CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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Outline

1. Lower bound
2. Interpolation Search
3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

Lower bound for search

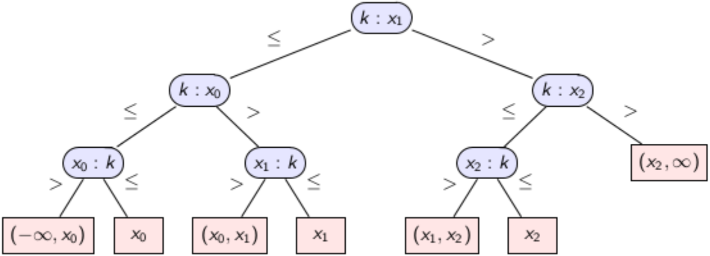
The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Lower bound for search

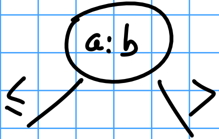
The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Theorem: In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof: via decision tree



But can we beat the lower bound for special keys?



Remark: could also use

Claim: an algorithm (in the comparison model) to do search in a size-n dictionary $\Rightarrow$ a decision tree with at least $n+1$ leaves.

$x_0 < x_1$.

```
if   k ≤ x0
  if  x0 ≤ k
     return  "found", 0
  else
     return  "not found"
else
  if  k ≤ x1
    if  x1 ≤ k
      return  "found", 1
    else
       return  "not found"
  else
    return "not found"
```

$k \le x_0$

$k < x_0$

$k > x_0$

$x_0 < k \le x_1$

$x_0 < k < x_1$

$k > x_1$

$x_0 < x_1$.

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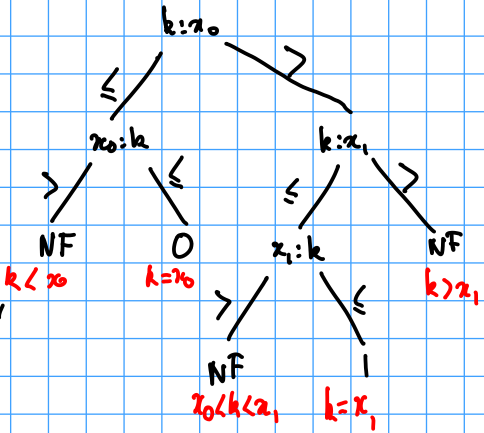
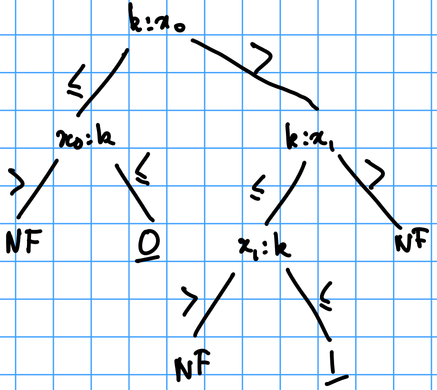




at least $3 = 2 + 1$ leaves

0, 1 NF

Let h be the worst-case # of comparisons that we do (for n keys)

→ any possible input k reaches a leaf after doing at most h comparisons

→ in the decision tree, there are at least $n+1$ leaves of depth $\leq h$.

In any binary tree, the number of leaves of depth $\leq h$ is at most 2^h. Proof: induction on h.

$n+1 \leq$ # of leaves of $\leq 2^h$

depth $\leq h$

$\rightarrow \quad n+1 \leq 2^h$

$\rightarrow \quad \log(n+1) \leq h.$

Binary Search

Recall the run-times in a *sorted array*:

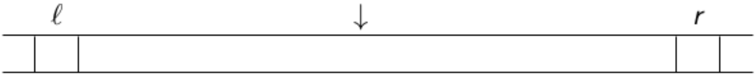
- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

```
binary-search(A, n, k)
A: Sorted array of size n, k: key
    ℓ ← 0, r ← n − 1
    while (ℓ ≤ r)
        m ← ⌊(ℓ+r)/2⌋
        if (A[m] < k) then ℓ = m + 1
        else if (k < A[m]) then r = m − 1
        else return "found at A[m]"
    return "not found, but would be between A[ℓ−1] and A[ℓ]"
```

r ℓ=r+1

Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$



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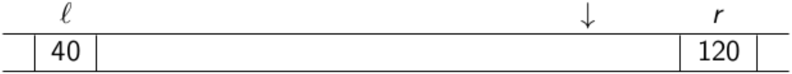

Question: If keys are *numbers*, where would you expect key $k = 100$?

