S	

$(d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*
3. Keep merging the next runs to complete one round. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - after one round of merging, number of sorted runs reduced by a factor of d

d-Way Mergesort in External Memory

- External $(B = 2)$

5	10	22	28	29	33	37	39	8	21	30	31	40	45	52	54	11	12	13	35	36	42	49	53	3	4	9	14	15	27	32	44

sorted run | sorted run | sorted run | sorted run

5	8	10	11	12	13	21	22	28	29	30	31	33	35	36	37	39	40	42	45	49	52	53	54	3	4	9	14	15	27	32	44

sorted run | sorted run

S_1 S_2 S_3 S

($d = 3$, priority queue not shown)

1. Create $\frac{n}{M}$ sorted runs of length M. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - bring consecutive chunks of size M into internal memory
   - sort each chunk with an efficient sorting algorithm
2. Merge first $d \approx \frac{M}{B} - 1$ sorted runs using *d-way-Merge*
3. Keep merging the next runs to complete one round. Takes is $\Theta\left(\frac{n}{B}\right)$ block transfers
   - after one round of merging, number of sorted runs reduced by a factor of d
4. Keep doing rounds until we get just one sorted run

d-Way Mergesort in External Memory: Running time

- Have $\log_d \frac{n}{M}$ rounds of merging
  - $\frac{n}{M}$ runs after initialization
  - each round decreases the number of sorted runs by a factor of d
  - $\frac{n}{M}/d$ runs after one round
  - $\frac{n}{M}/d^k$ runs after k rounds
  - stop when $\frac{\frac{n}{M}}{d^k} = 1 \Longrightarrow k = \log_d \frac{n}{M}$

since $d \approx \frac{M}{B} - 1$

- Each round takes $\Theta\left(\frac{n}{B}\right)$ block transfers
- Total number of bock transfers is proportional to $\frac{n}{B} \cdot \log_d \frac{n}{M} \in O\left(\frac{n}{B} \cdot \log_{M/B} \frac{n}{M}\right)$
- One can prove lower bound in external memory model for comparison sorting

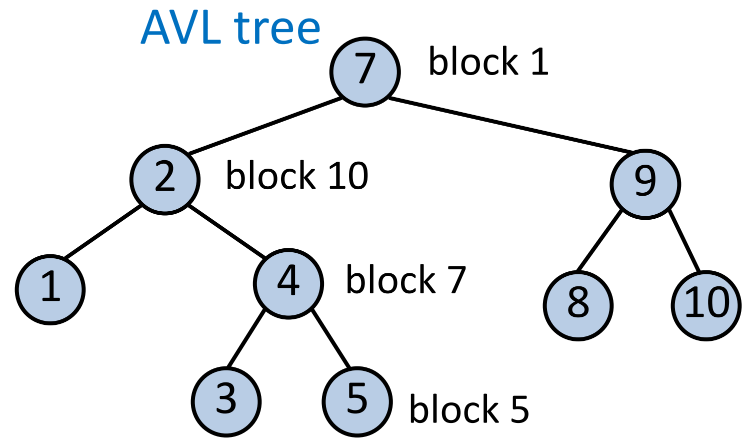
$$\Omega\left(\frac{n}{B} \cdot \log_{M/B} \frac{n}{M}\right)$$

- Thus d-way mergesort is optimal (up to constant factors)

Outline

- External Memory
  - Motivation
  - Stream Based Algorithms
  - External sorting
  - External Dictionaries
    - 2-4 Trees
    - (*a*, *b*)-Trees
    - B-Trees

Dictionaries in External Memory: Motivation



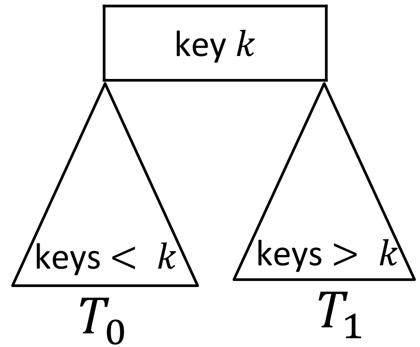
- AVL tree based dictionary implementations have poor *memory locality*
    - tree nodes are in non-contiguous memory locations
    - for any tree path, each node is usually in a different block
- In an AVL tree $\Theta(\log n)$ blocks are loaded in the worst case
- Idea: define multi-way tree
    - one node stores many KVPs
    - for multi-way trees, $b-1$ KVPs $\Leftrightarrow$ b subtrees
- To allow insert/delete, we permit a varying number of KVPs in nodes
- This gives much smaller height than AVL-trees
    - smaller height implies fewer block transfers
- First consider a special case: *2-4 trees*
    - 2-4 trees also used for dictionaries in internal memory
    - may be even faster than AVL-trees

Outline

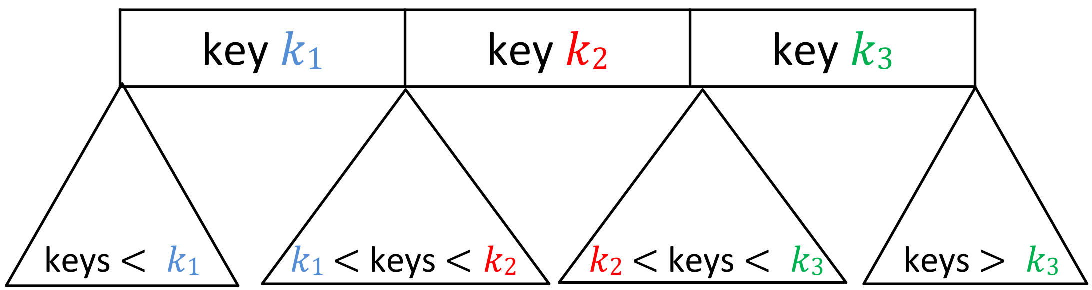
- External Memory
  - Motivation
  - Stream based algorithms
  - External sorting
  - External dictionaries
    - 2-4 Trees
    - (*a*, *b*)-Trees
    - B-Trees

2-4 Trees Motivation

- Binary Search Tree supports efficient search with special key ordering

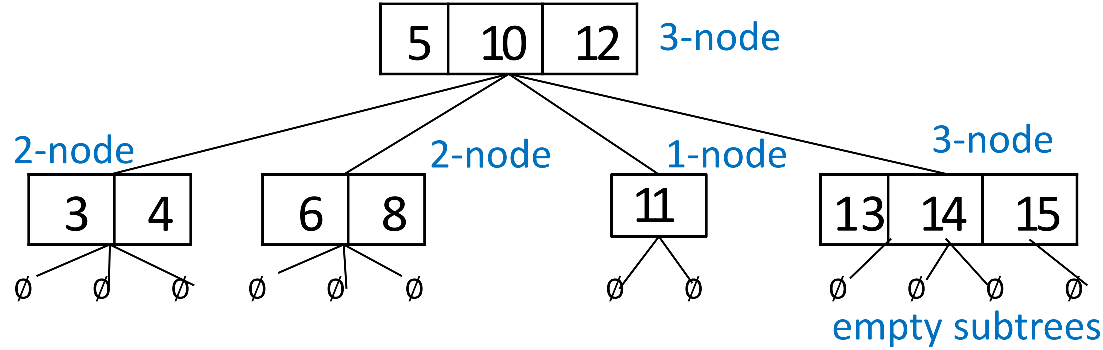


- Need nodes that store more than one key
    - how to support efficient search?

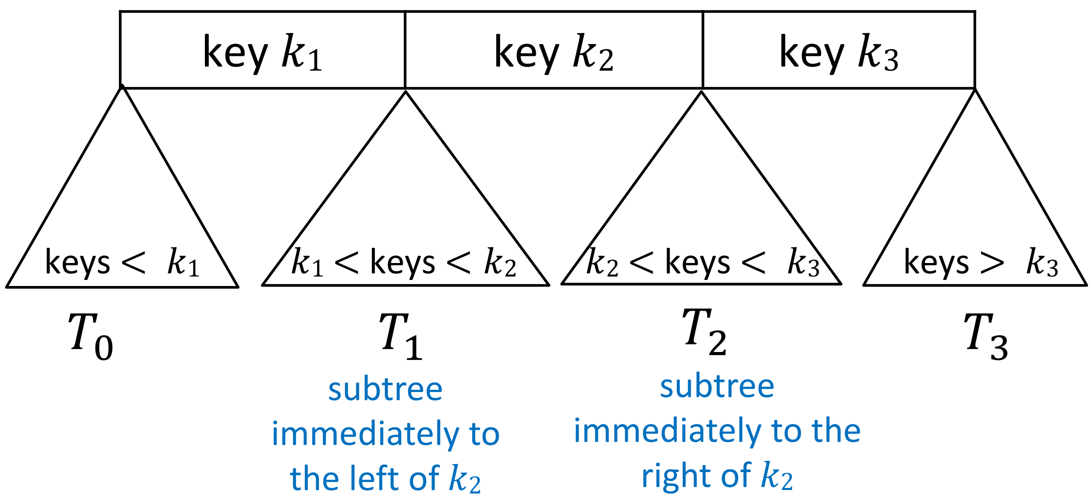


- Need additional properties to ensure tree is balanced and therefore *insert*, *delete* are efficient

2-4 Trees



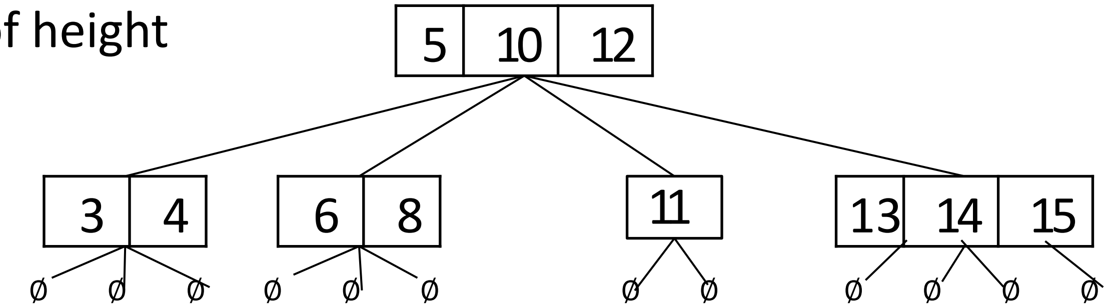
- **Structural properties**
  - Every node is either
    - 1-node: *one KVP* and *two subtrees* (possibly empty), or
    - 2-node: *two KVPs* and *three subtrees* (possibly empty), or
    - 3-node: *three KVPs* and *four subtrees* (possibly empty)
    - allowing 3 types of nodes simplifies insertion/deletion
  - All empty subtrees are at the same level
    - necessary for ensuring height is logarithmic in the number of KVP stored
- **Order property:** keys at any node are between the keys in the subtrees



key-subtree list of the node
$< T_0, k_1, T_1, k_2, T_2, k_3, T_3, k_1 >$

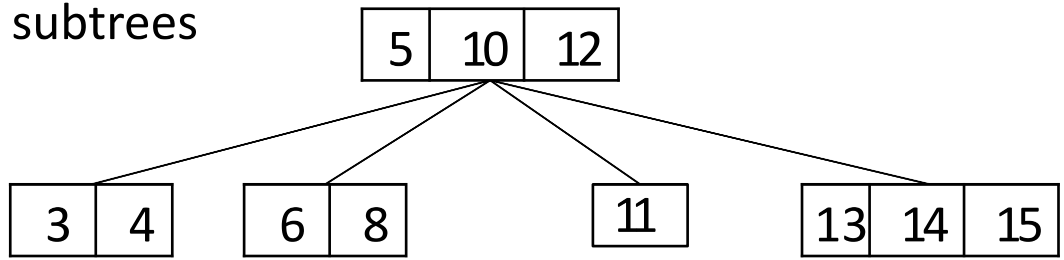
2-4 Tree Example

- Empty subtrees are not part of height computation



tree of height 1

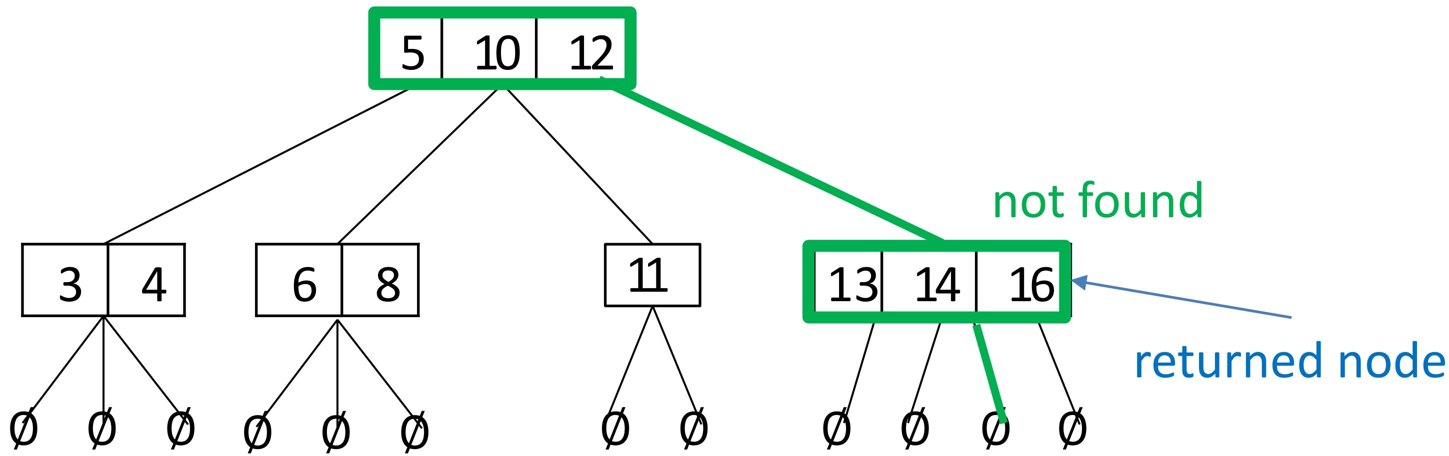
- Often do not even show empty subtrees



- Will prove height is $O(\log n)$ later, when we talk about (a,b)-trees
  - 2-4 tree is a special type of (a,b)-tree

2-4 Tree: Search Example

- **Search**
  - similar to search in BST
  - search(k) compares key k to k_1, k_2 , k_3, and either finds k among k_1, k_2 , k_3 or figures out which subtree to recurse into
  - if key is not in tree, search returns parent of empty tree where search stops
    - key can be inserted at that node
  - search(15)



2-4 Tree operations

24Tree::search$(k, v \leftarrow$root$, p \leftarrow$empty subtree)

k: key to search, v: node where we search; p: parent of v

if v represents empty subtree

return "not found, would be in p"

let $< T_0, k_1, \ldots, k_d, T_d >$ be key-subtrees list at v

if $k \geq k_1$

$i \leftarrow$ maximal index such that $k_i \leq k$

if $k_i = k$

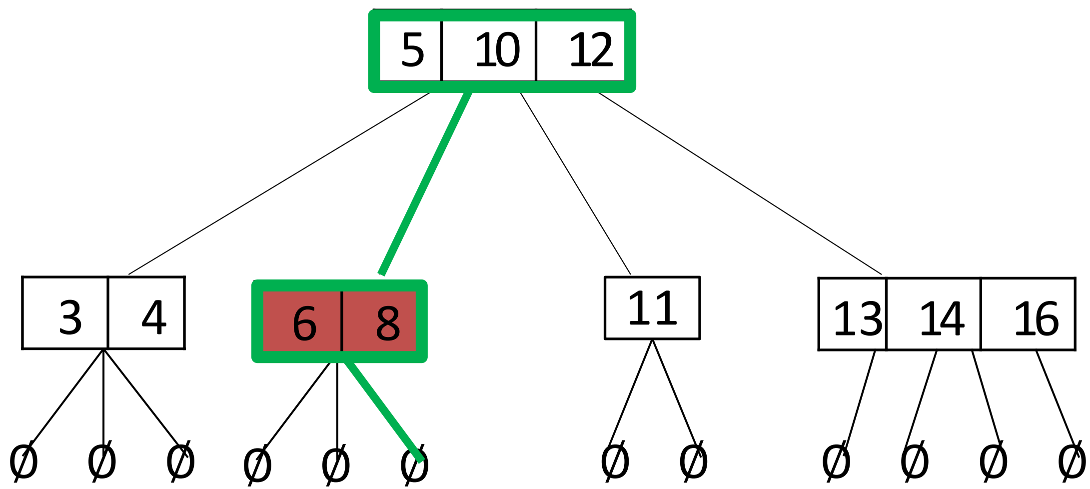
return "at ith key in v"

else *24Tree::search*(k, T_i, v)

else *24Tree::search*(k, T_0, v)

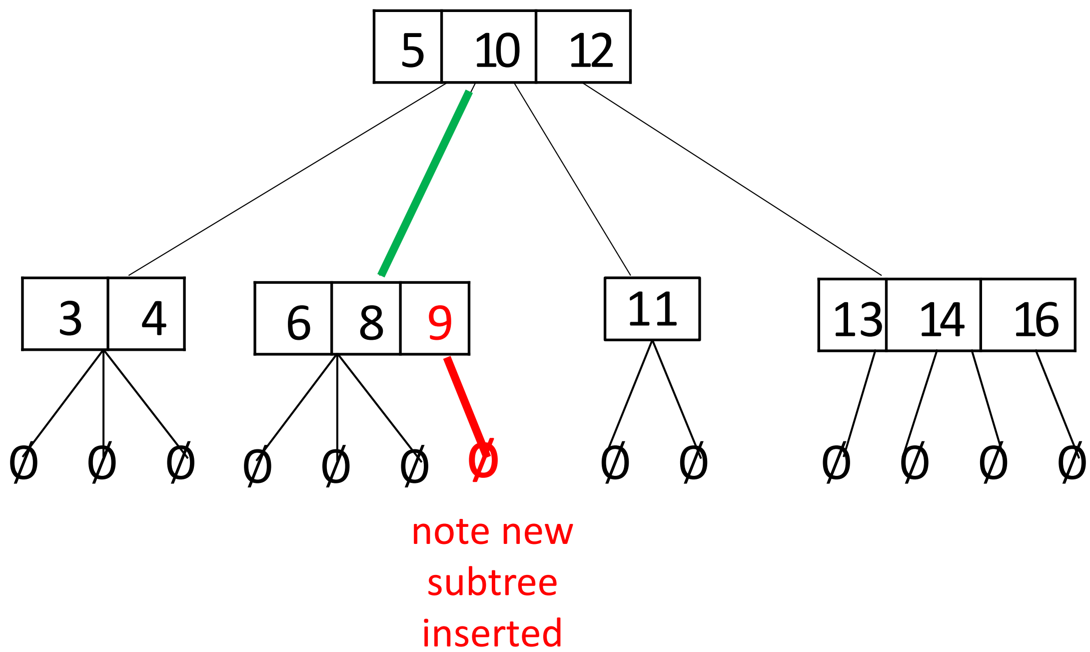
Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(9)
  - first step is 24Tree::search(9)
  - insert at the leaf node returned by search



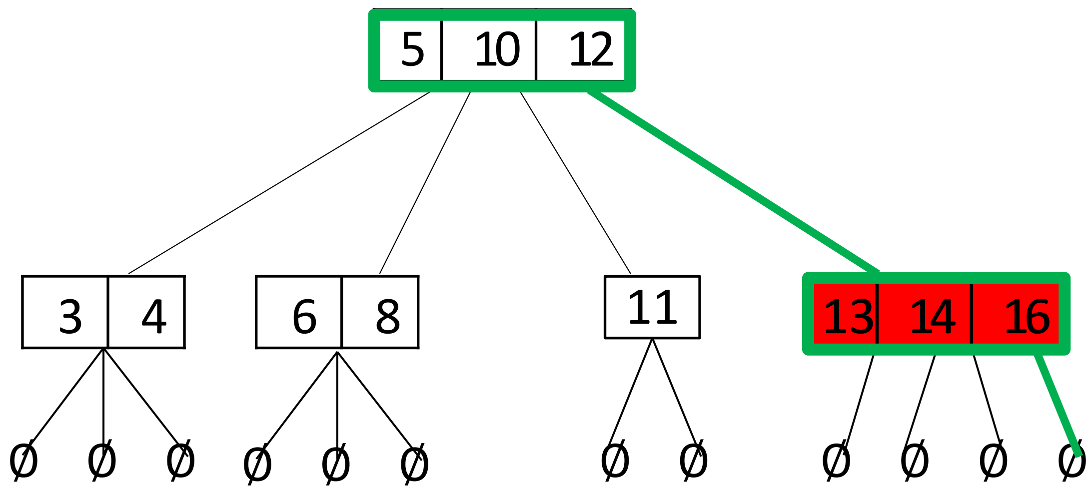
Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(9)
  - first step is 24Tree::search(9)
  - insert at the leaf node returned by search
  - node stays valid, it now has 3 KVPs, which is allowed