Interpolation Search: Motivation

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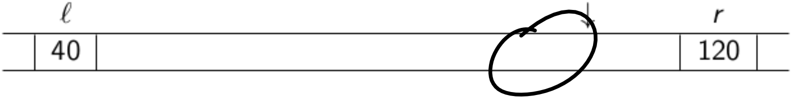
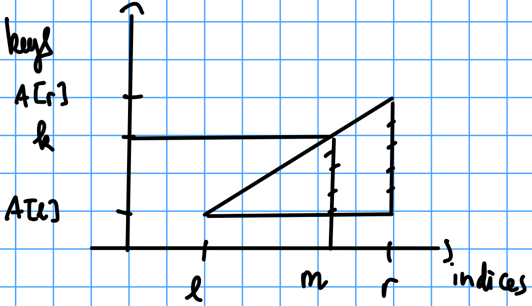


Question: If keys are *numbers*, where would you expect key $k = 100$?

interpolation-search$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]}(r-\ell) \right\rfloor$



$$\frac{A[r]-A[l]}{r-l} = \frac{k-A[l]}{m-l}$$

$$\frac{m-l}{r-l} = \frac{k-A[l]}{A[r]-A[l]} \rightsquigarrow m = l + \frac{k-A[l]}{A[r]-A[l]}(r-l).$$

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

interpolation-search(A[0..10],449):

Interpolation Search Example

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ℓ		$\uparrow$								r

interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$

Interpolation Search Example

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- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$

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- $\ell = 3,\ r = 4,\ m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$, found at $A[4]$

Interpolation Search Example

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Works well if keys are *uniformly* distributed:

- Can show: Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

But: Worst case performance $\Theta(n)$

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$$T(n) = T(\sqrt{n}) + c$$

$$2 \to 2^2 \to 2^4 \to 2^8 \to 2^{16} \to \dots \to \underline{2^{2^i}} \to \dots$$

$$\to T(2^{2^i}) = T(2^{2^{i-1}}) + c$$

$$= T(2^{2^{i-2}}) + 2c$$

$$= T(2^{2^{i-3}}) + 3c$$

$$= \dots = T(2^{2^{i-i}}) + ic = T(2) + ic$$

$\to$ for $n = 2^{2^i}$, $T(n) = T(2) + c \cdot \log(\log(n))$.

Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash during computation of m.

interpolation-search(A, n, k)
A: Sorted array of size n, k: key

$\ell \leftarrow 0$, $r \leftarrow n-1$
while $(\ell \leq r)$
  if $(k < A[\ell]$ or $k > A[r])$ **return** "not found"
  if $(A[\ell] = A[r])$ **then return** "found at $A[\ell]$"
  $m \leftarrow \ell + \lfloor \frac{k - A[\ell])}{A[r]-A[\ell]} \cdot (r - \ell) \rfloor$
  if $(A[m] < k)$ **then** $\ell = m + 1$
  else if $(k < A[m])$ **then** $r = m - 1$
  else return "found at $A[m]$"
// We always return from somewhere within while-loop

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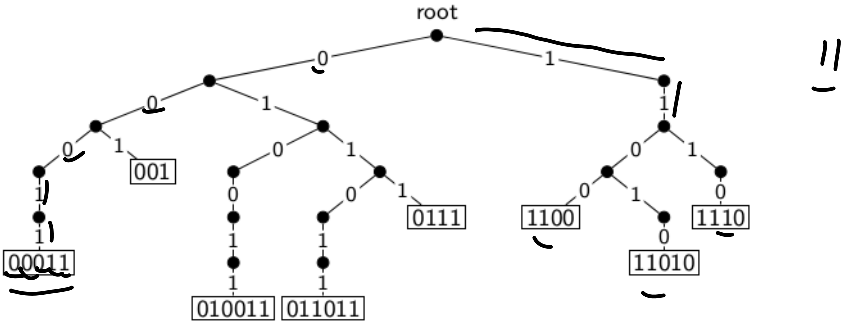
Tries: Introduction

Trie (also know as **radix tree**): A dictionary for bitstrings.

(Should know: string, word, $|w|$, alphabet, prefix, suffix, comparing words,....)

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key

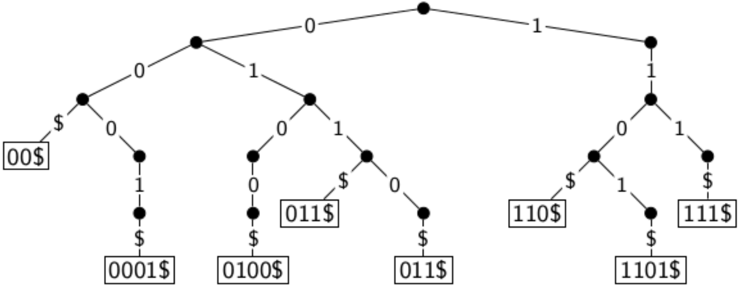


More on tries

Assumption: Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with 'end-of-word' character \$.

Example: A trie for $\{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}$



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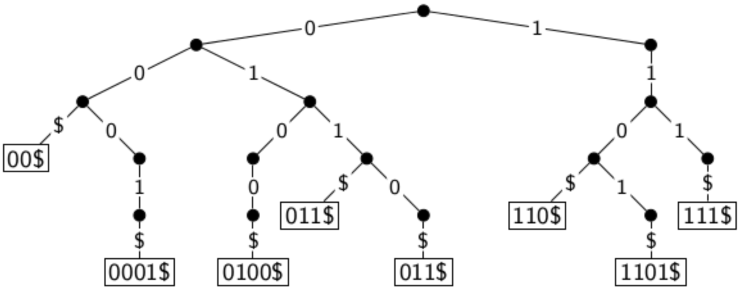


Then items (keys) are stored *only* in the leaf nodes

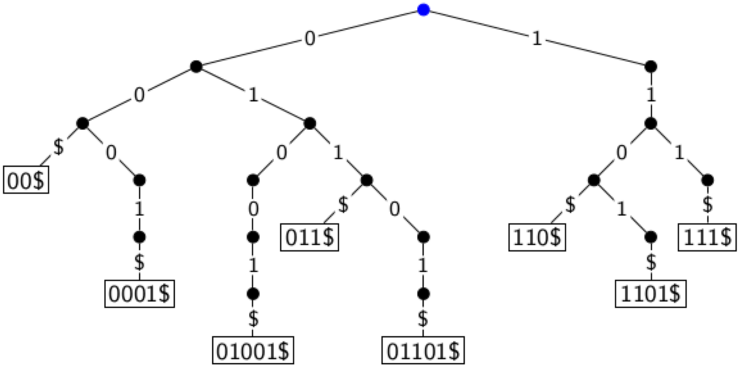
Tries: Search

- start from the root and the most significant bit of x
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- return success if we reach a leaf (it must store x)
- else recurse on the new node and the next bit of x

```
Trie::search(v ← root, d ← 0, x)
v: node of trie; d: level of v, x: word stored as array of chars
    if v is a leaf
        return v
    else
        let v′ be child of v labelled with x[d]
        if there is no such child
            return "not found"
        else Trie::search(v′, d + 1, x)
```

Tries: Search Example

Example: Trie::search(011$)



Tries: Search Example

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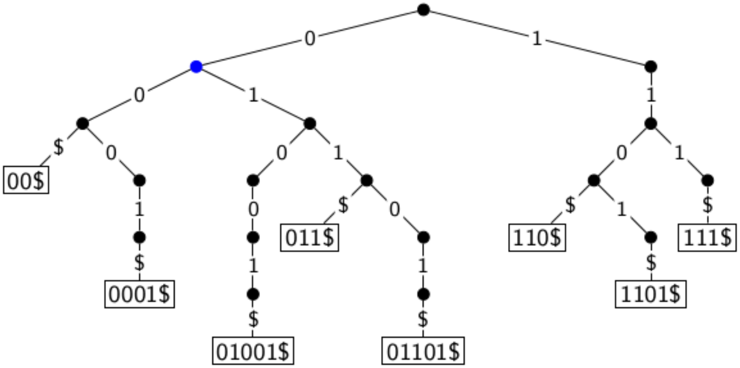