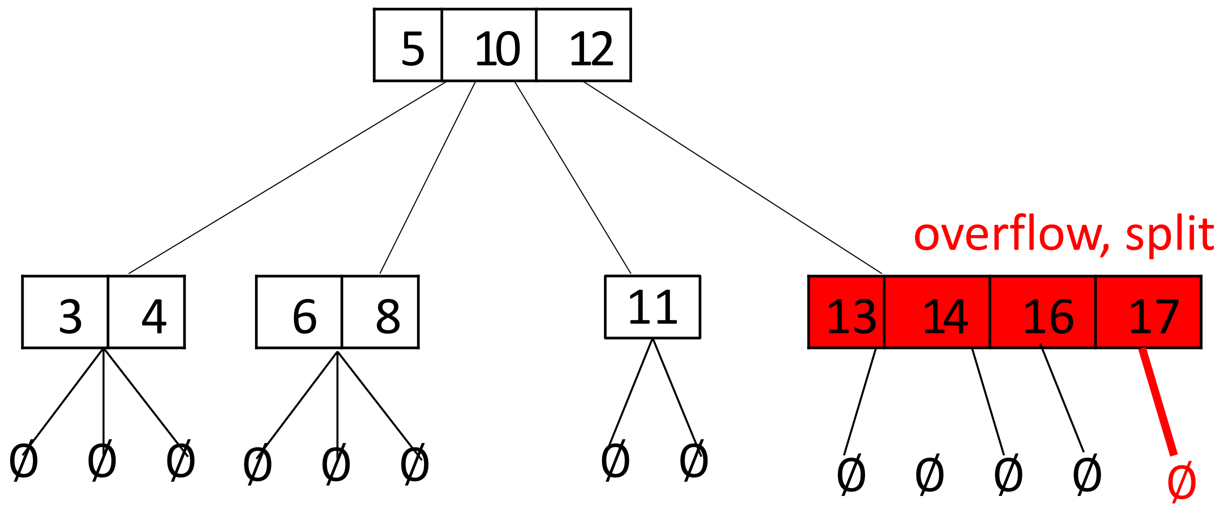
Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(17)
  - first step is 24Tree::search(17)
  - insert at the leaf node returned by search



Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(17)
  - now leaf has 4 KVPs, not allowed, have to fix this



Example: 2-4 tree Insert

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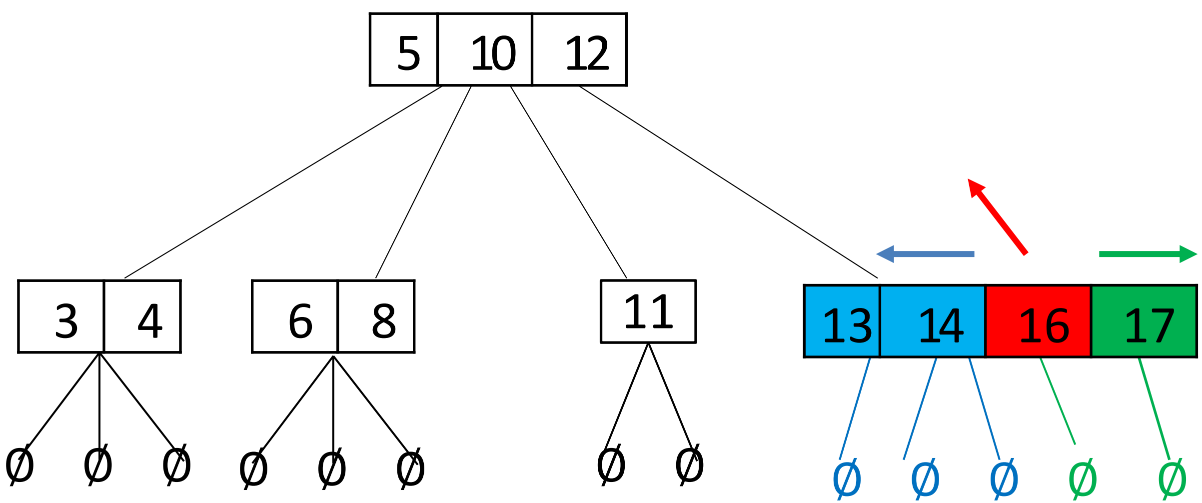
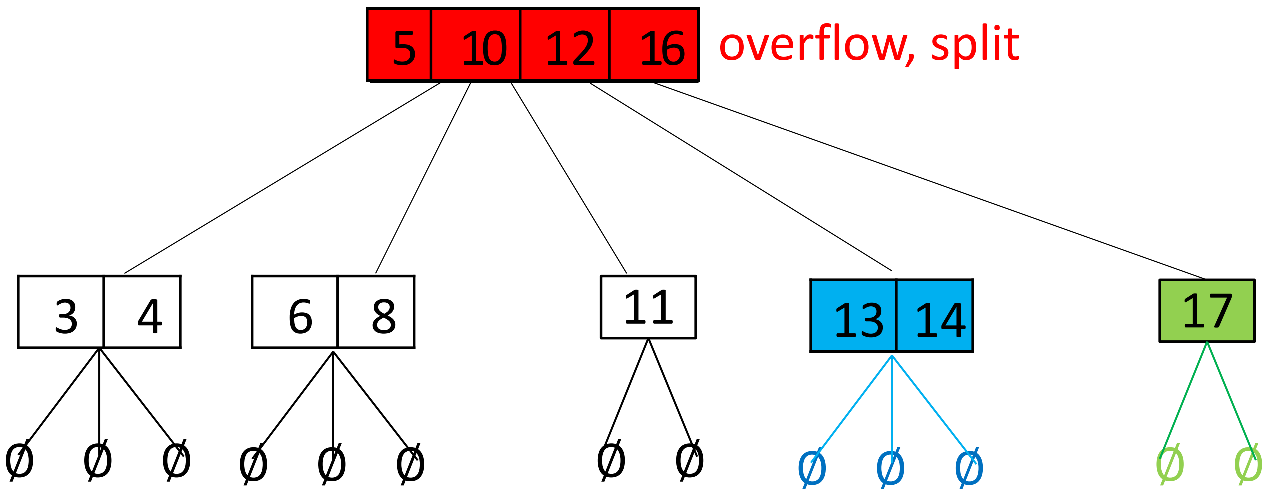


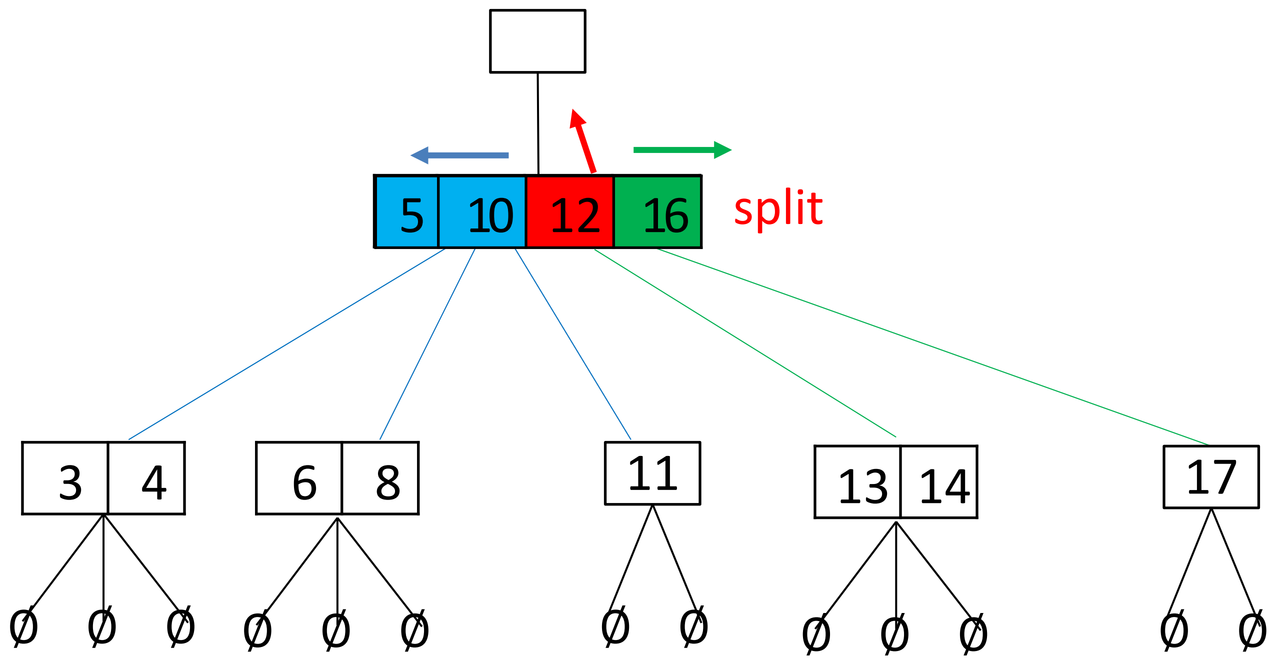
Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(17)
  - overflow propagates to the parent of split node



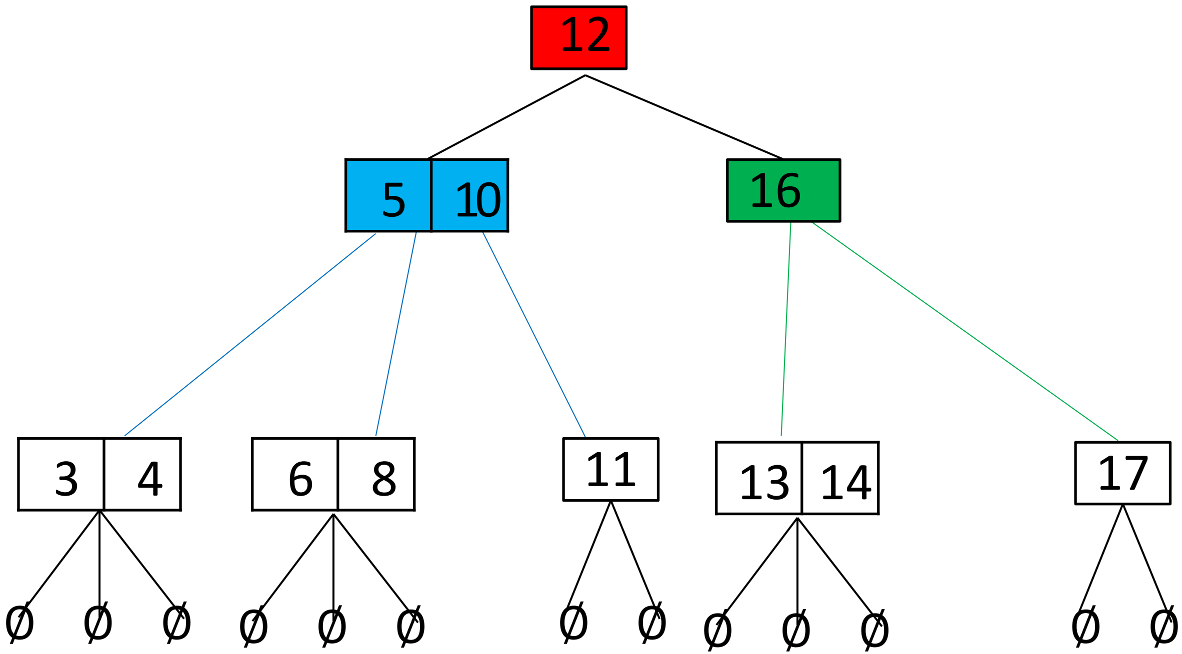
Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(17)
  - when splitting the root node, need to create new root



Example: 2-4 tree Insert

- **Example**: *24TreeInsert*(17)



2-4 Tree Insert Pseudocode

24Tree::insert(k)

$v \leftarrow$ *24Tree::search*(k) //leaf where *k* should be

add k and an empty subtree in key-subtree-list of v

while v has 4 keys (overflow → node split)

let $< T_0, k_1, \ldots, k_4, T_4 >$ be key-subtrees list at v

if v has no parent

create an empty parent of v

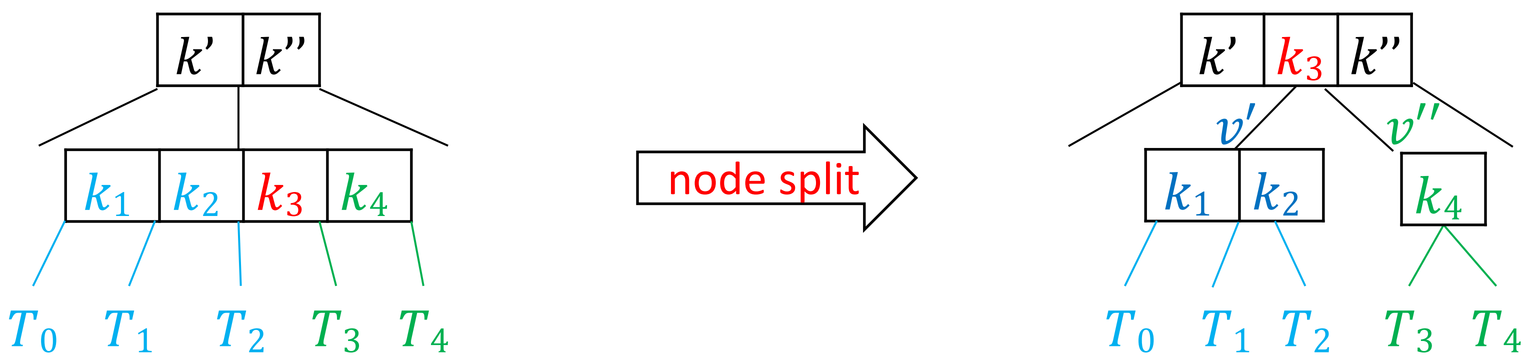
$p \leftarrow$ parent of v

$v' \leftarrow$ new node with keys k_1, k_2 and subtrees T_0, T_1, T_2

$v'' \leftarrow$ new node with key k_4 and subtrees T_3, T_4

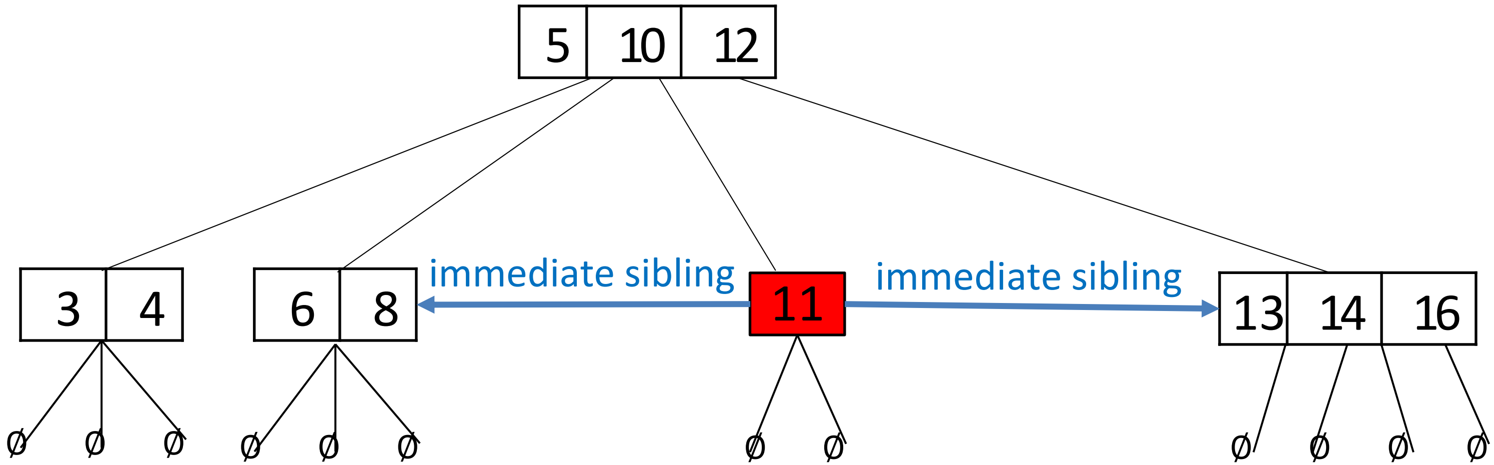
replace $< v >$ by $< v', k_3, v'' >$ in key-subtree-list of p

$v \leftarrow p$ //continue checking for overflow upwards

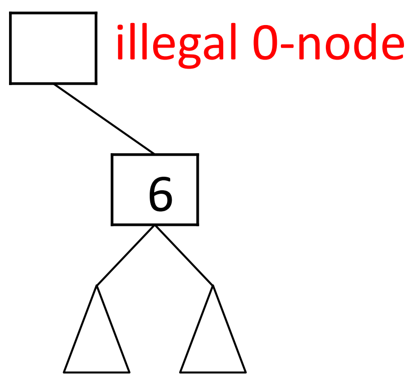


2-4 Tree: Immediate Sibling

- A node can have an *immediate* left sibling, immediate right sibling, or both

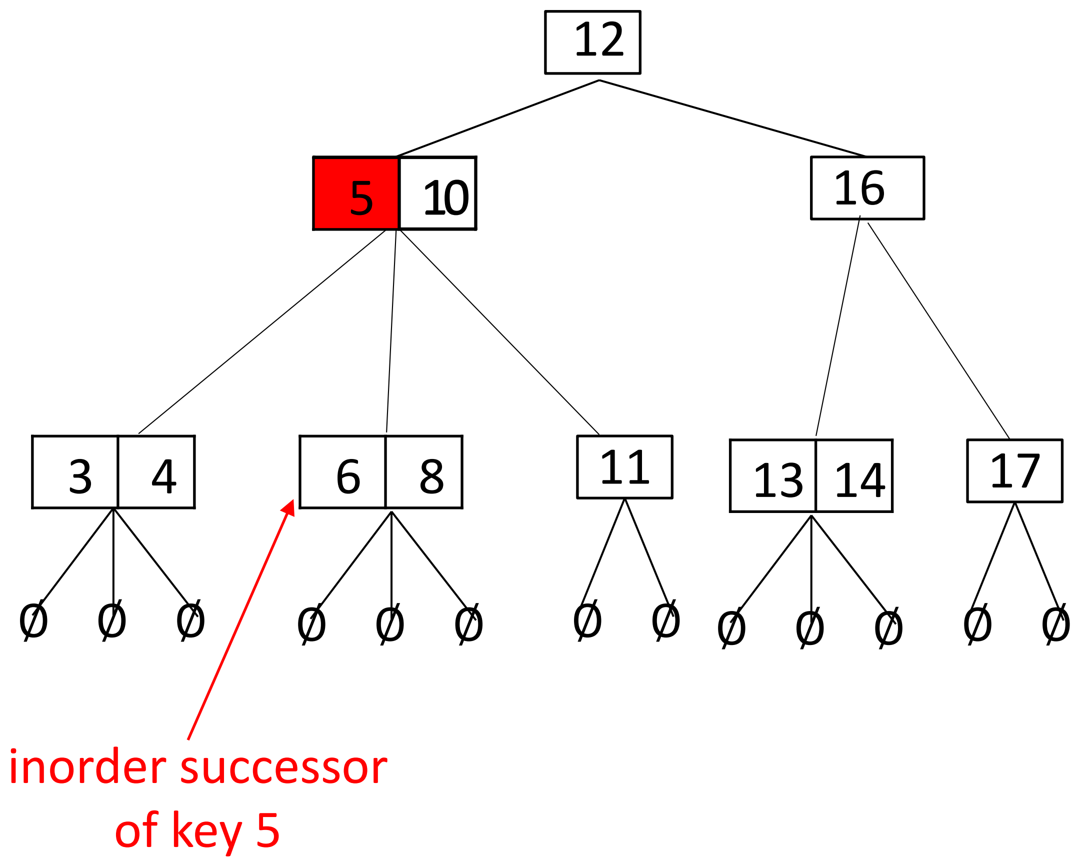


- Any node except the root must have an immediate sibling



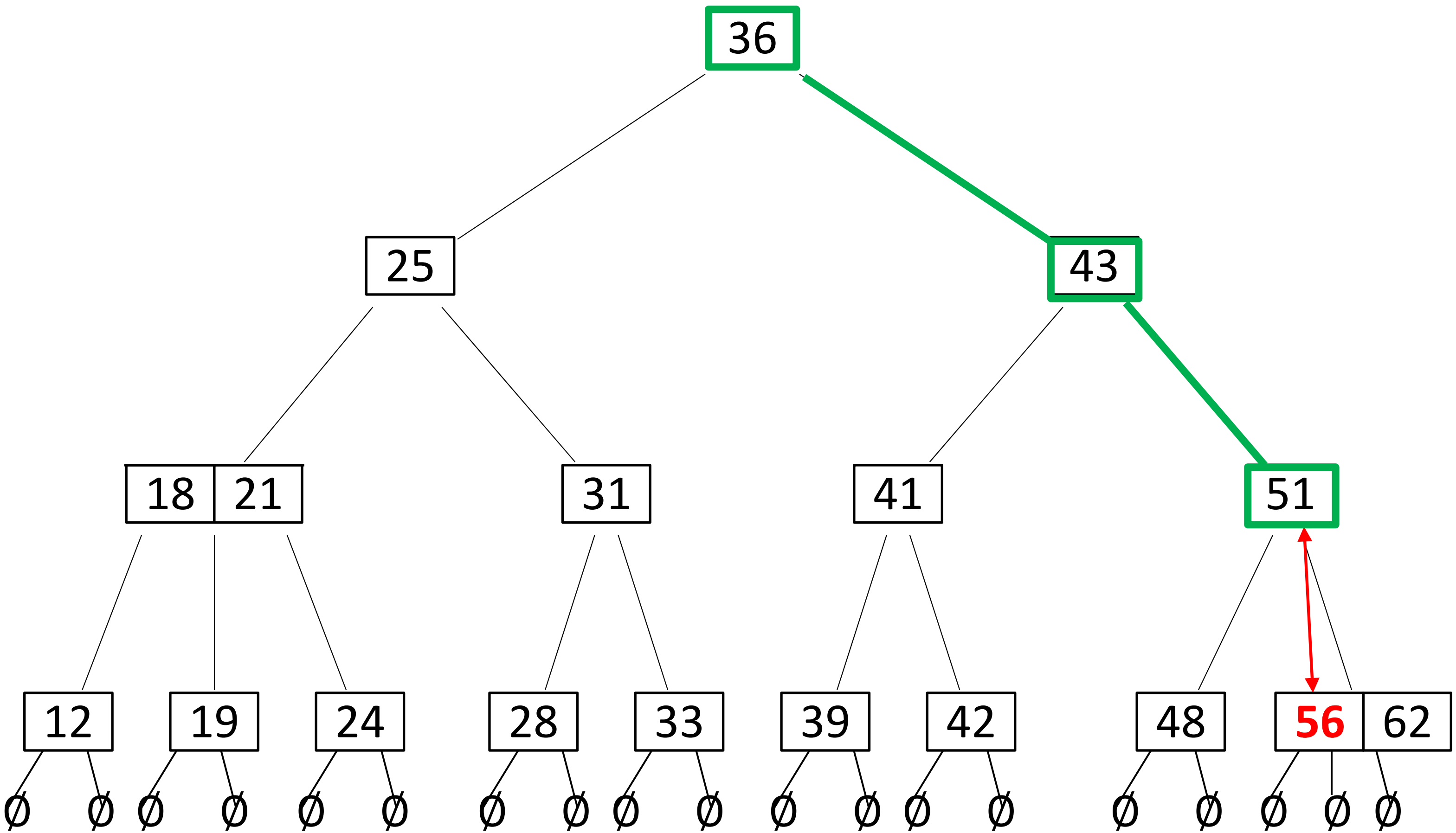
2-4 Tree: Inorder Successor

- Inorder successor of key k is the smallest key in the subtree immediately to the right of k