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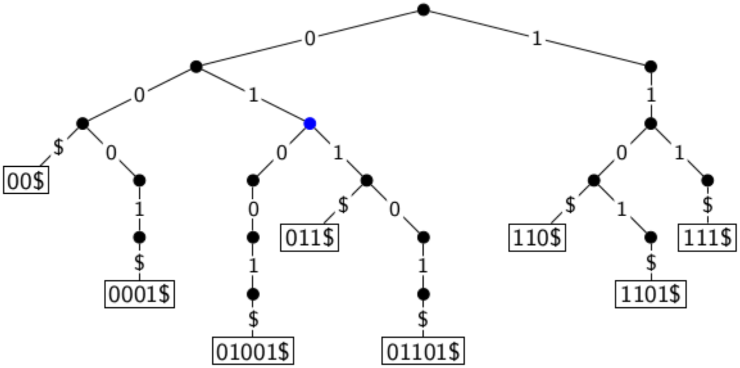
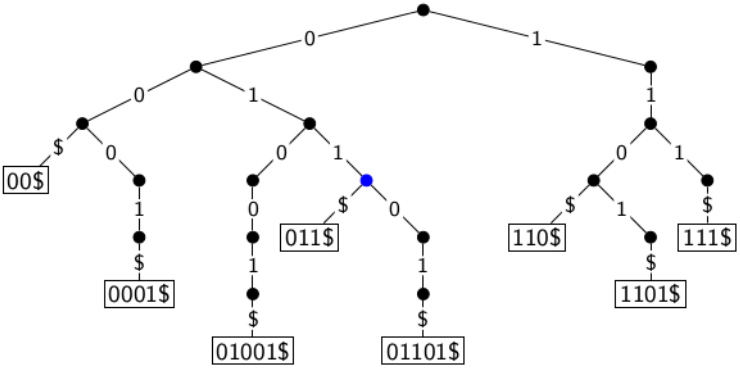


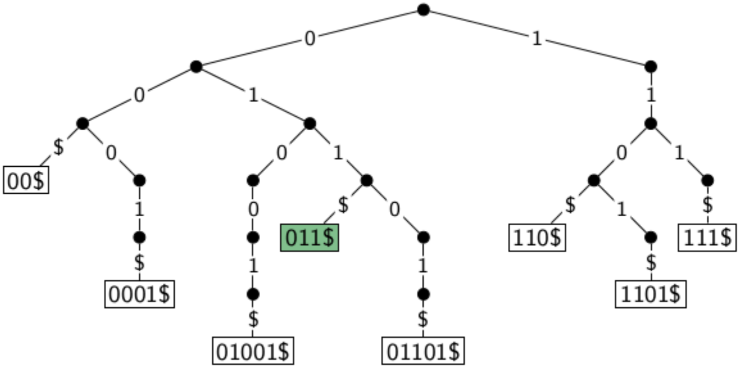
Tries: Search Example

Example: Trie::search(011$)



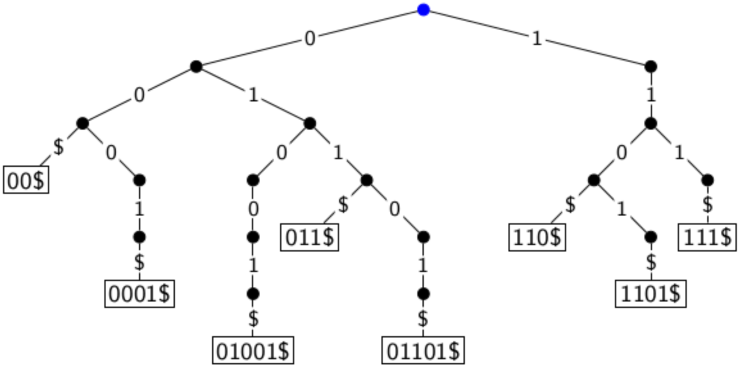
Tries: Search Example

Example: Trie::search(011$) successful



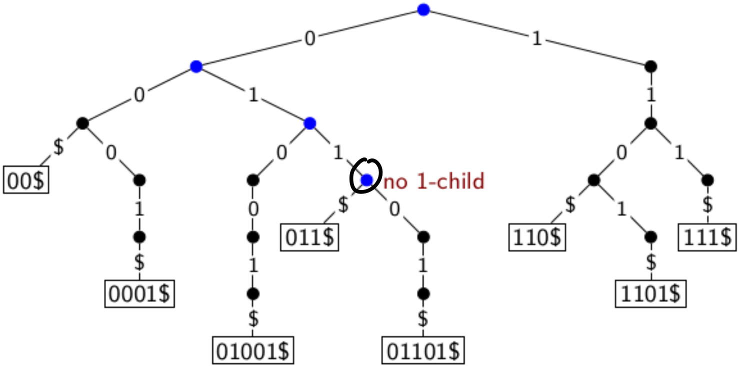
Tries: Search Example

Example: Trie::search(0111$)