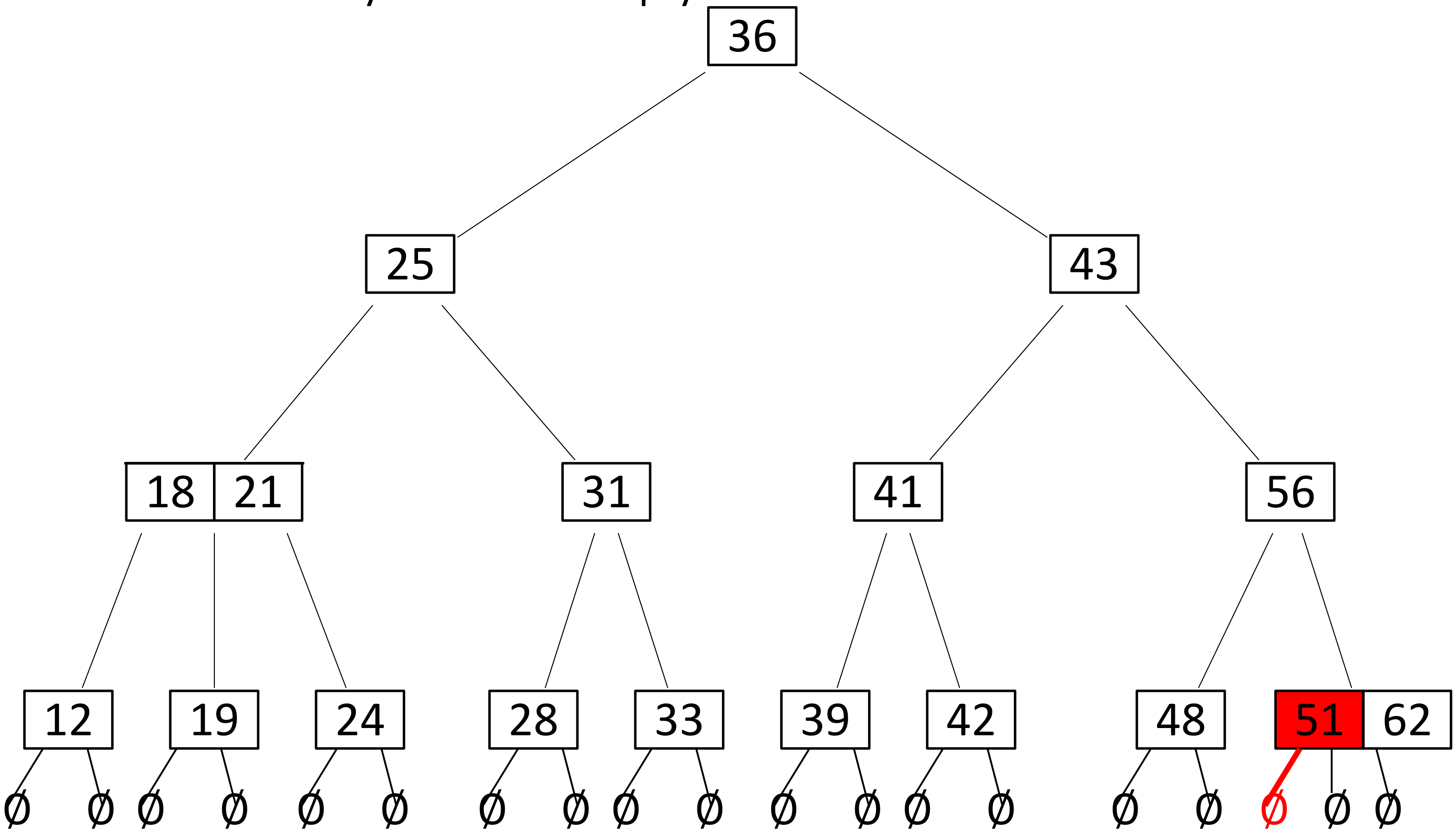
2-4 Tree Delete

- Example: *delete*(51)
- Search for key to delete
  - can delete keys only from a leaf node
  - replace key with inorder successor



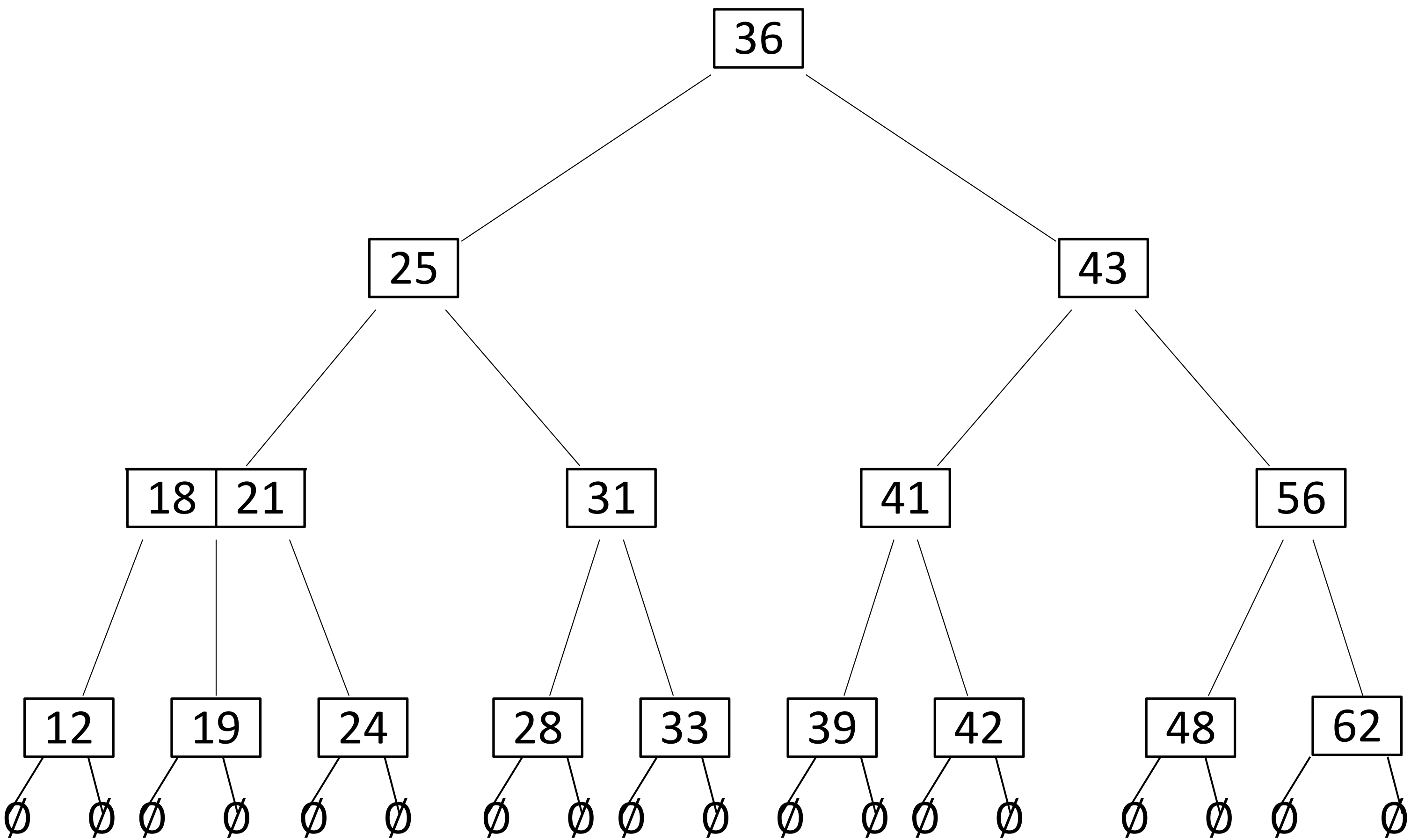
2-4 Tree Delete

- Example: *delete*(51)
- Search for key to delete
  - can delete keys only from a leaf node
  - replace key with in-order successor
  - delete key 51 and an empty subtree



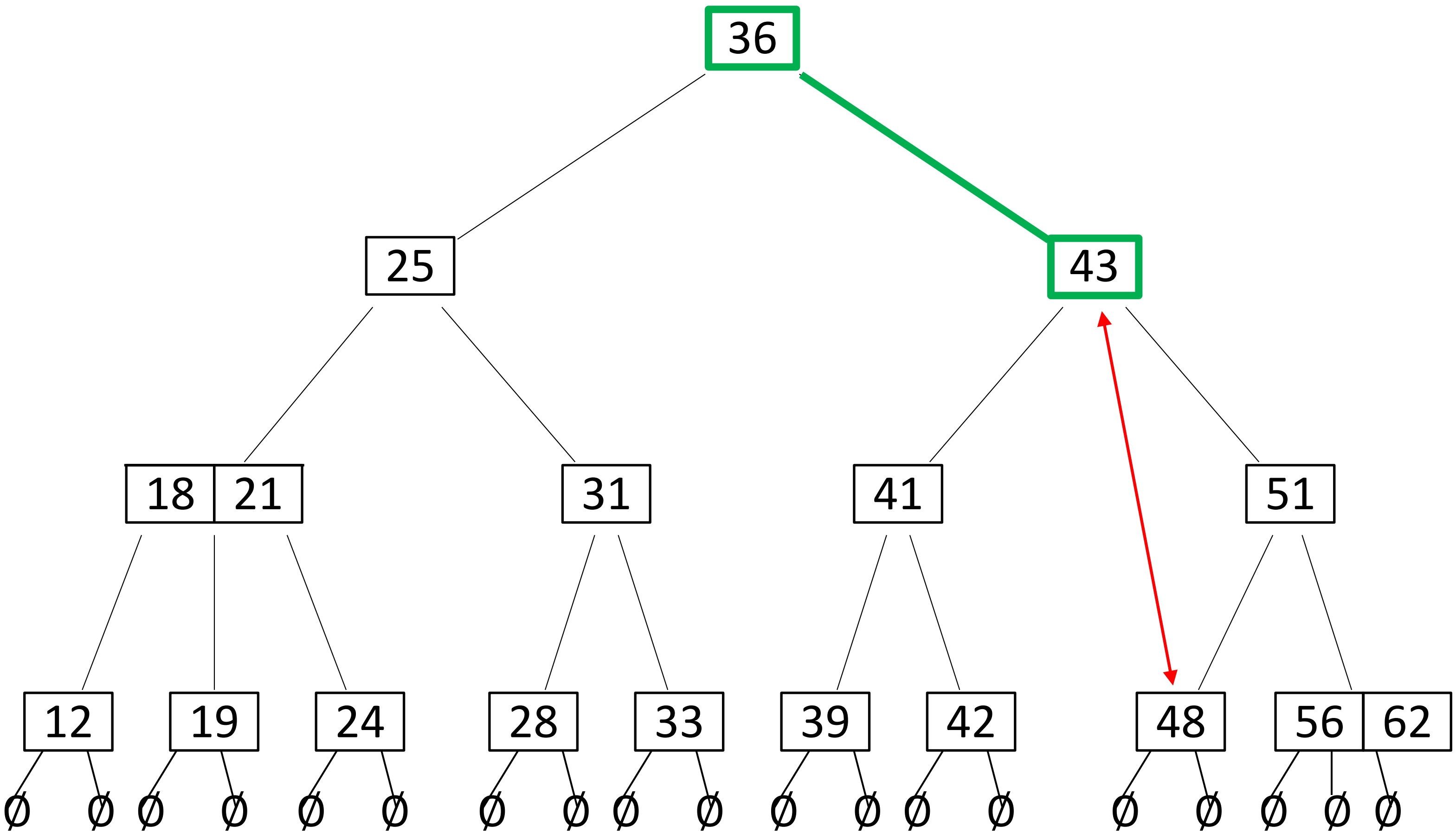
2-4 Tree Delete

- Example: *delete*(51)
- Search for key to delete
- Done!



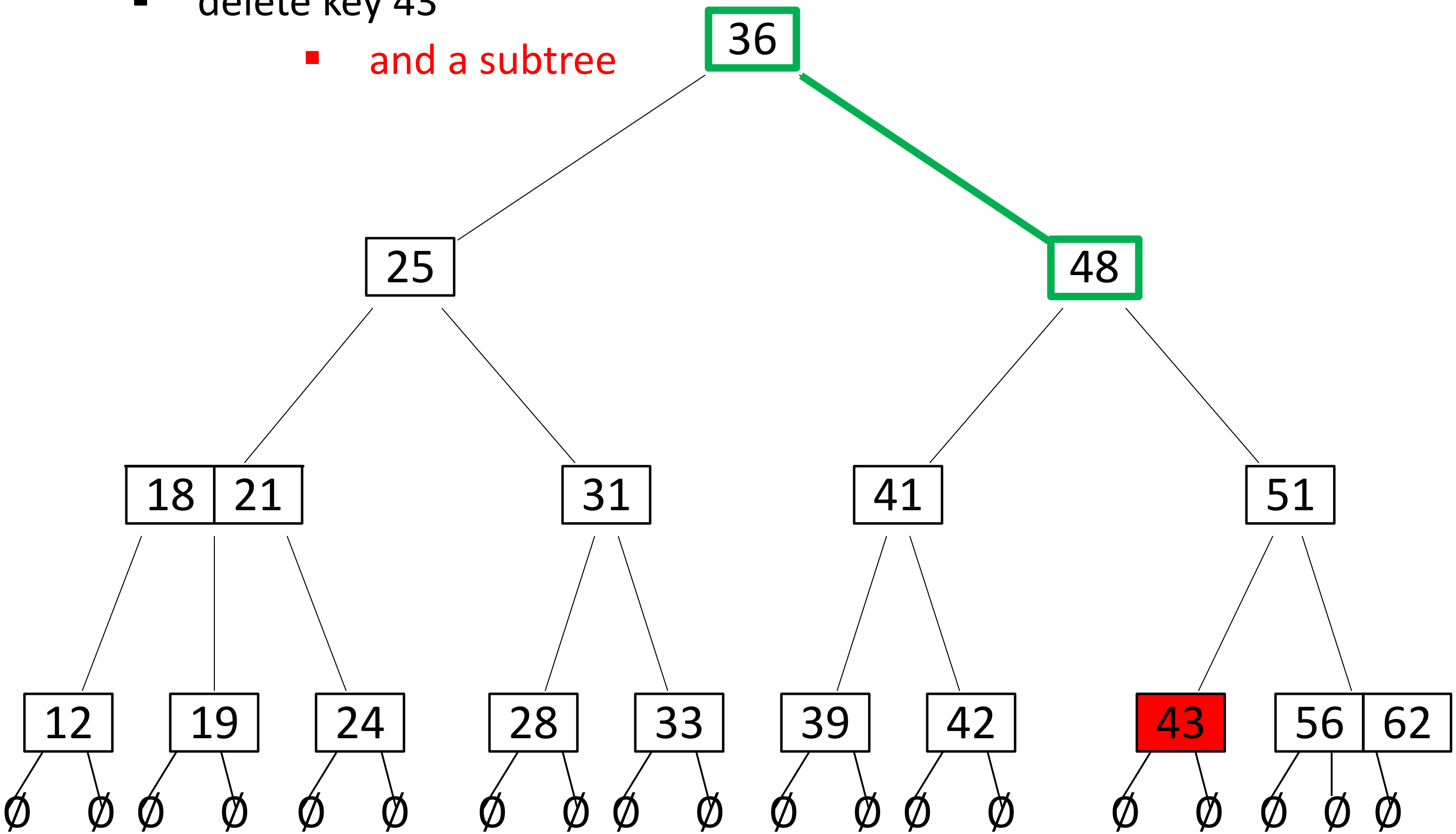
2-4 Tree Delete

- Example: *delete*(43)
- Search for key to delete
    - can delete keys only from a leaf node
    - replace key with in-order successor



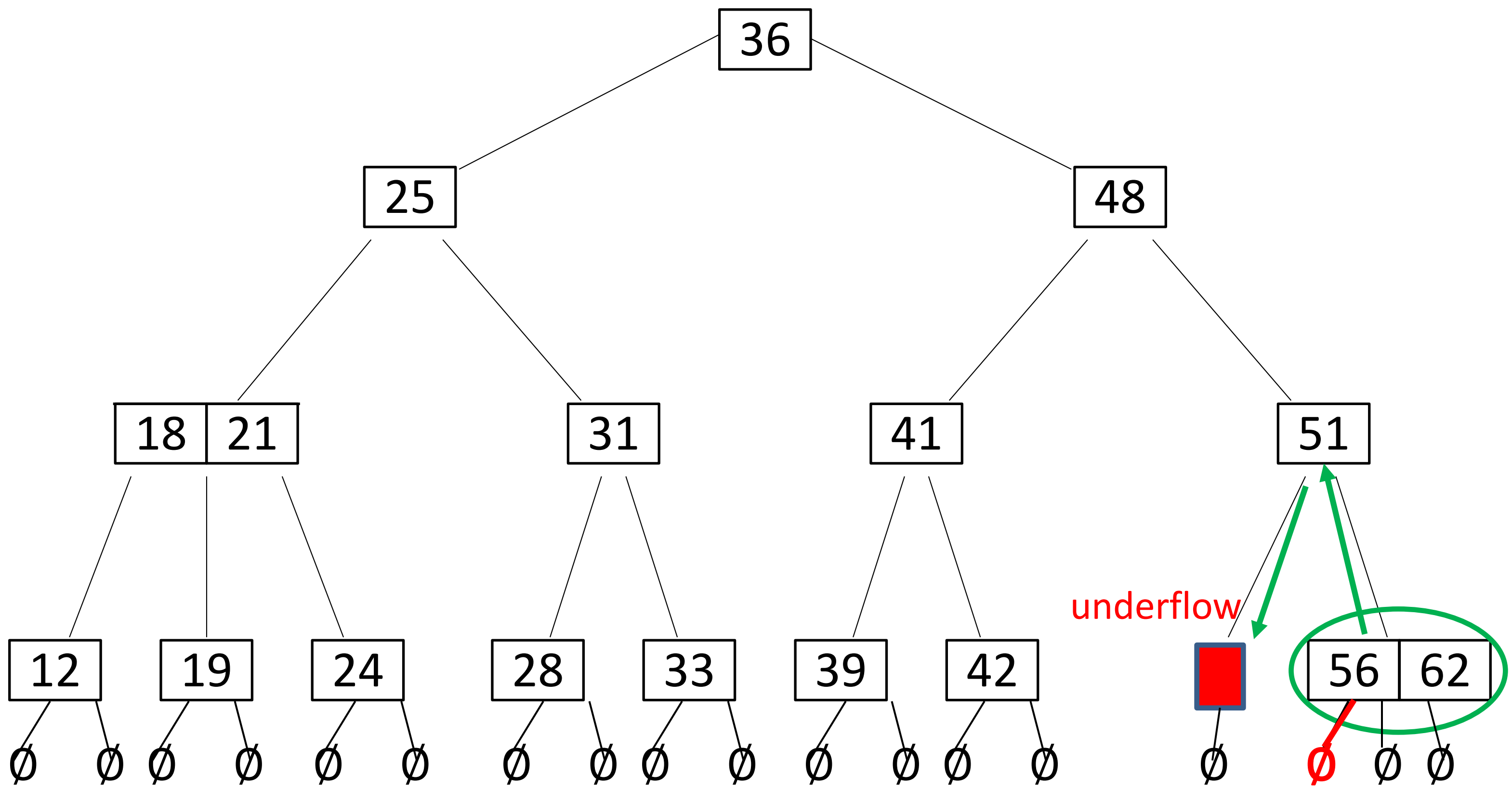
2-4 Tree Delete

- Example: *delete*(43)
- Search for key to delete
  - can delete keys only from a leaf node
  - replace key with in-order successor
  - delete key 43
    - and a subtree