Tries: Search Example

Example: Trie::search(0111$) unsuccessful

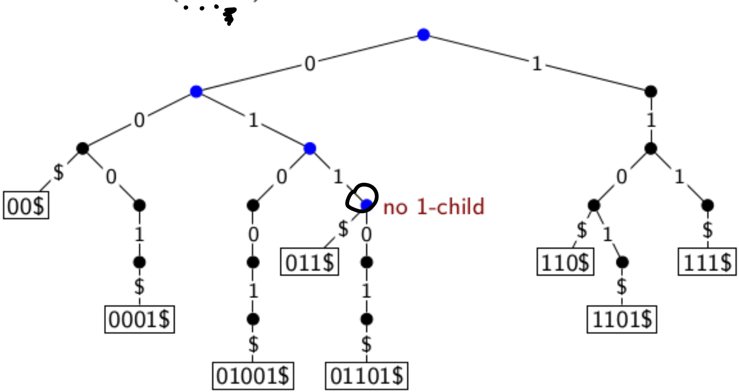


Tries: Insert & Delete

- *Trie::insert(x)*
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child. Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.
- *Trie::delete*(x)
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.
- **Time Complexity** of all operations: $\Theta(|x|)$
  $|x|$: length of binary string x, i.e., the number of bits in x

Tries: Insert Example

Example: *Trie::insert*(0111$)



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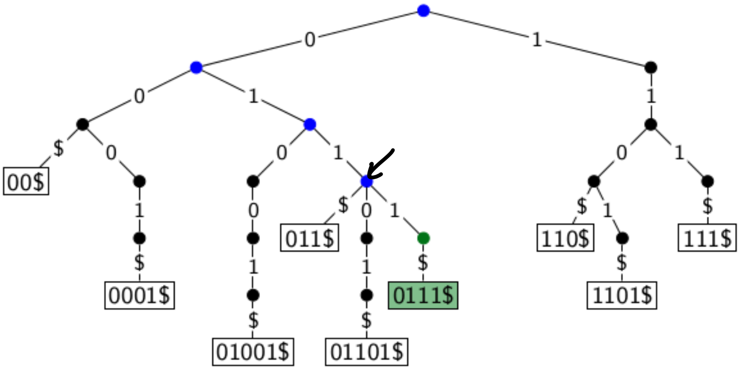
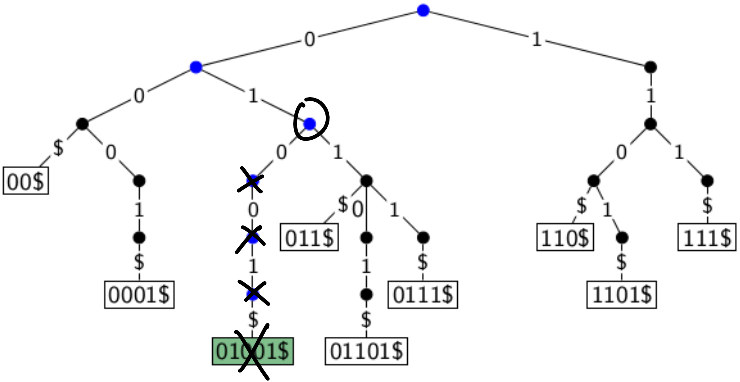


Tries: Delete Example

Example: *Trie::delete*(01001$)