2-4 Tree Delete

- Example: *delete*(43)
  - *rich* immediate sibling, transfer key from sibling, with help from the parent
    - sibling is *rich* if it is a 2-node or 3-node
    - adjacent subtree from sibling is also transferred



2-4 Tree Delete

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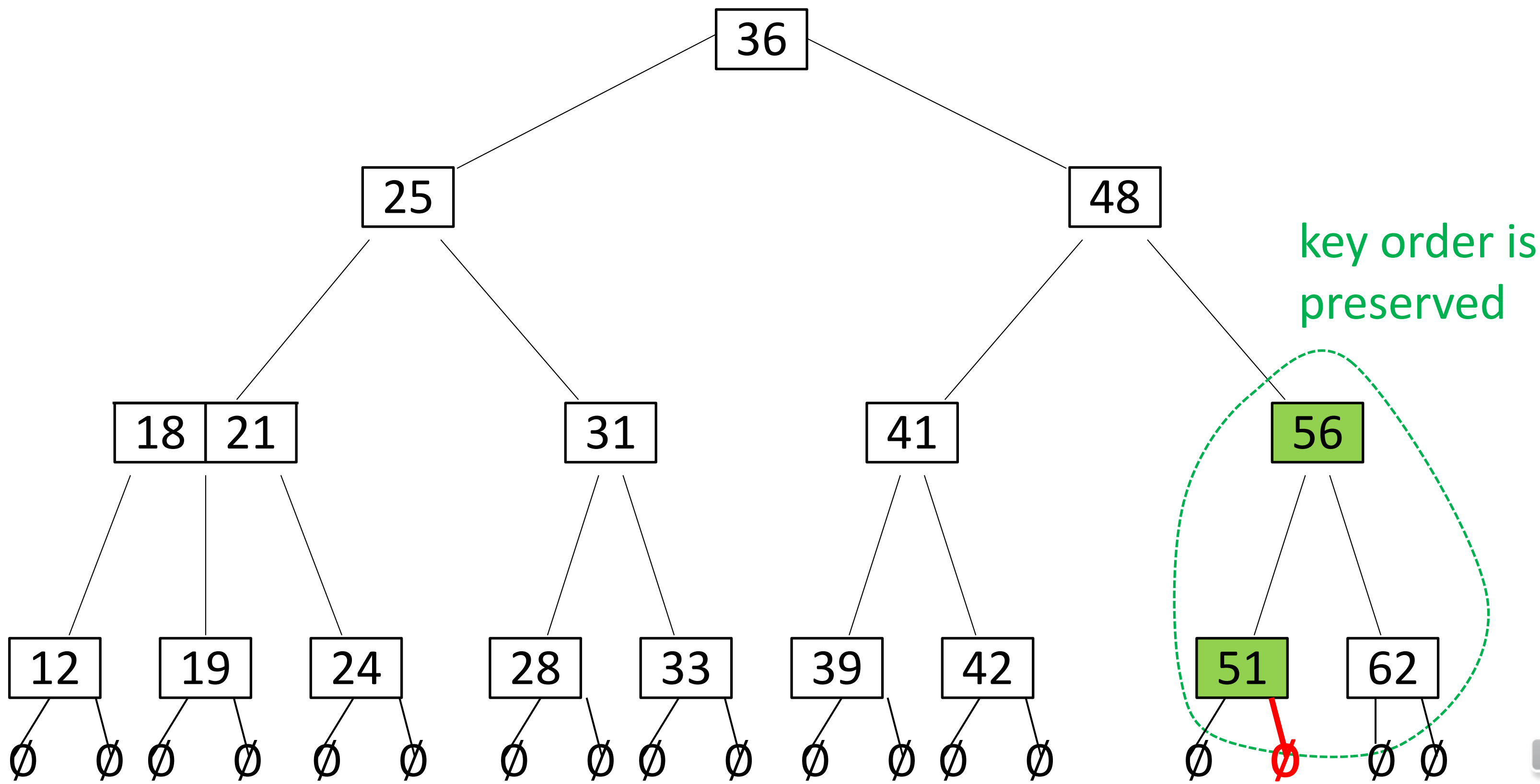
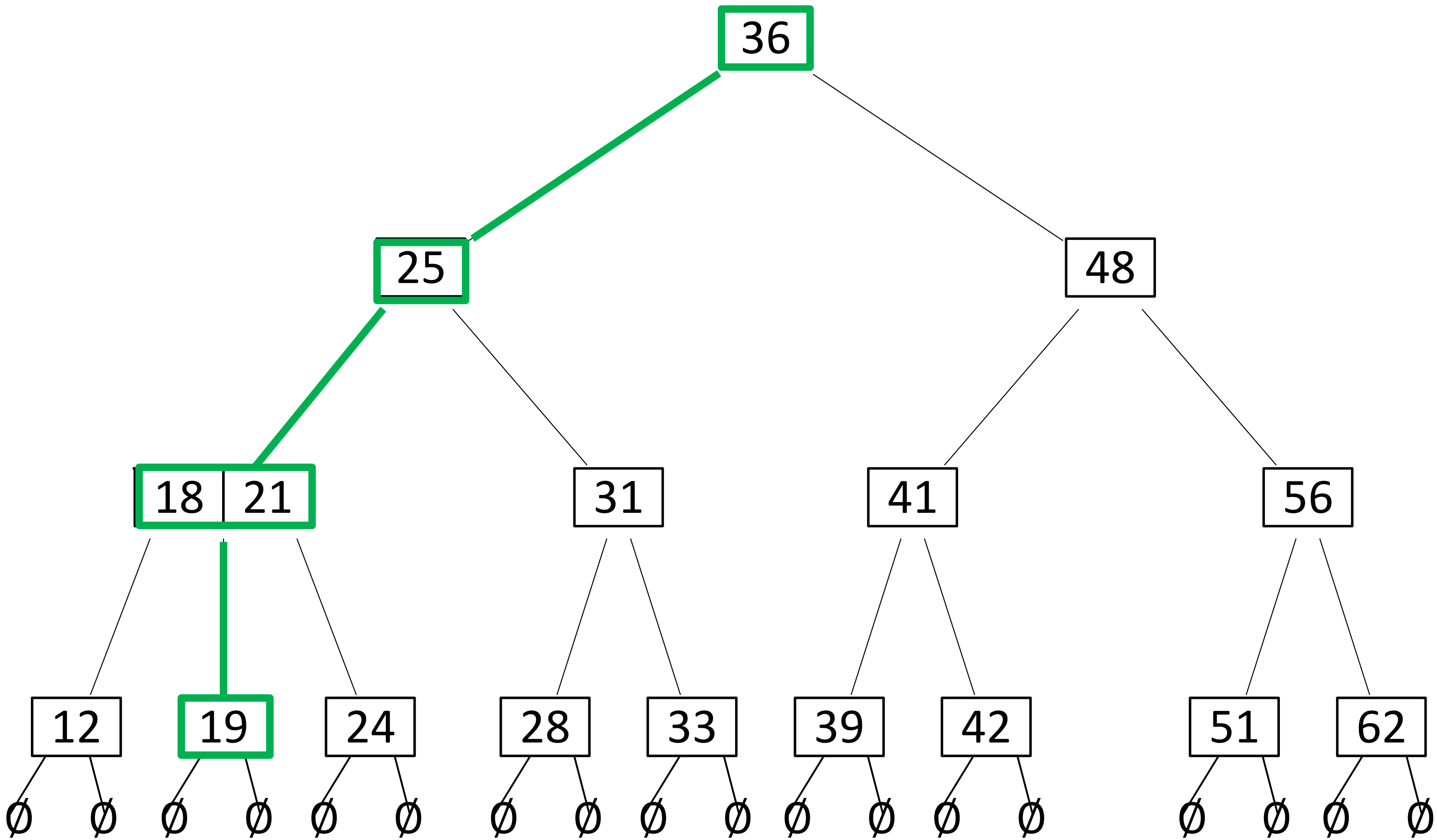


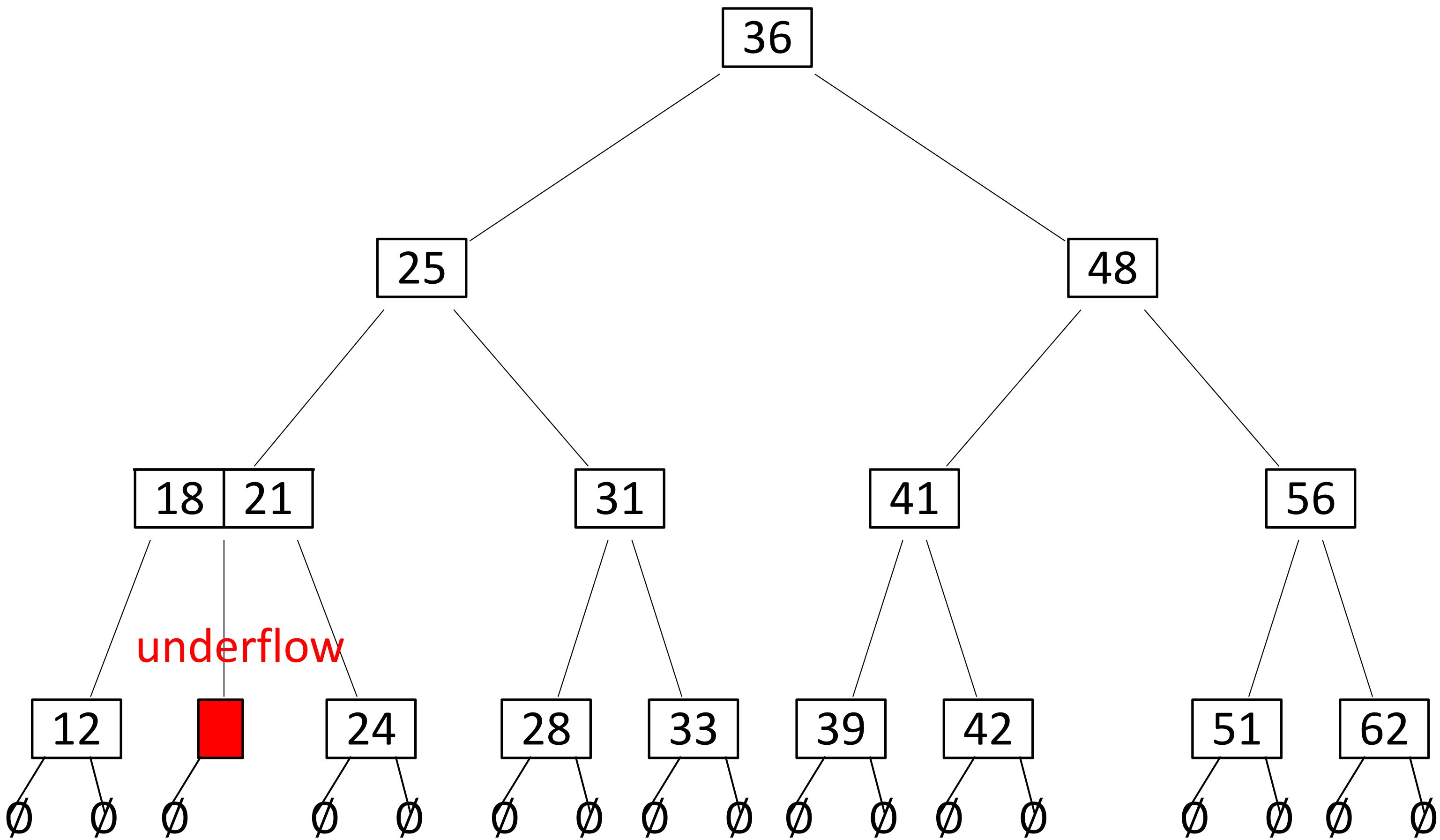
2-4 Tree Delete

- Example: *delete*(19)
  - first search(19)



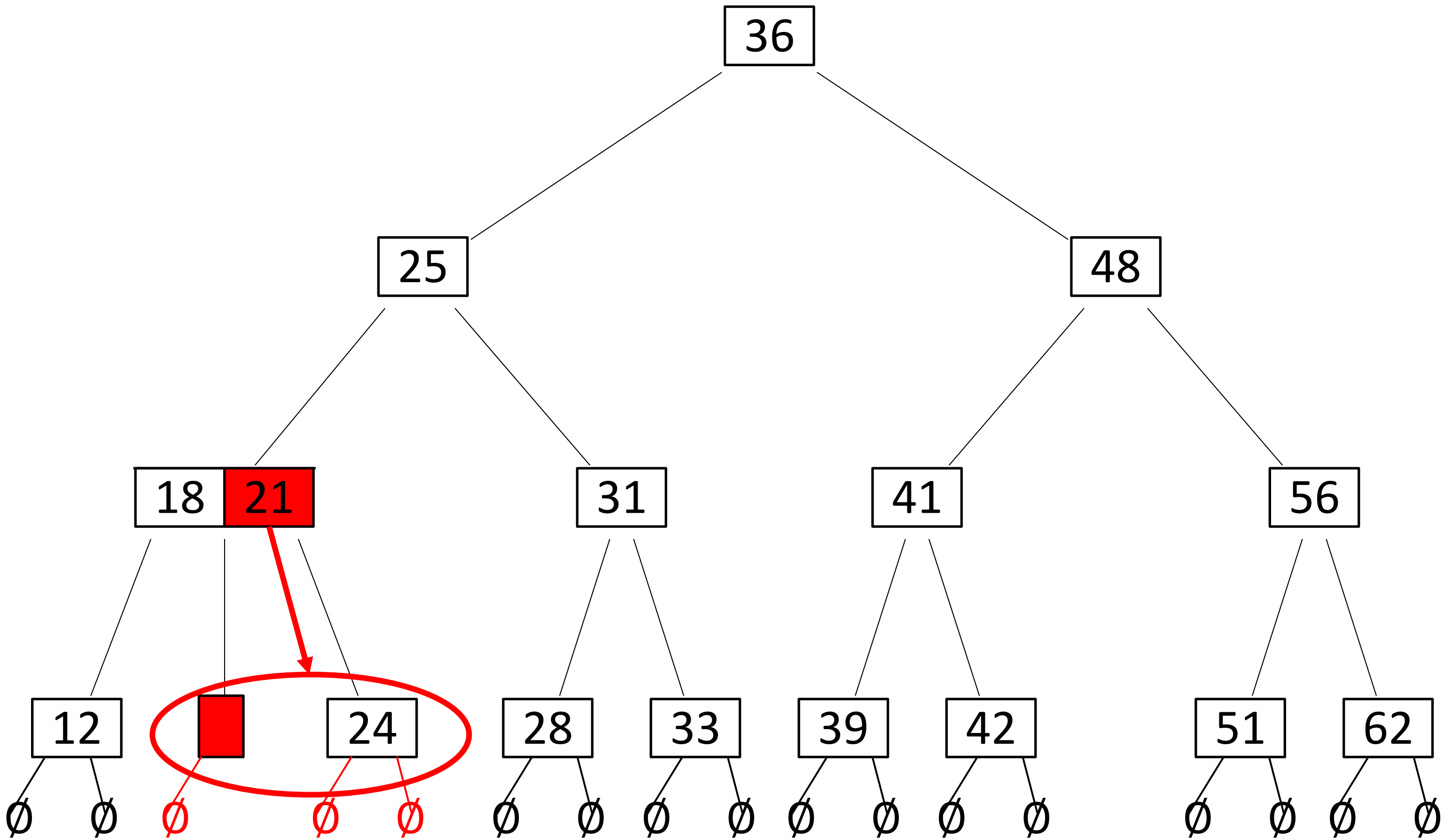
2-4 Tree Delete

- Example: *delete*(19)
    - first search(19)
    - then delete key 19 (and an empty subtree) from the node
    - immediate siblings exist, but not rich, cannot transfer



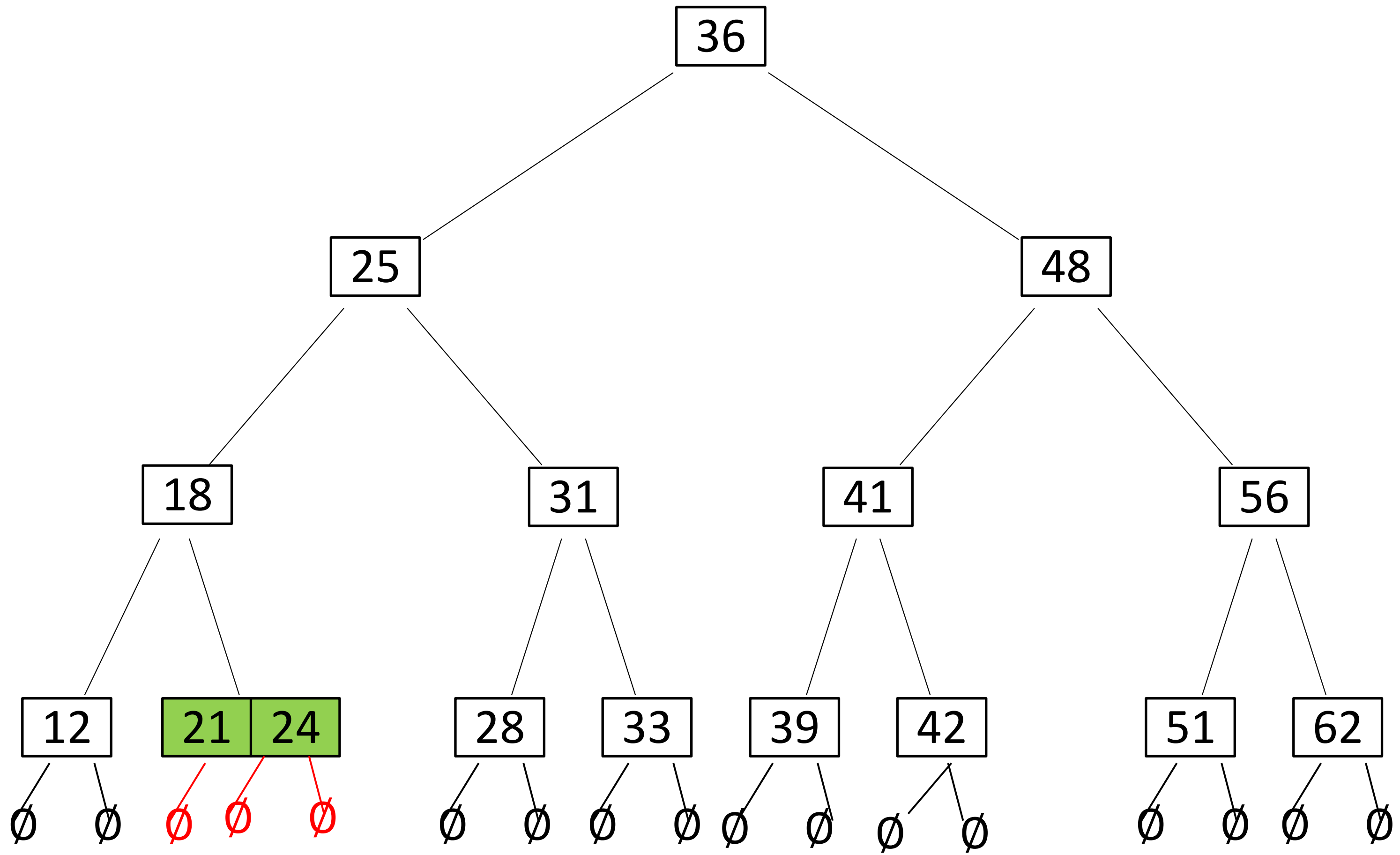
2-4 Tree Delete

- Example: *delete*(19)
  - immediate siblings exist, but not rich, cannot transfer
    - merge with right immediate sibling with help from parent