Tries: Delete Example

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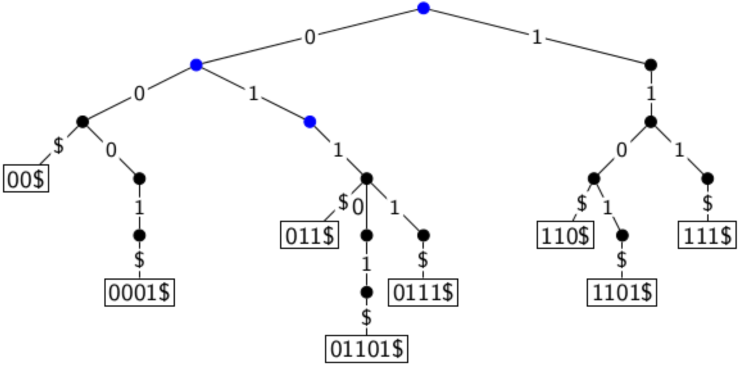
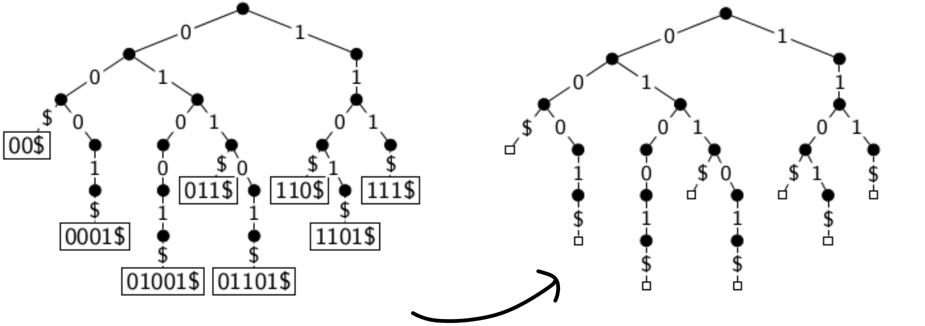


Outline

Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

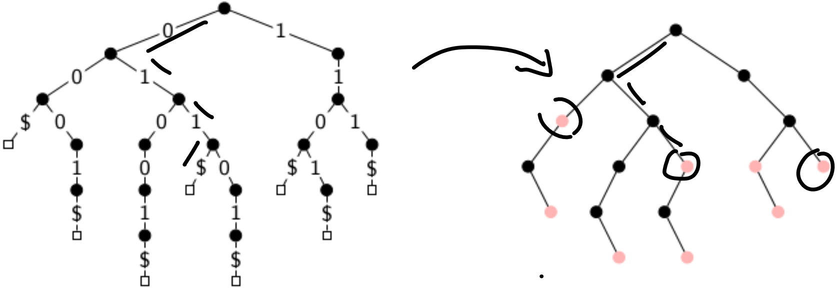
- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.

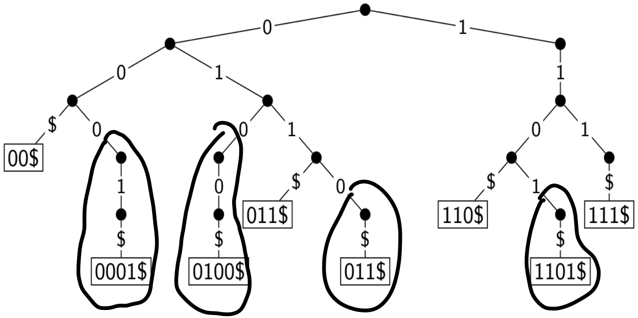


Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys.
  Use a *flag* to indicate such nodes.
- No need for end-of-word character $
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.



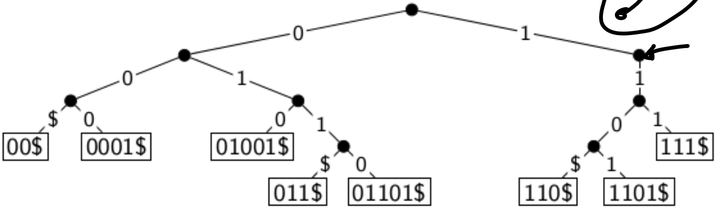


0
1
0
1
1
$
0
0
1
0
1
00$
1
0
$
0
$
1
$
011$
110$
111$
$
$
$
$
0001$
0100$
011$
1101$

Variations 3 of Tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)



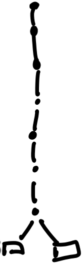
This is in practice the most efficient version of tries, but the operations get a bit more complicated.