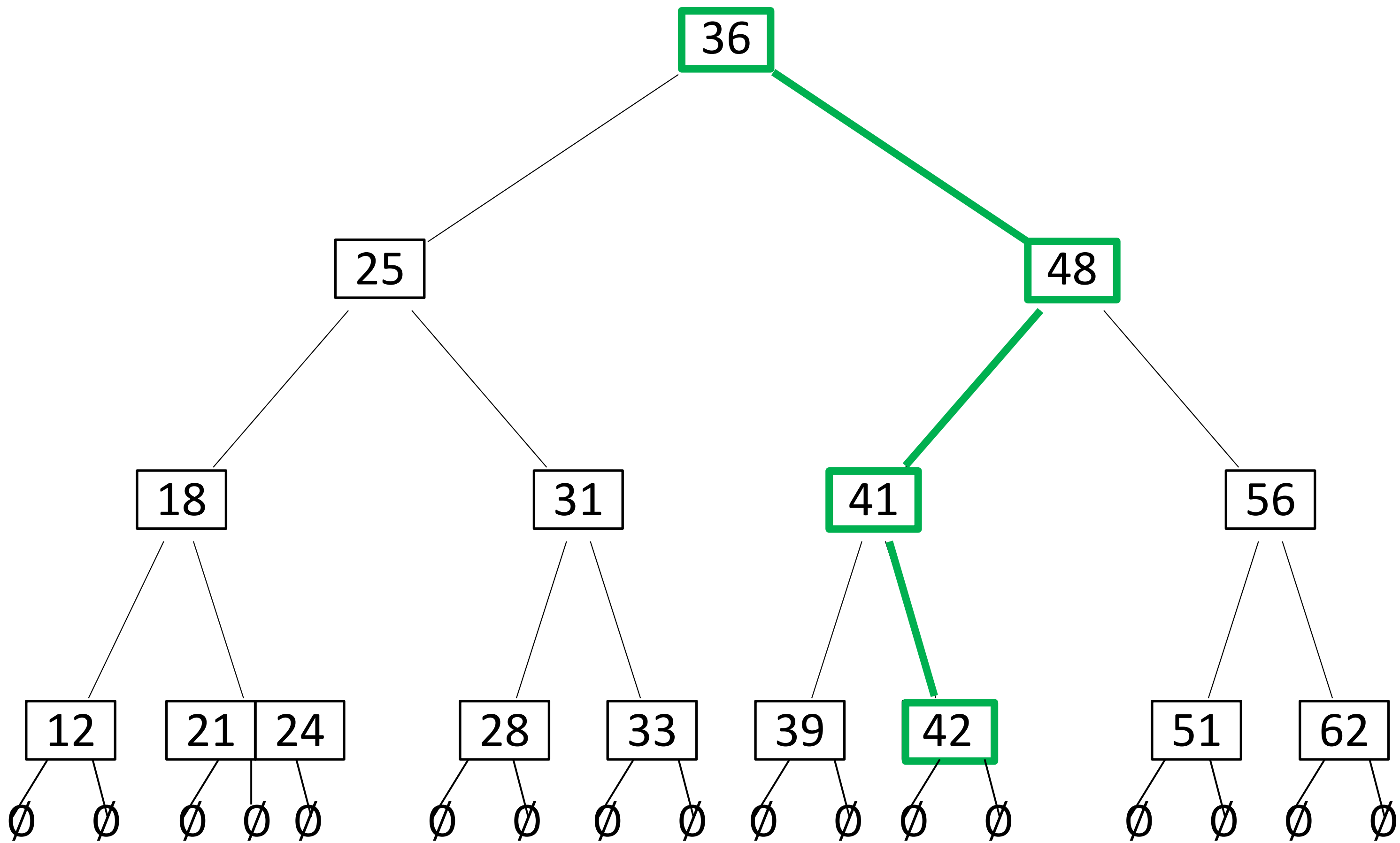
2-4 Tree Delete

- Example: *delete*(19)
    - immediate siblings exist, but not rich, cannot transfer
        - merge with right immediate sibling with help from parent
            - all subtrees merged together as well



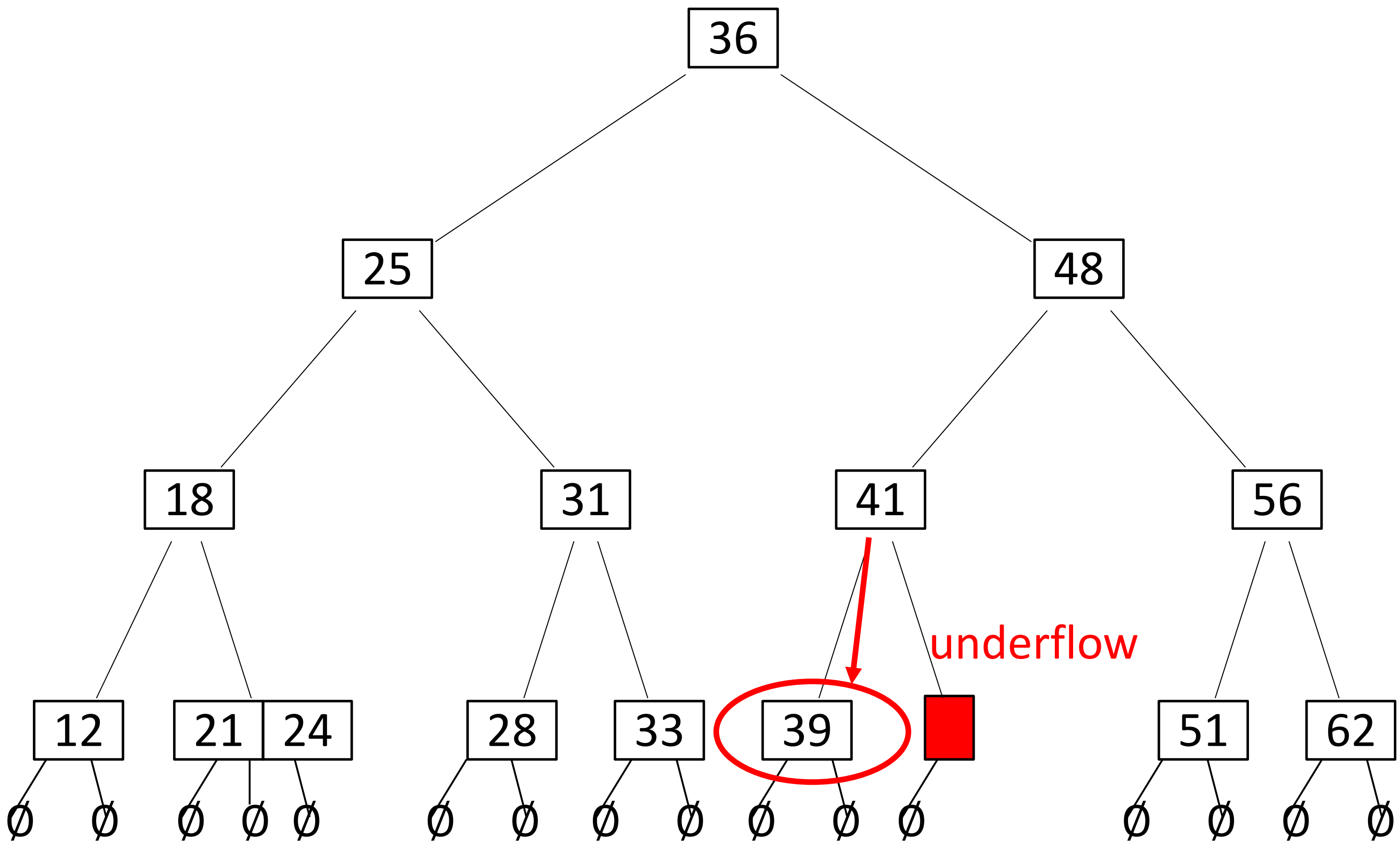
2-4 Tree Delete

- Example: *delete*(42)
  - first search(42)
  - delete key 42 with one empty subtree



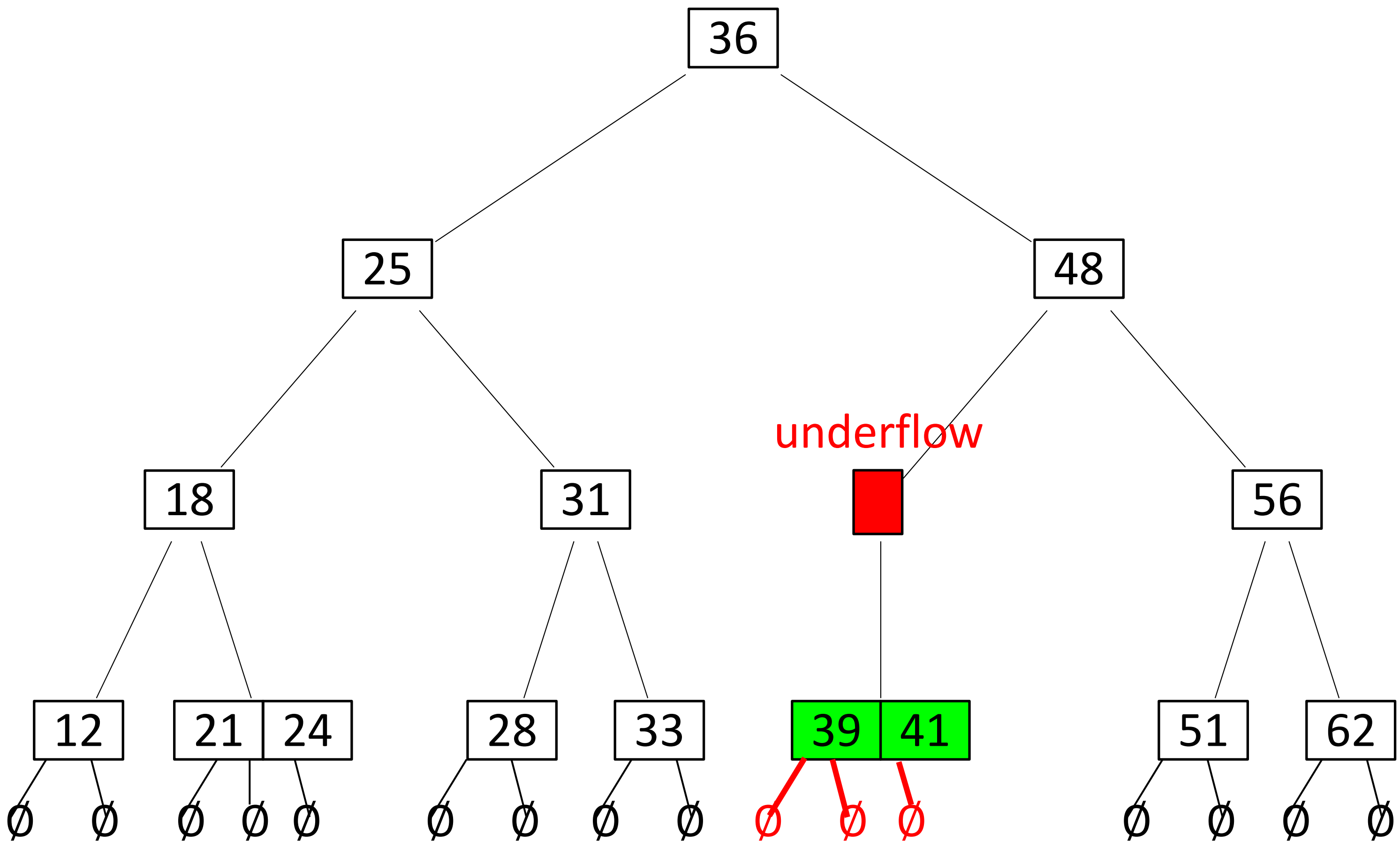
2-4 Tree Delete

- Example: *delete*(42)
    - first search(42)
    - the only immediate sibling is not rich, perform merge



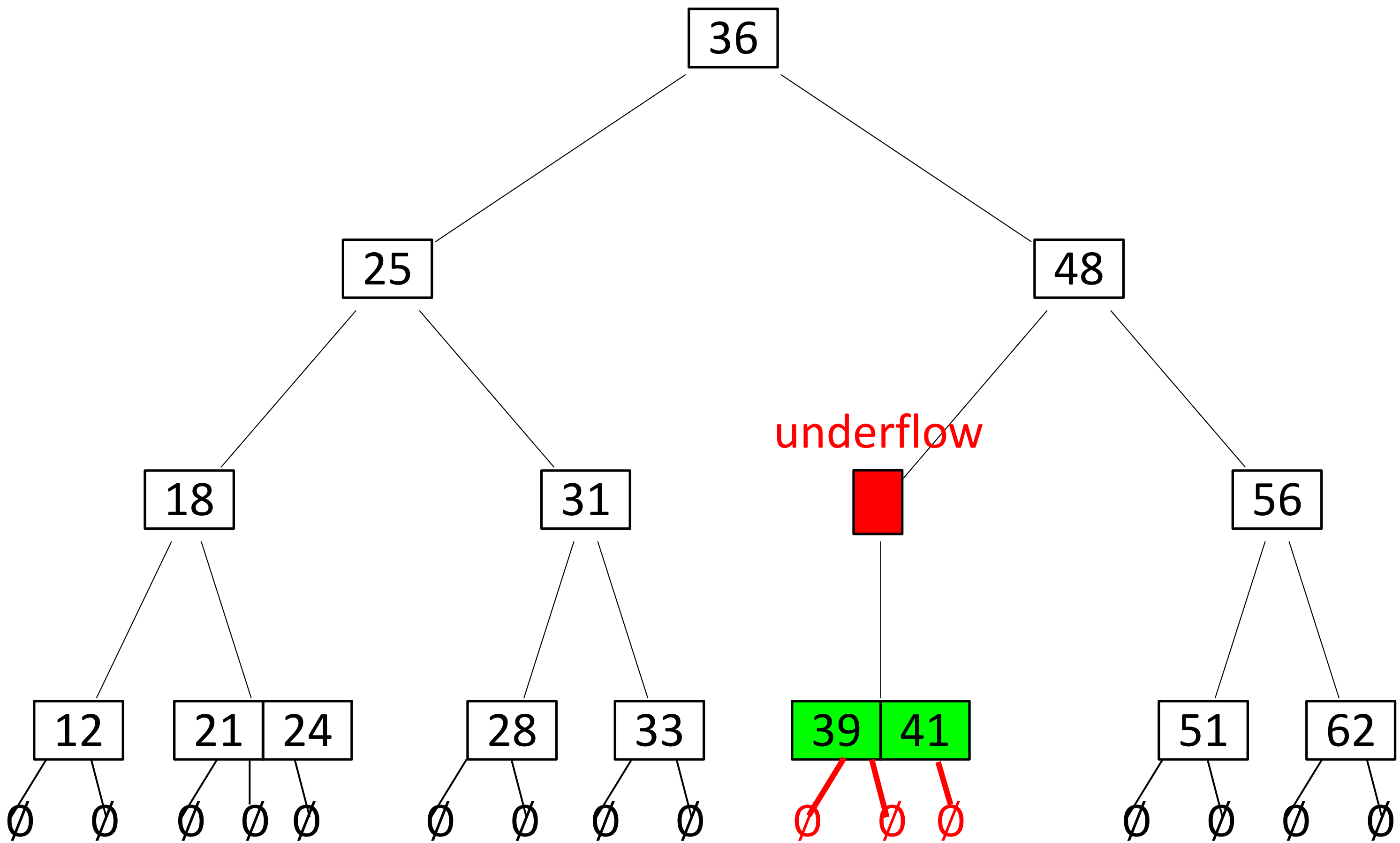
2-4 Tree Delete

- Example: *delete*(42)
    - first search(42)
    - the only immediate sibling is not rich, perform merge
        - all subtrees merged together as well



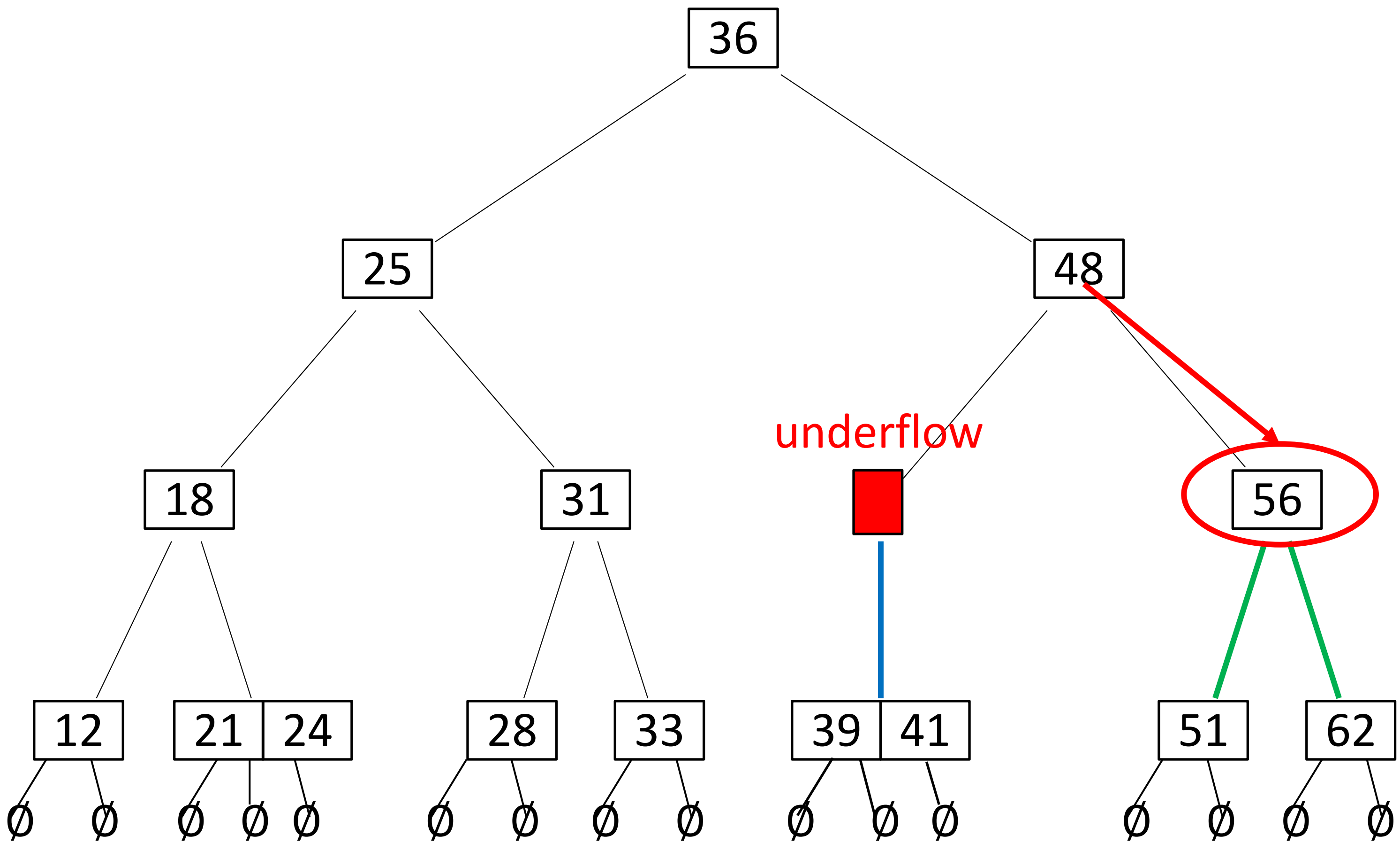
2-4 Tree Delete

- Example: *delete*(42)
  - merge operation can cause underflow at the parent node
  - while needed, continue fixing the tree upwards
    - possibly all the way to the root



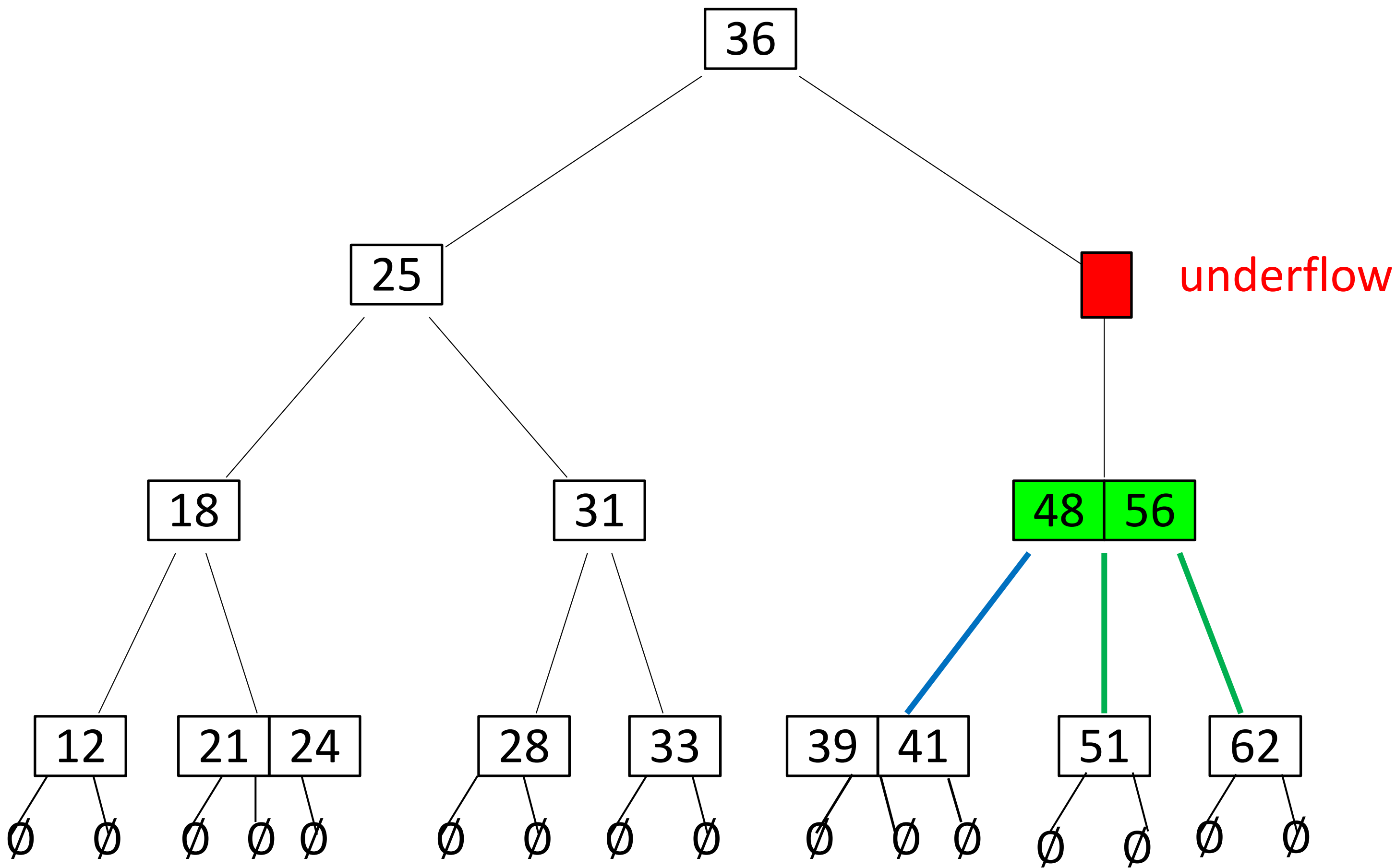
2-4 Tree Delete

- Example: *delete*(42)
    - the only sibling is not rich, perform a merge



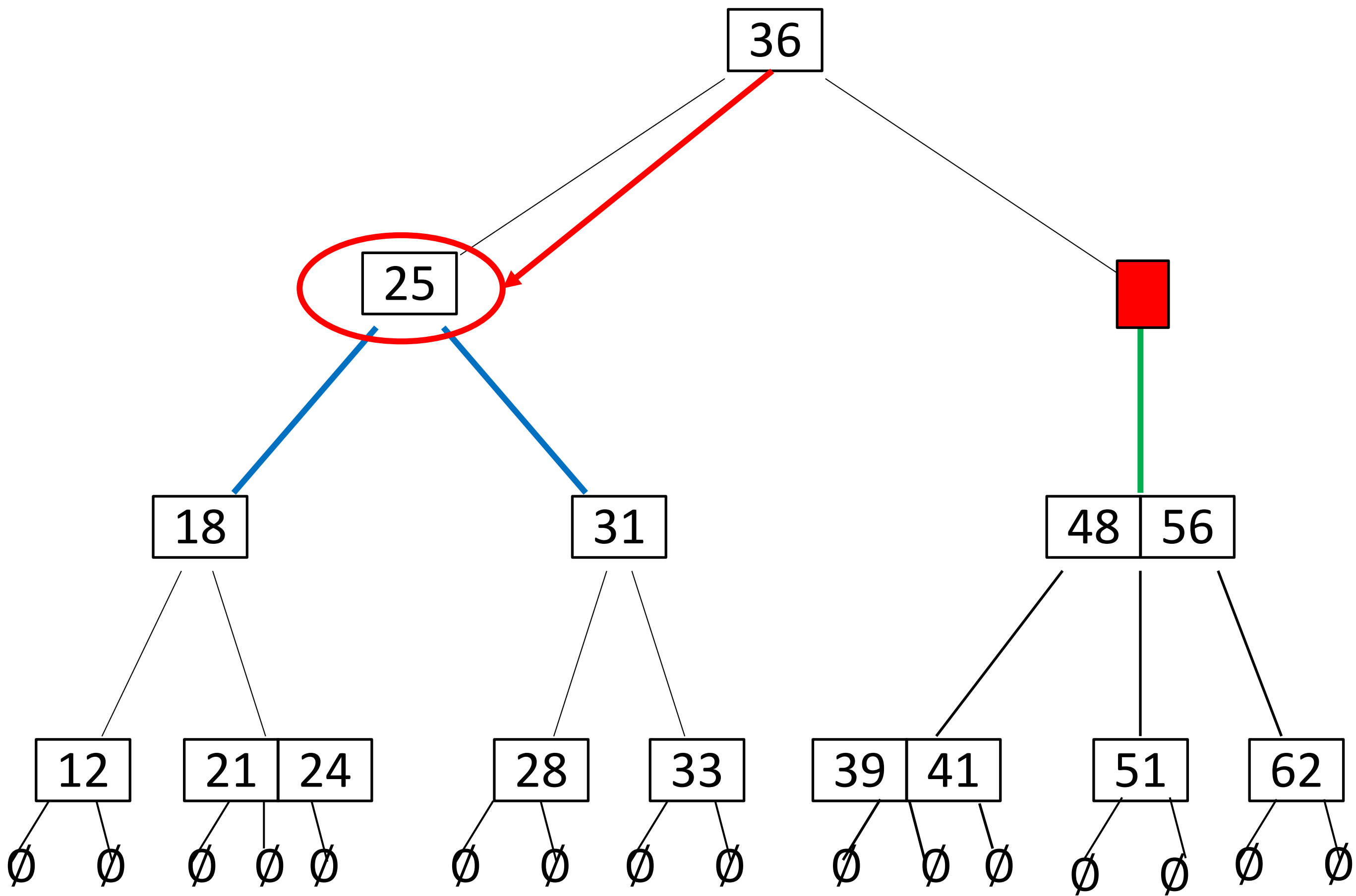
2-4 Tree Delete

- Example: *delete*(42)
  - the only sibling is not rich, perform a merge
  - subtrees are merged as well
  - continue fixing the tree upwards



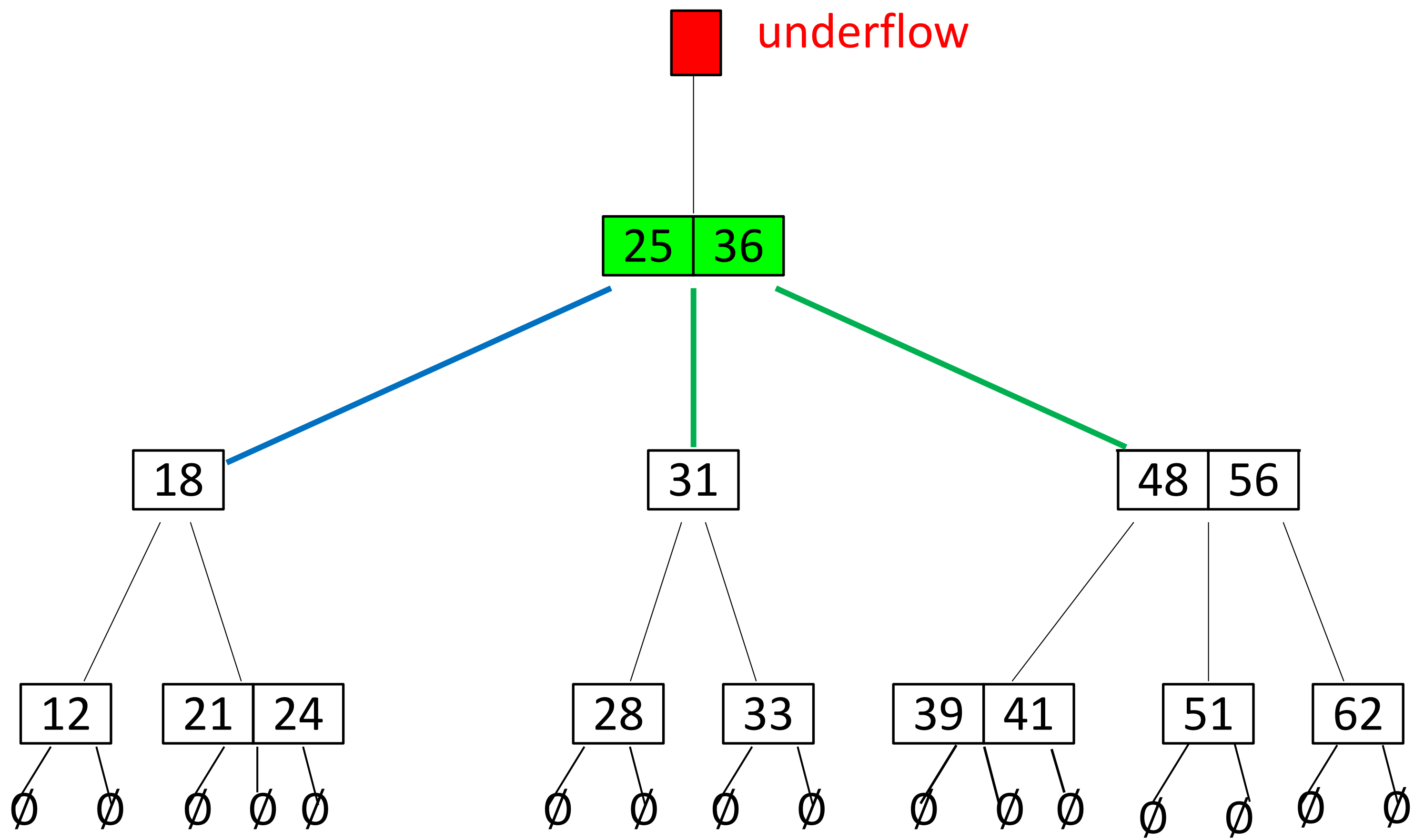
2-4 Tree Delete

- Example: *delete*(42)
  - the only sibling is not rich, perform a merge



2-4 Tree Delete

- Example: *delete*(42)
    - the only sibling is not rich, perform merge
    - underflow at parent node
        - it is the root, delete root



2-4 Tree Delete

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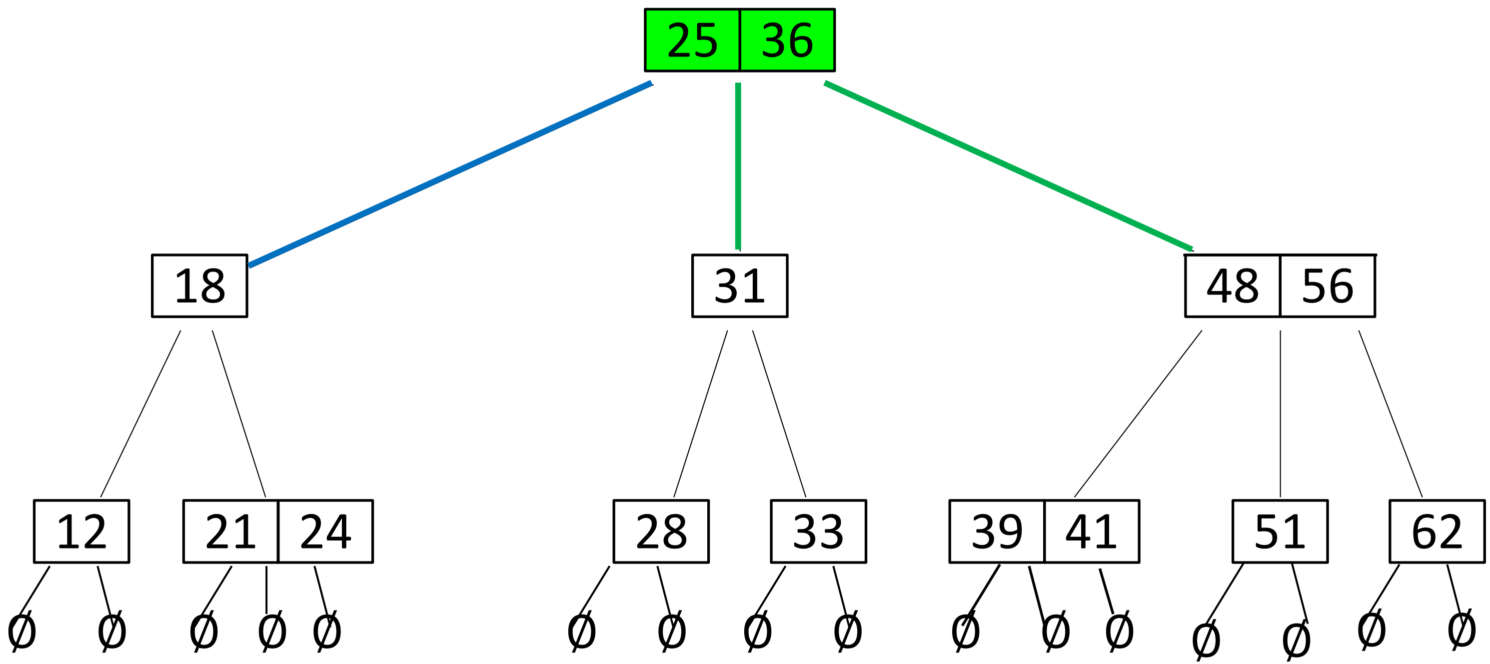
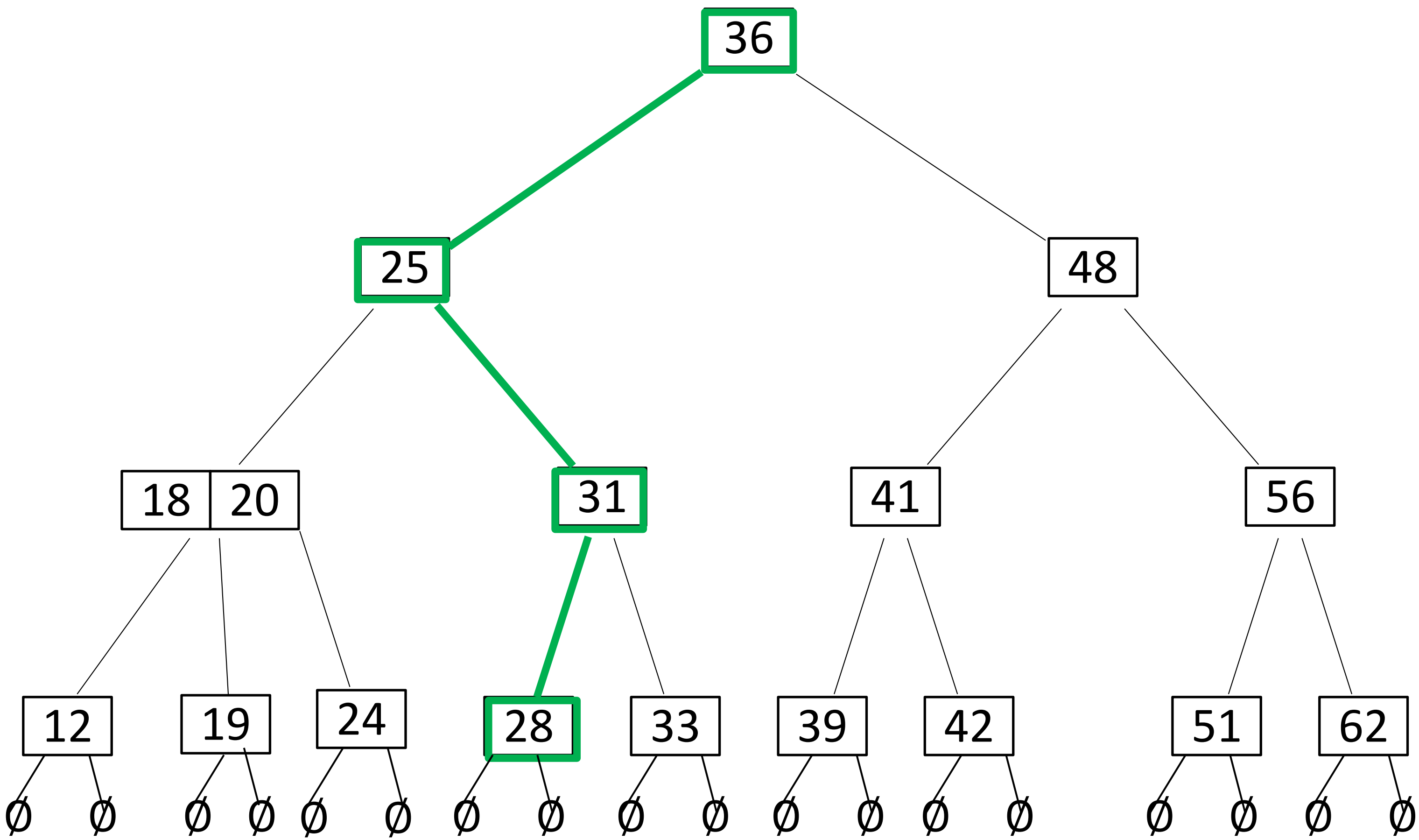
    - the only sibling is not rich, perform merge
    - underflow at parent node
        - it is the root, delete root



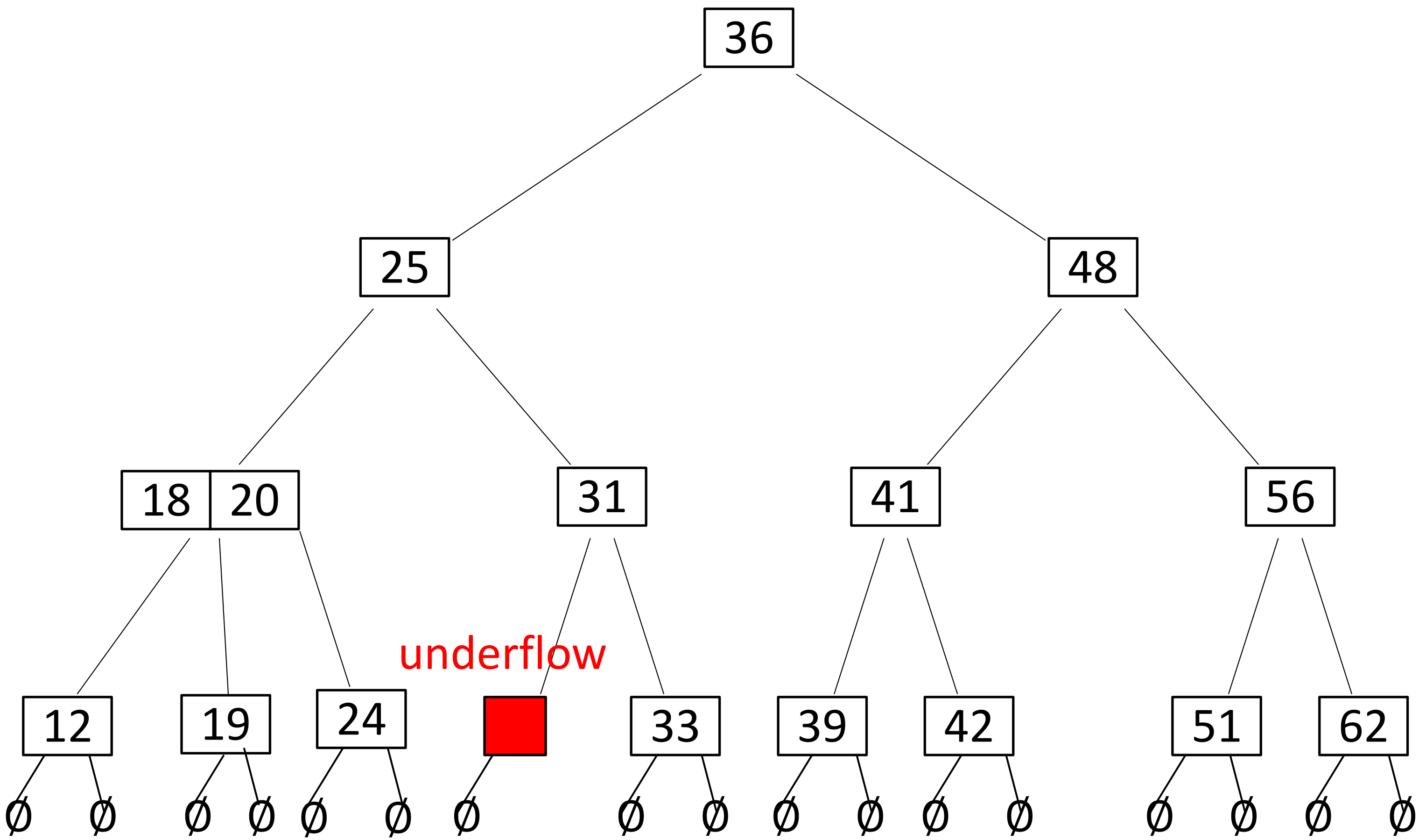
2-4 Tree Delete

- Example: *delete*(28)
    - first search(28)
    - delete key 28 with one empty subtree



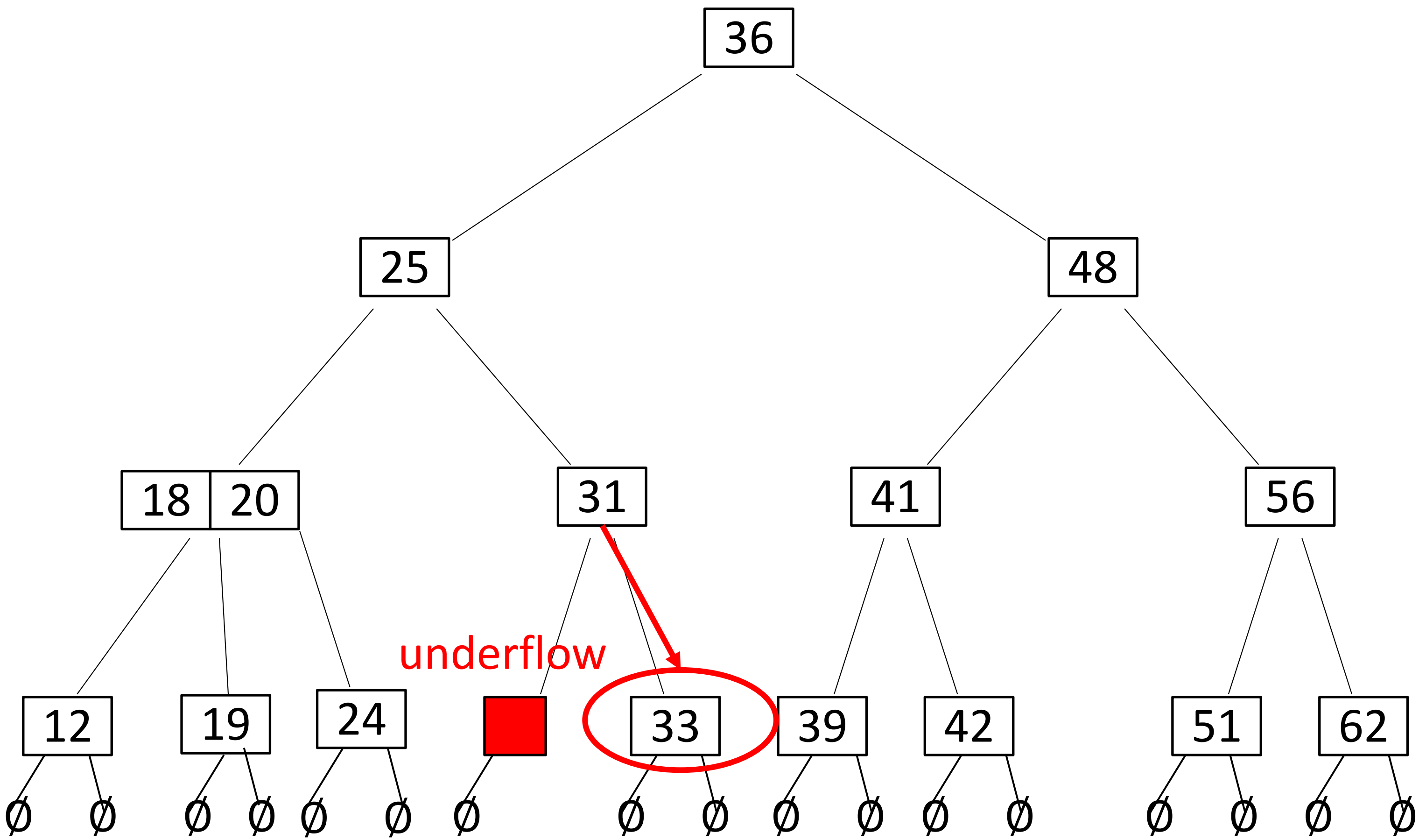
2-4 Tree Delete

- Example: *delete*(28)
    - first search(28)
    - delete key 28 with one empty subtree