Outline

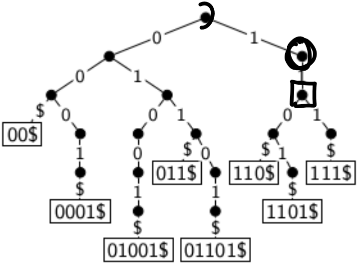
Variation 4 of Tries

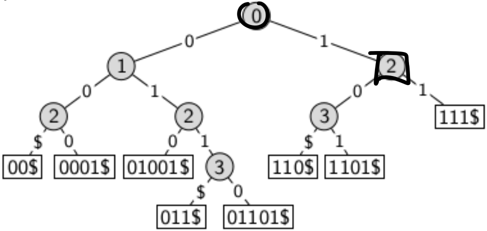
Compressed Trie: compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with n keys has at most $n-1$ internal nodes



→ at most 2n−1 nodes





Also known as **Patricia-Tries**:

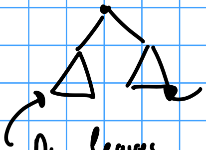
Practical Algorithm to Retrieve Information Coded in Alphanumeric

Proof: by induction on the height.

1) if $h = 0$

$n = 1$, no internal nodes $\to$ OK.

2) Suppose true for $0 \dots h-1$; prove it for a trie of height h.



n_2 leaves
at most $n_2 - 1$ internal nodes.

n_1 leaves
at most $n_1 - 1$ internal nodes

in the whole tree:

- # leaves $= n_1 + n_2$
- # internal nodes = # internal nodes on the left *

  + # internal nodes on the right **

  +1

  $\leq n_1 - 1 + n_2 - 1 + 1 = \underbrace{n_1 + n_2}_{\text{\#leaves}} - 1$

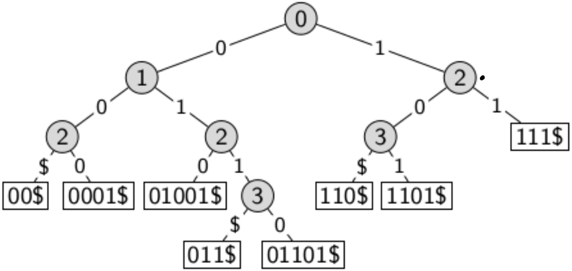
Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- if we reach a leaf, expicitly check whether word stored at leaf is x
- else recurse on the new node and the next bit of x

```
CompressedTrie::search(v ← root, x)
v: node of trie; x: word
    if v is a leaf
        return strcmp(x, v.key)
    d ← index stored at v
    if x has at most d bits
        return "not found"
    v′ ← child of v labelled with x[d]
    if there is no such child
        return "not found"
    CompressedTrie::search(v′, x)
```

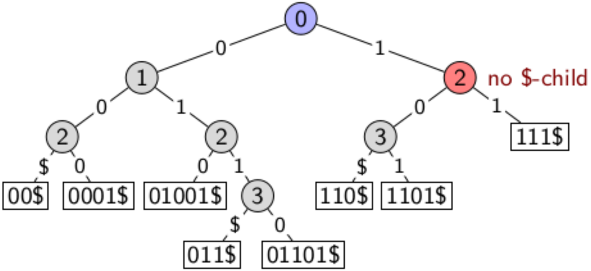
Compressed Tries: Search Example

Example: CompressedTrie::search(10$)



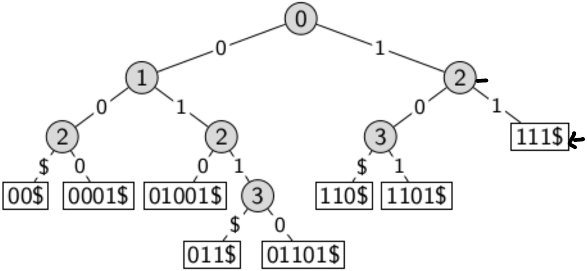
Compressed Tries: Search Example

Example: CompressedTrie::search(10$) unsuccessful



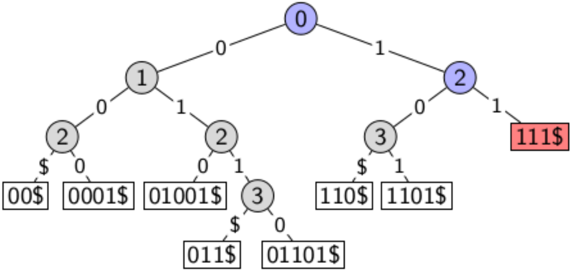
Compressed Tries: Search Example

Example: CompressedTrie::search(101$)