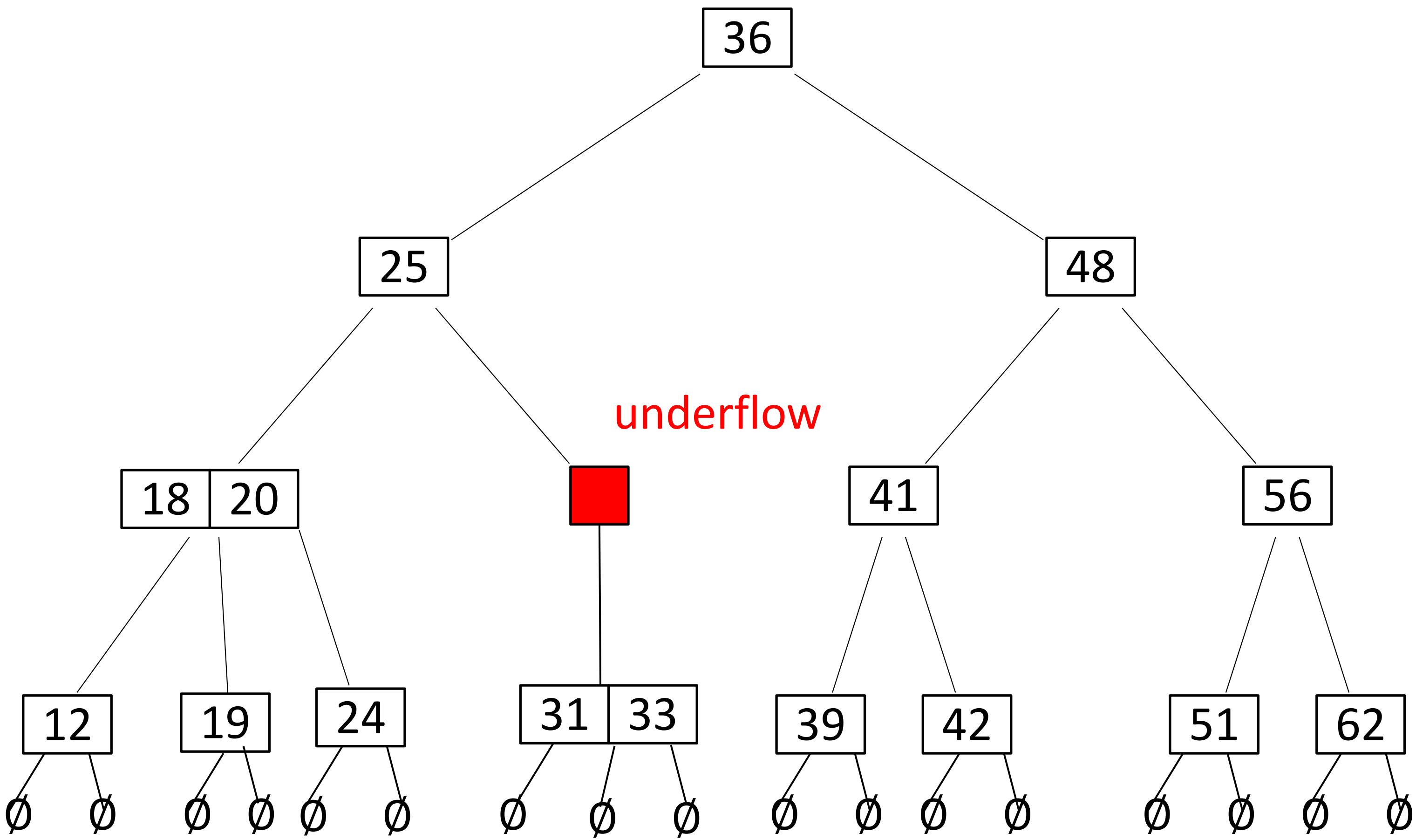
2-4 Tree Delete

- Example: *delete*(28)
  - first search(28)
  - delete key 28 with one empty subtree
  - merge with the only immediate sibling, who is not rich



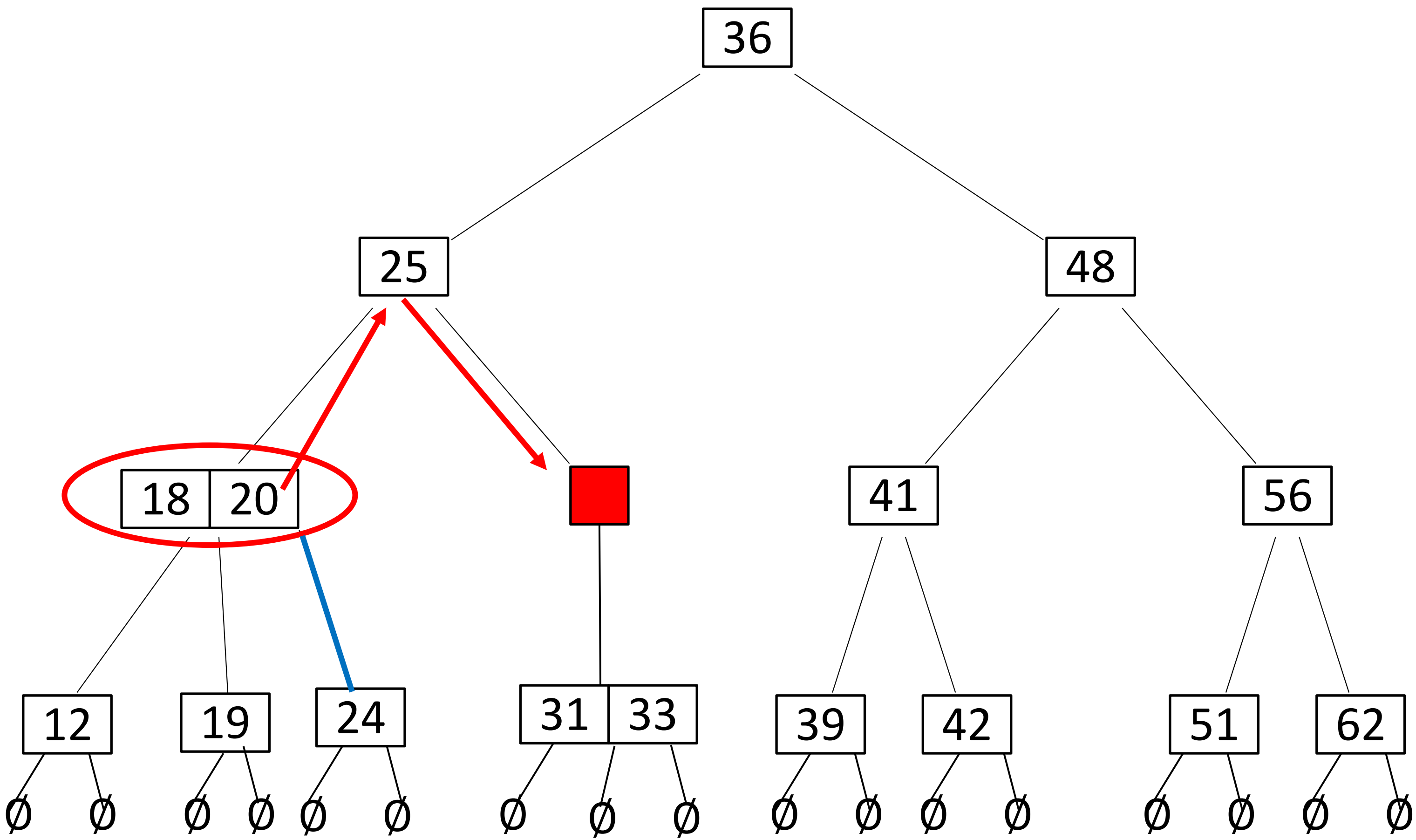
2-4 Tree Delete

- Example: *delete*(28)
  - first search(28)
  - delete key 28 with one empty subtree
  - merge with the only immediate sibling, who is not rich



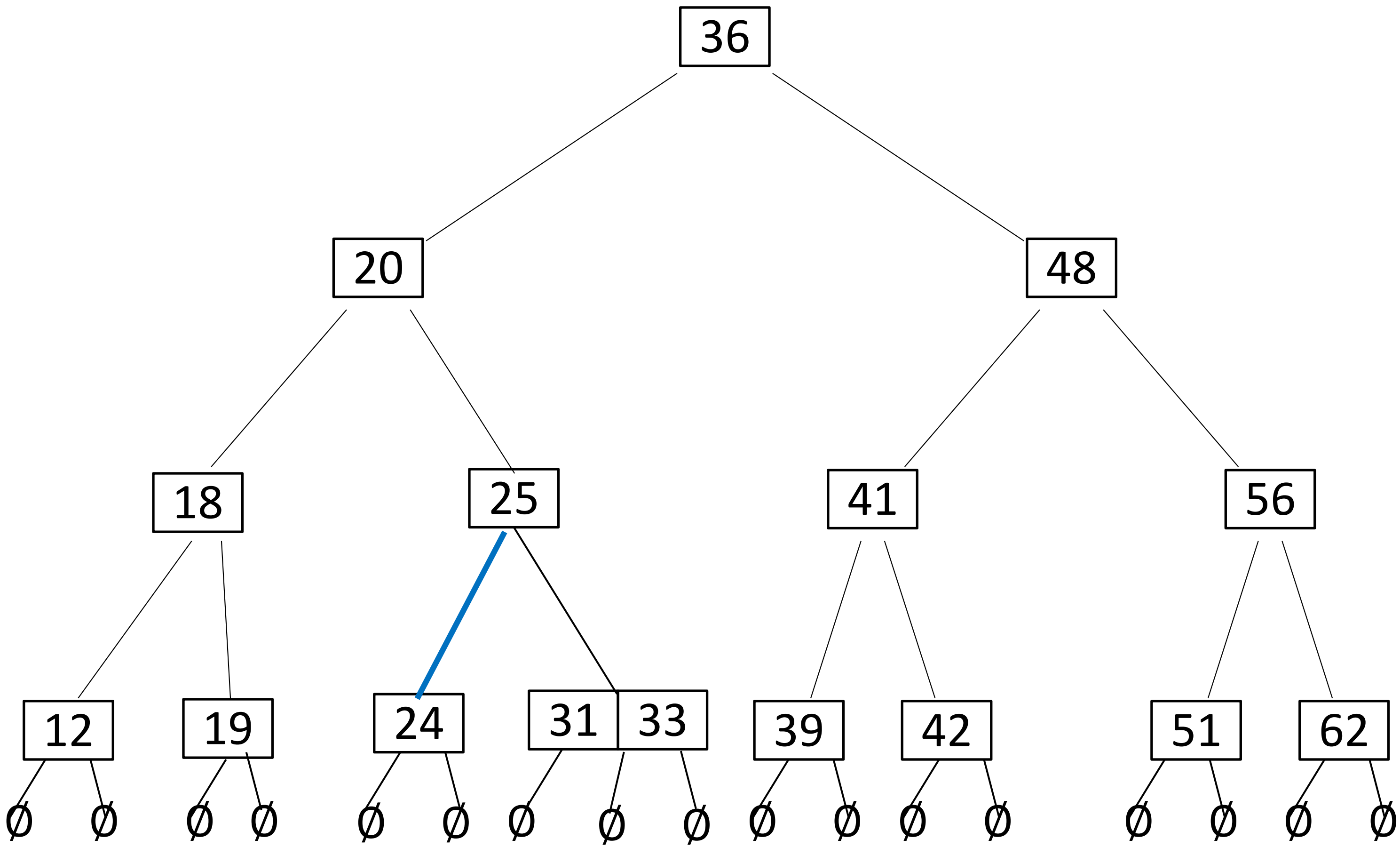
2-4 Tree Delete

- Example: *delete*(28)
    - transfer from a rich immediate sibling



2-4 Tree Delete

- Example: *delete*(28)
    - transfer from a rich immediate sibling
        - together with a subtree



2-4 Tree Delete Summary

- If key not at a leaf node, swap with inorder successor (guaranteed at leaf node)
- Delete key and one empty subtree from the leaf node involved in swap
- If underflow
    - If there is an immediate sibling with more than one key, transfer
        - no further underflows caused
            - do not forget to transfer a subtree as well
        - convention: if two siblings have more than one key, transfer with the right sibling
    - If all immediate siblings have only one key, merge
        - there must be at least one sibling, unless root
            - if root, delete
        - convention: if two immediate siblings with one key, merge with the right one
        - merge may cause underflow at the parent node, continue to the parent and fix it, if necessary

Deletion from a 2-4 Tree

```
24Tree::delete(k)
    v ← 24Tree::search(k)  //node containing k
    if v is not a leaf
            swap k with its inorder successor k′
            swap v with leaf that contained k′
    delete k  and one empty subtree in key-subtree-list of v
    while v has 0 keys // underflow
        if v is the root, delete v and break
        if v has immediate sibling u with 2 or more KVPs  // transfer, then done!
            transfer the key of u that is nearest to v to p
            transfer the key of p between u and v to v
            transfer the subtree of u that is nearest to  v to v
            break
        else // merge and repeat
            u ← immediate sibling of v
            transfer the key of p between u and v to u
            transfer the subtree of v to u
            delete node v
            v ← p
```

2-4 Tree Summary

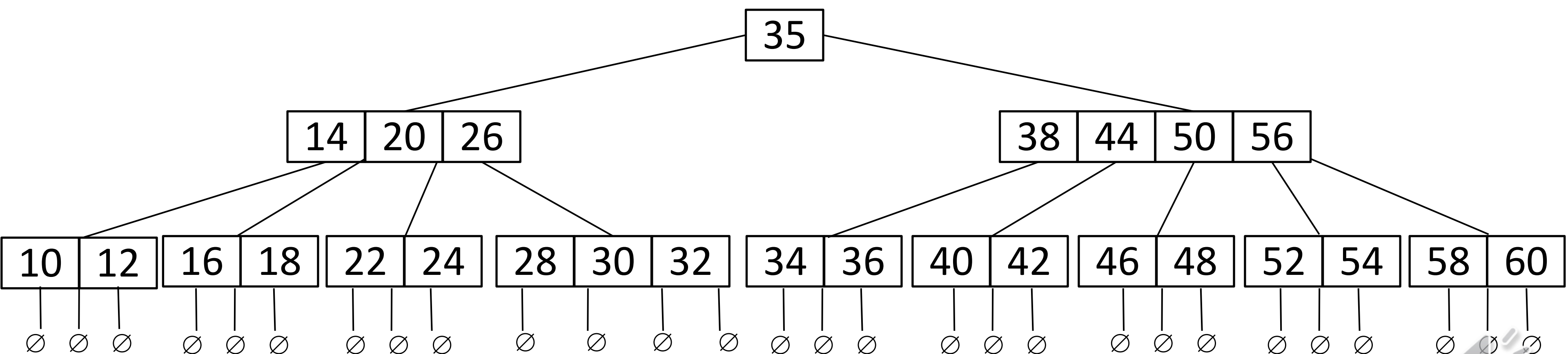
- 2-4 tree has height $O(\log n)$
  - in internal memory, all operations have run-time $O(\log n)$
  - this is no better than AVL-trees in theory
  - but 2-4 trees are faster than AVL-trees in practice, especially when converted to binary search trees called red-black trees
    - no details
- 2-4 tree has height $\Omega(\log n)$
  - tree of height h has at most $n = 4^{h+1} - 1$ KVPs
  - thus h is $\Omega(\log n)$
- So 2-4 tree is not significantly better than AVL-tree wrt block transfers
- But can generalize the concept to decrease the height

Outline

- External Memory
  - Motivation
  - Stream based algorithms
  - External sorting
  - External dictionaries
    - 2-4 Trees
    - (*a*, *b*)-Trees
    - B-Trees

(a,b)-Trees

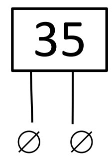
- 2-4 Tree is a specific type of (a,b)-tree
- (a,b)-tree satisfies
  - each node has at least a subtrees, unless it is the root
    - root must have at least 2 subtrees
  - each node has at most b subtrees
  - if node has k subtrees, then it stores $k-1$ key-value pairs (KVPs)
  - all empty subtrees are at the same level
  - keys in the node are between keys in the corresponding subtrees
  - requirement: $a \le \left\lceil \frac{b}{2} \right\rceil = \lfloor (b+1)/2 \rfloor$



(3, 5)-tree, also a valid (3, 6)-tree

(a, b)-Trees: Root

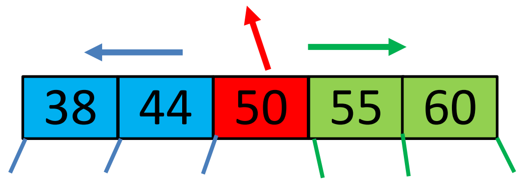
- Why special condition for the root?
- Needed for (a,b)-tree storing very few KVP
- (3,5) tree storing only 1 KVP



- Could not build it if forced the root to have at least 3 children
    - remember # keys at any node is one less than number of subtrees

(a, b)-Trees

- Because $a \leq \lfloor (b+1)/2 \rfloor$ *search*, *insert*, *delete* work just like for 2-4 trees
  - straightforward redefinition of underflow and overflow
- For example, for $(3,5)$-tree
  - at least 3 children, at most 5
    - each node is at least a 2-node, at most a 4-node
  - during insert, overflow if get a 5-node



  - split results in two 2-nodes, and 2-nodes are smallest allowed nodes

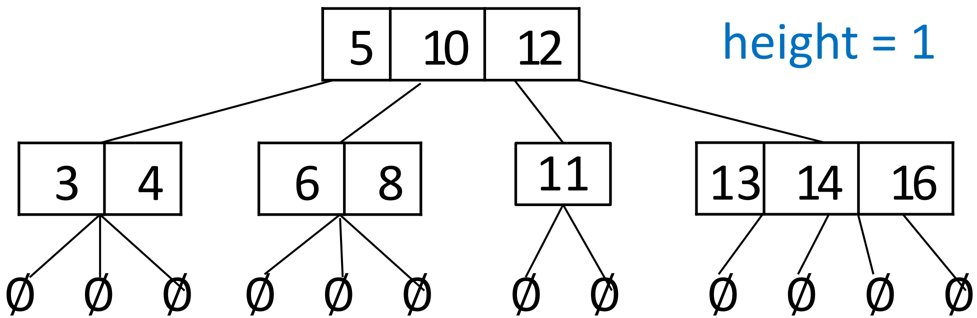
- If $a > \lceil b/2 \rceil$, for example if allow $(4,5)$-tree, cannot split like before
  - equal (best possible) split results in two 2 nodes, which is not allowed
- In general, overflow means node has $b + 1$ subtrees
  - node split in the middle means new nodes have at least $\lfloor (b+1)/2 \rfloor$ subtrees
  - since $a \leq \lfloor (b+1)/2 \rfloor$, each new node has at least a subtrees, as required

(a, b)-Trees delete

- For example, for $(3{,}5)$-tree
    - at least 3 children, at most 5
        - each node is at least a 2-node, at most a 4-node
    - during insert, underflow if get a 1-node
    - if we have an immediate sibling which is rich (3 or 4-node), do transfer
    - otherwise, do merge
        - guaranteed to have at least one sibling which is a 2-node

Height of (a, b)-tree

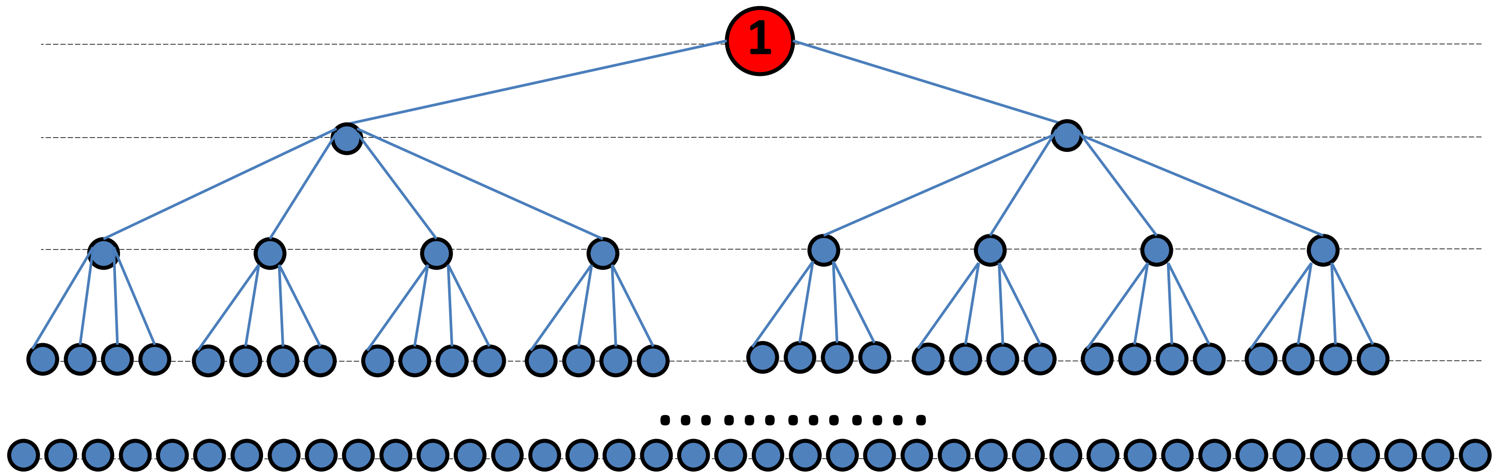
- Height = number of levels **not** counting empty subtrees



Height of (a, b)-tree

- Consider (a,b)-tree with the *smallest number* of KVP and of height h
    - red node (the root) has 1 KVP, blue nodes have $(a-1)$ KVP

level	**# of nodes**
0	1
1	$2a^0$
2	$2a^1$
3	$2a^2$
h	$2a^{h-1}$



$$\text{\# of KVPs} = \mathbf{1} + \sum_{i=0}^{h-1} 2a^i(a-1) = \mathbf{1} + 2(a-1)\underbrace{\boxed{\sum_{i=0}^{h-1} a^i}}_{\frac{a^h-1}{a-1}} = 2a^h - 1$$

- Let n the number of KVP in *any* (a, b)-tree of height h

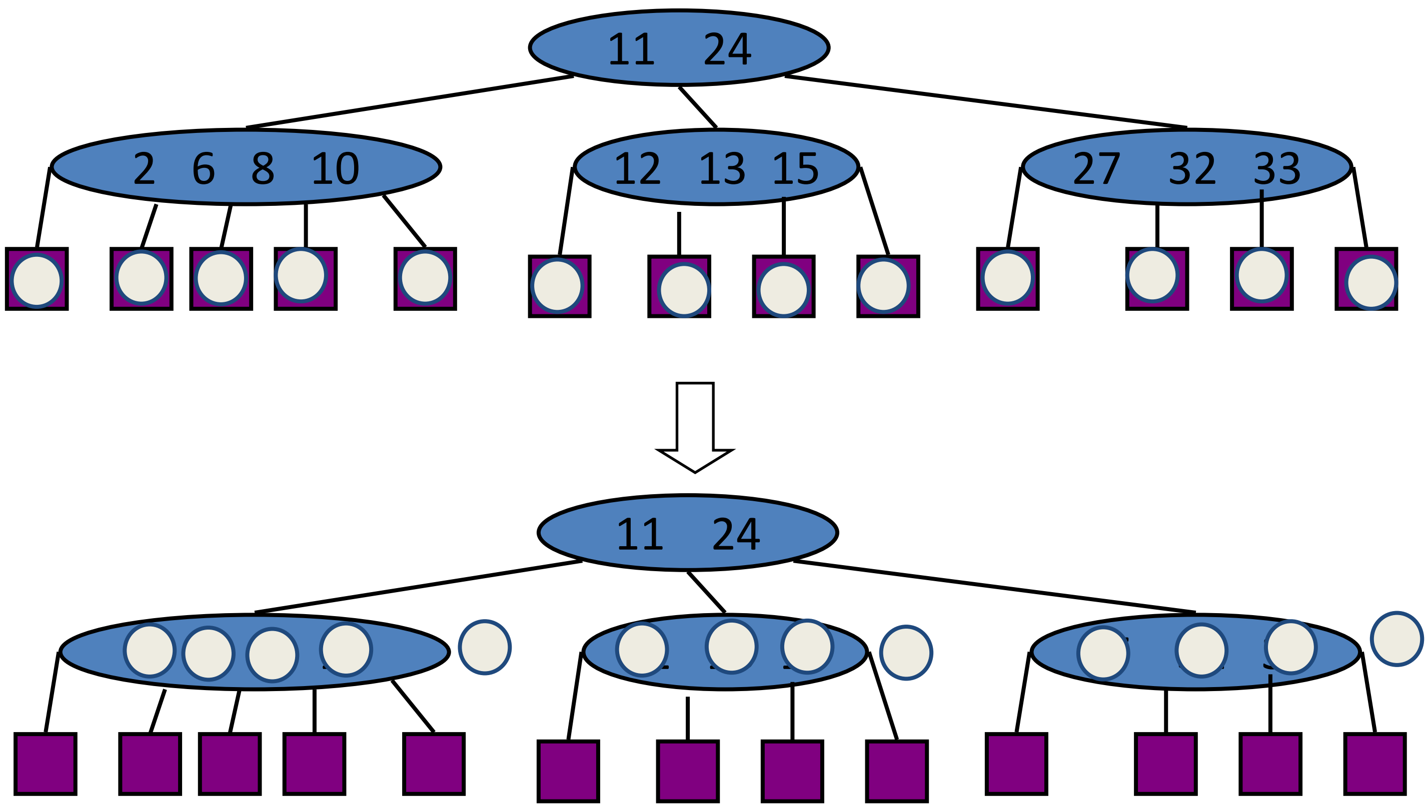
$$n \geq 2a^h - 1\text{, therefore, } \log_a \frac{n+1}{2} \geq h$$

- Height of tree with n KVPs is $O(\log_a n) = O(\log n / \log a)$

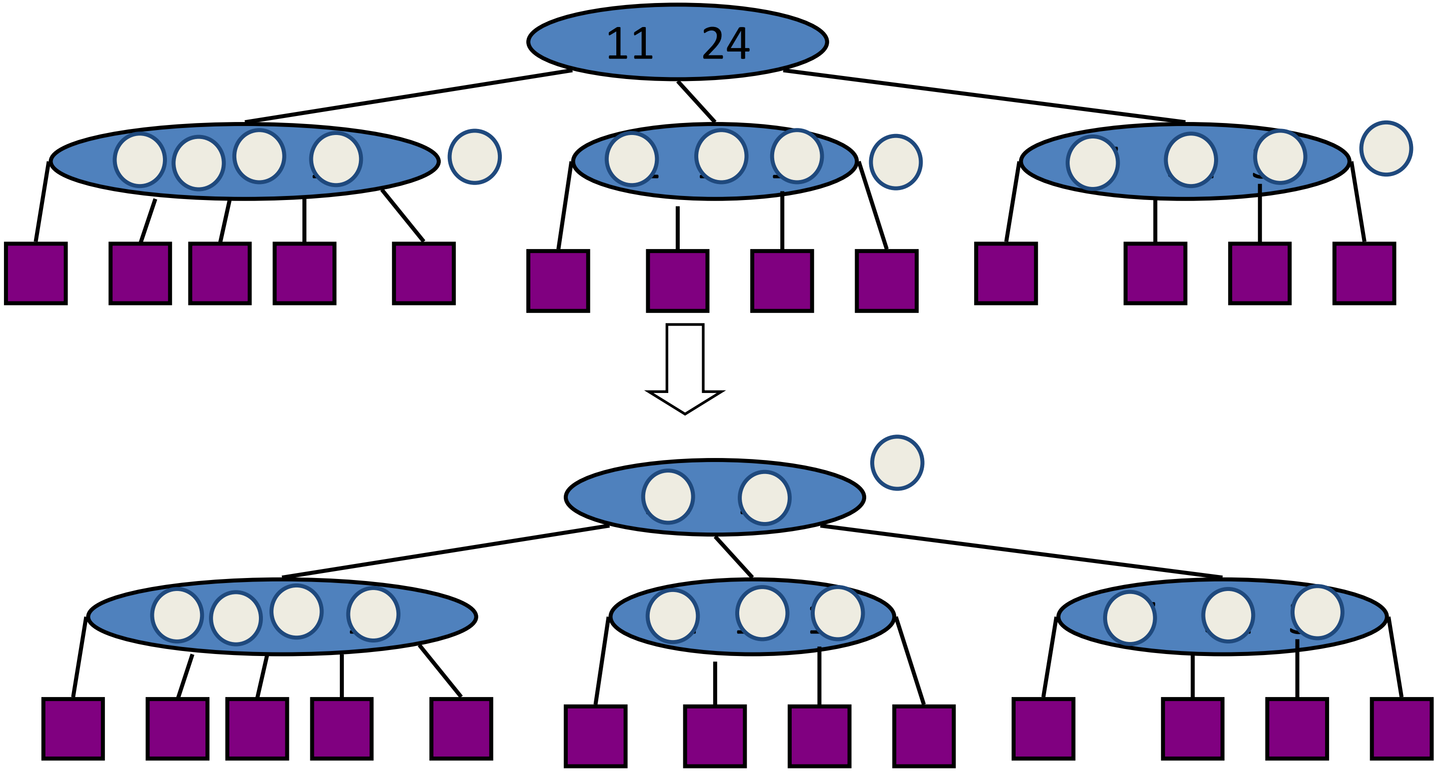
Useful Fact about (a, b)-trees

- number of of KVP = number of empty subtrees – 1 in any (a, b)-tree

Proof: Put one stone on each empty subtree and pass the stones up the tree. Each node keeps 1 stone per KVP, and passes the rest to its parent. Since for each node, #KVP = # children – 1, each node will pass only 1 stone to its parent. This process stops at the root, and the root will pass 1 stone outside the tree. At the end, each KVP has 1 stone, and 1 stone is outside the tree.



Useful Fact about (a, b)-trees



(a, b)-Tree Analysis in Internal Memory

- Search, insert, delete each require visiting $\Theta(height)$ nodes
- Height is $O(\log n / \log a)$
- Recall that $a \le \left\lceil \frac{b}{2} \right\rceil$ is required for insert and delete to work correctly
- Therefore, chose $a = \left\lceil \frac{b}{2} \right\rceil$ to minimize the height
- Work at a node can be done in $O(\log b)$ time
- Total cost

$$O\left(\frac{\log n}{\log a} \cdot \log b\right) = O\left(\frac{\log b}{(\log b) - 1} \cdot \log n\right) = O(\log n)$$

- This is not better than AVL-trees in internal memory
- But the main motivation for (a,b)-tree is external memory

Outline

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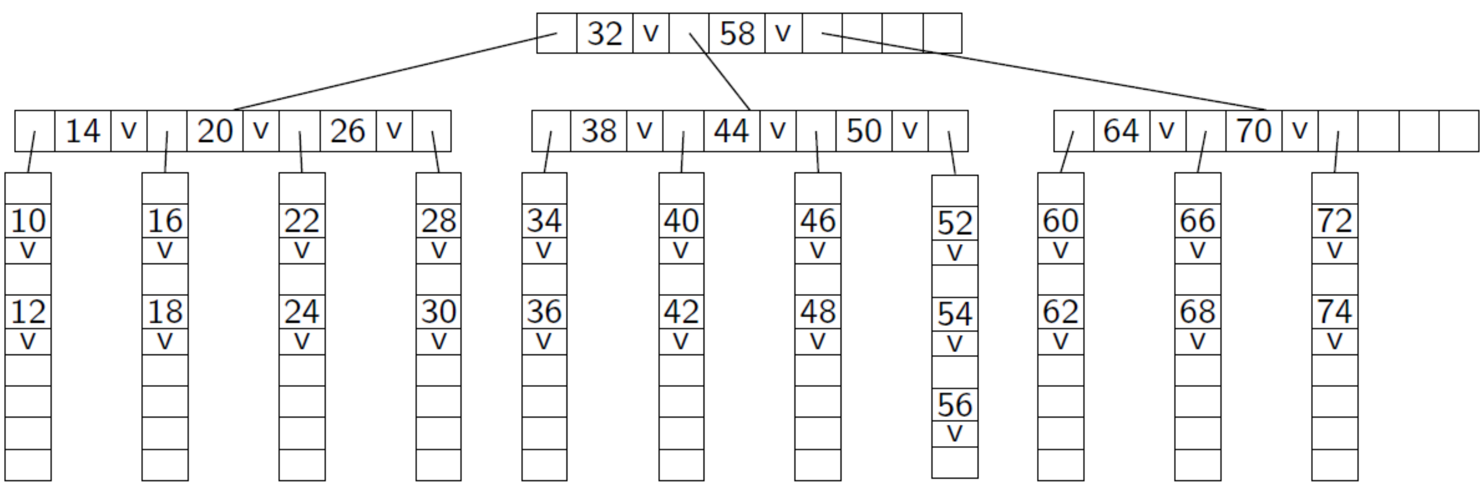
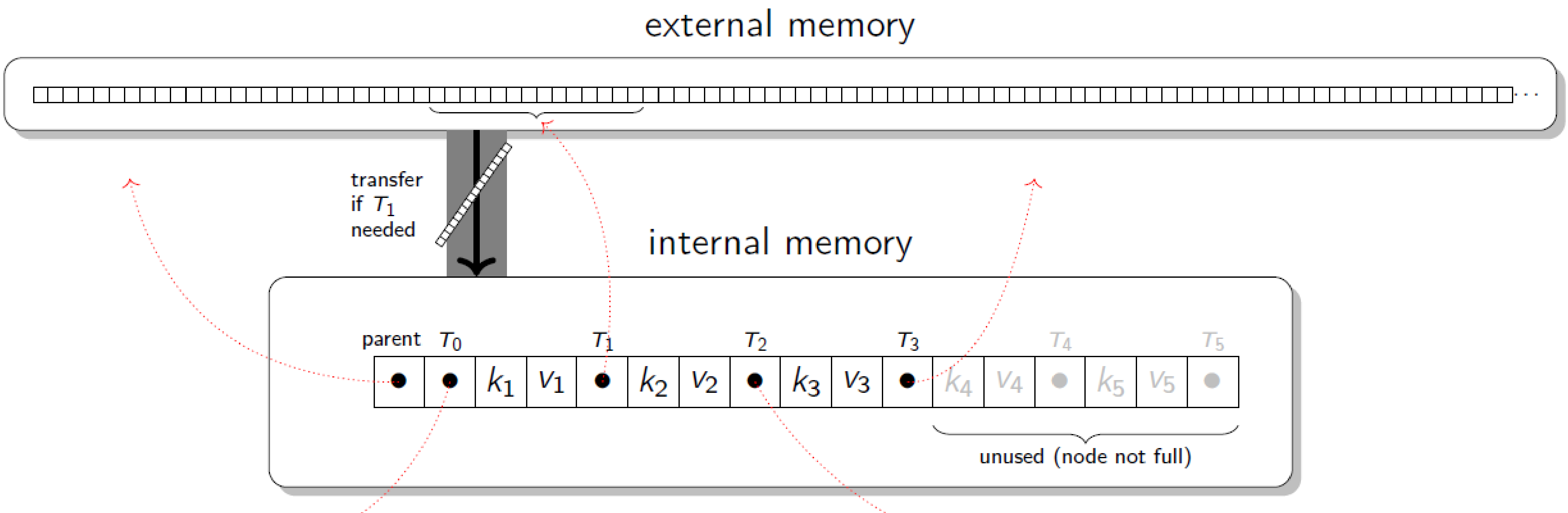
B-trees

- B-tree is a type of (a, b)-tree tailored to the external memory model
- In B-tree, $a = \lceil b/2 \rceil$
- Thus we usually specify B-tree by giving b
    - b is called the order of B-tree
    - B-tree or order b is a $(\lceil b/2 \rceil, b)$-tree
    - typically $b \in \Theta(B)$
- Every node is one block (size B) of memory
- Choose b so that a node with $b - 1$ KVPs (and hence $b - 1$ value references and b subtree references) fits into one block



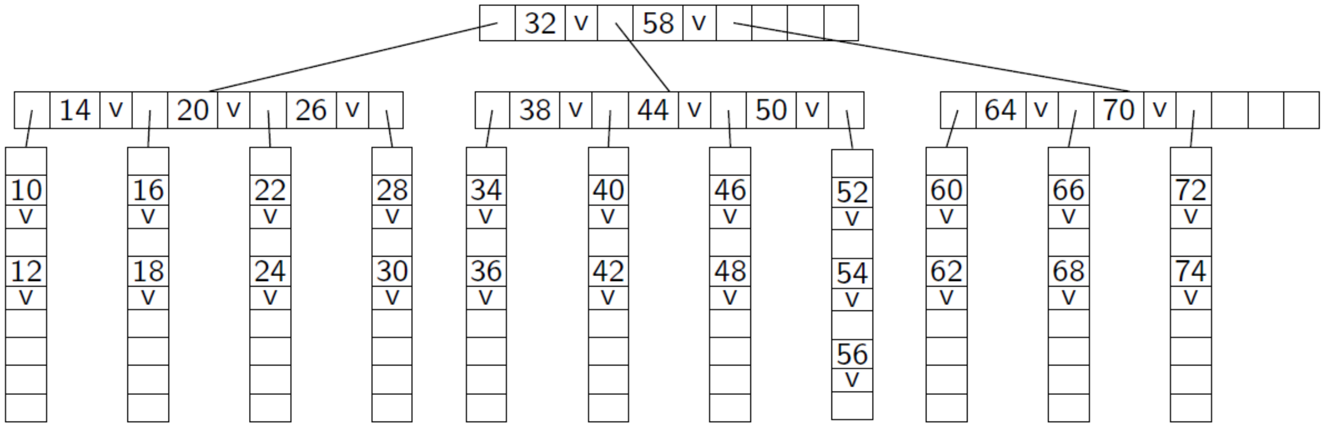
B-trees in External Memory

- Close-up on one node in one block



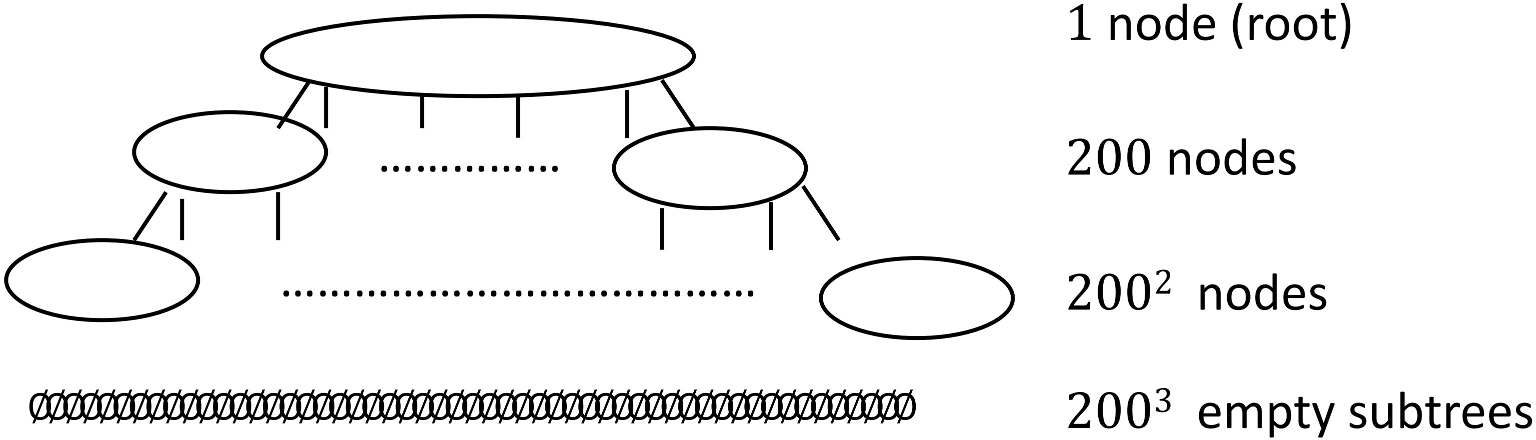
- In this example, 17 references fit into one block, so B-tree can have order 6
- Note that each block is at least half full
  - since each node is at least $\lceil b/2 \rceil$-node

B-tree Analysis in External Memory



- Search, insert, and delete each requires visiting $\Theta(height)$ nodes
  - $\Theta(height)$ block transfers
- Work within a node is done in internal memory, no block transfers
- The height is $\Theta(\log_b n) = \Theta(\log_B n)$
  - since $b \in \Theta(B)$
- So all operations require $\Theta(\log_B n)$ block transfers
  - this is asymptotically optimal
- There are variants that are even better in practice
- B-trees are hugely important for storing databases (cs448)

Example of B-tree usage



- B-tree of order 200
  - B-tree of order 200 and height 2 can store up to $200^3 - 1$ KVPs
    - from the ‘useful fact’ proven before
  - if we store root in internal memory, then only 2 block reads are needed to retrieve any item