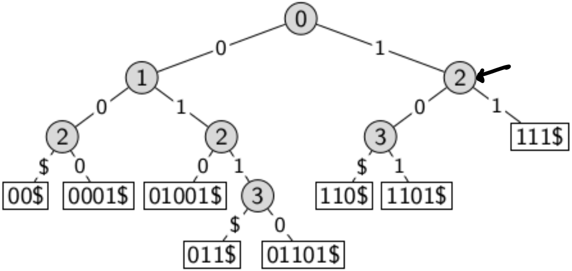
Compressed Tries: Search Example

Example: CompressedTrie::search(101$) unsuccessful



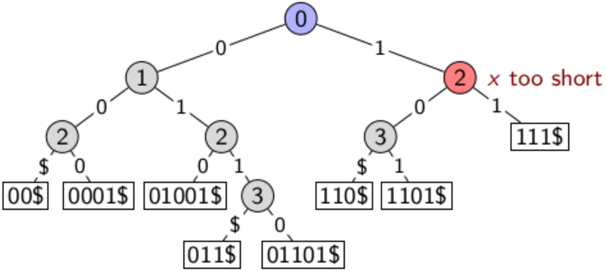
Compressed Tries: Search Example

Example: CompressedTrie::search(1$)



Compressed Tries: Search Example

Example: CompressedTrie::search(1\$) unsuccessful



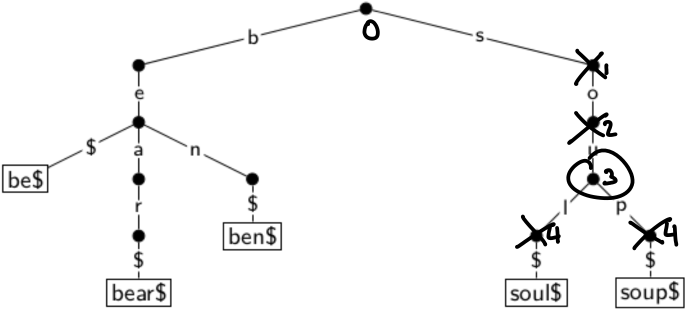
Compressed Tries: Insert & Delete

- *CompressedTrie::delete*(x):
  - Perform *search*(x)
  - Remove the node v that stored x
  - Compress along path to v whenever possible.
- *CompressedTrie::insert*(x):
  - Perform *search*(x)
  - Let v be the node where the search ended.
  - Conceptually simplest approach:
    - Uncompress path from root to v.
    - Insert x as in an uncompressed trie.
    - Compress paths from root to v and from root to x.

    But it can also be done by only adding those nodes that are needed, see the textbook for details.
- All operations take $O(|x|)$ time.

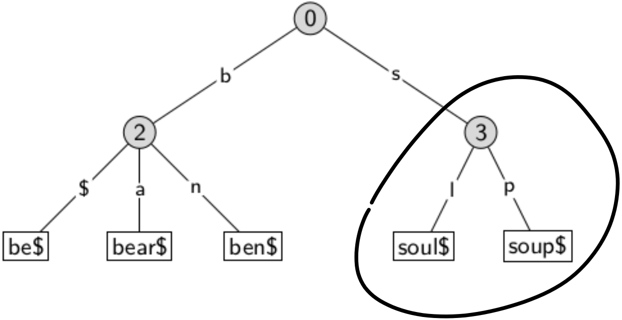
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



Multiway Tries: Summary

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- Run-time $O(|x| \cdot \text{(time to find the appropriate child)})$

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Solution 1: Array of size $|\Sigma| + 1$ for each node.
Complexity: $O(1)$ time to find child, $O(|\Sigma|)$ space per node.

Solution 2: List of children for each node.
Complexity: $O(|\Sigma|)$ time to find child, $O(\#\text{children})$ space.

Solution 3: Dictionary (AVL-tree?) of children for each node.
Complexity: $O(\log(\#\text{children}))$ time, $O(\#\text{children})$ space.

Best in theory, but not worth it in practice unless $|\Sigma|$ is huge.

In practice, use *hashing* (keys are in (typically small) range Σ).

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