CS 240 – Data Structures and Data Management

Module 3: Sorting and Randomized Algorithms

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Based on lecture notes by many previous cs240 instructors

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Outline

- Sorting and Randomized Algorithms
  - QuickSelect
  - Randomized Algorithms
  - QuickSort
  - Lower Bound for Comparison-Based Sorting
  - Non-Comparison-Based Sorting

Outline

- Sorting and Randomized Algorithms
  - QuickSelect
  - Randomized Algorithms
  - QuickSort
  - Lower Bound for Comparison-Based Sorting
  - Non-Comparison-Based Sorting

Selection Problem

- Given array A of n numbers, and $0 \leq k < n$, find the element that would be at position k if A was sorted
  - 'select k'
  - k elements are smaller or equal, $n-1-k$ elements are larger or equal

0	1	2	3	4	5	6	7	8	9
30	60	10	0	50	80	90	20	40	70

select(2)

- Special case: *median finding* $(k = \lfloor \frac{n}{2} \rfloor)$
- Heap-based selection can be done in $\Theta(n + k \log n)$
  - this is $\Theta(n \log n)$ for median finding
    - the same cost as our best sorting algorithms
- **Question**: can we do selection in linear time?
  - yes, with quick-select (average case analysis)
  - subroutines for quick-select also useful for sorting algorithms

Crucial Subroutines

0	1	2	3	$p = 4$	5	6	7	8	9
30	60	10	0	v =50	80	90	20	40	70

- *quick-select* and related algorithm *quick-sort* rely on two subroutines
  - *choose-pivot*(A)
    - return an index p in A
    - use *pivot-value* $v \leftarrow A[p]$ to rearrange the array

0	1	2	3	4	$i = 5$	6	7	8	9
30	10	0	20	40	v =50	60	80	90	70

- *partition* (A, p) rearranges A so that
  - all items in $A\ [0, \ldots, i-1]$ are $\leq v$
  - pivot-value v is in $A[i]$
  - all items in $A\ [i+1, \ldots, n-1]$ are $\geq v$
  - index i is called *pivot-index* i
  - *partition*(A, p) returns *pivot-index* i
    - i is a correct location of v in sorted A
    - if we were interested in select(i), then v would be the answer

Choosing Pivot

- Simplest idea for *choose-pivot*
  - always select rightmost element in array

choose-pivot1(A)
    return A.size() – 1

0	1	2	3	4	5	6	7	8	$p = 9$
30	60	10	0	50	80	90	20	40	$v =$ **70**

- Will consider more sophisticated ideas later

Partition Algorithm

partition(A, p)

A: array of size n, p: integer s.t. $0 \leq p < n$

create empty lists $small, equal$ and $large$

$v \leftarrow A[p]$

for each element x **in** A

if $x < v$ **then** $small.append(x)$

else if $x > v$ **then** $large.append(x)$

else $equal.append(x)$

$i \leftarrow small.size$

$j \leftarrow equal.size$

overwrite $A[0 \ldots i-1]$ by elements in $small$

overwrite $A[i \ldots i+j-1]$ by elements in $equal$

overwrite $A[i+j \ldots n-1]$ by elements in $large$

return *i*

- Easy linear-time implementation using extra (auxiliary) $\Theta(n)$ space
- More challenging: partition *in-place*, i.e. $O(1)$ auxiliary space

Efficient In-Place partition (Hoare)

i = -1 j = 9

30	60	10	0	50	80	90	20	40	v=70

i = 5 j = 8

30	60	10	0	50	**80**	**90**	**20**	**40**	v=70

i = 5 j = 8

30	60	10	0	50	**40**	90	20	**80**	v=70

i = 6 j = 7

30	60	10	0	50	40	**90**	**20**	80	v=70

i = 6 j = 7

30	60	10	0	50	40	**20**	**90**	80	v=70

j = 6 i = 7

almost done, just swap with pivot v

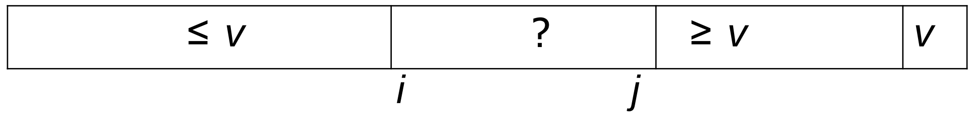
30	60	10	0	50	40	20	90	80	v=70

j = 6 **i = 7**

30	60	10	0	50	40	20	v=70	80	90

Efficient In-Place partition (Hoare)

- **Idea Summary**: Keep swapping the outer-most wrongly-positioned pairs



- One possible implementation

$$\textbf{do } i \leftarrow i+1 \textbf{ while } i < n \textbf{ and } A[i] \leq v$$
$$\textbf{do } j \leftarrow j-1 \textbf{ while } j > 0 \textbf{ and } A[j] \geq v$$

- More efficient (for quickselect and quicksort) when many repeating elements

$$\textbf{do } i \leftarrow i+1 \textbf{ while } i < n \textbf{ and } A[i] < v$$
$$\textbf{do } j \leftarrow j-1 \textbf{ while } j > 0 \textbf{ and } A[j] > v$$

- Can simplify the loop bounds

$$\textbf{do } i \leftarrow i+1 \textbf{ while } A[i] < v$$
$$\textbf{do } j \leftarrow j-1 \textbf{ while } j \geq i \textbf{ and } A[j] > v$$

Efficient In-Place partition (Hoare)

```
partition (A, p)
    A: array of size n
    p: integer s.t. 0 ≤ p < n
        swap(A[n − 1], A[p])
        i ← −1,      j ← n − 1,      v ← A[n − 1]
        loop
            do i ← i + 1 while A[i] < v
            do j ← j − 1 while j ≥ i and A[j] > v
            if i ≥ j then break
            else swap(A[i], A[j])
        end loop
        swap(A[n − 1], A[i])
        return i
```

- Running time is $\Theta(n)$

Efficient In-Place partition (Hoare)

partition (A, p)

A: array of size n

p: integer s.t. $0 \leq p < n$

swap$(A[n-1],\ A[p])$

$i \leftarrow -1, \qquad j \leftarrow n-1, \qquad v \leftarrow A[n-1]$

loop

do $i \leftarrow i+1$ **while** $A[i] < v$

do $j \leftarrow j-1$ **while** $j \geq i$ **and** $A[j] > v$

if $i \geq j$ **then break**

else *swap*$(A[i],\ A[j])$

end loop

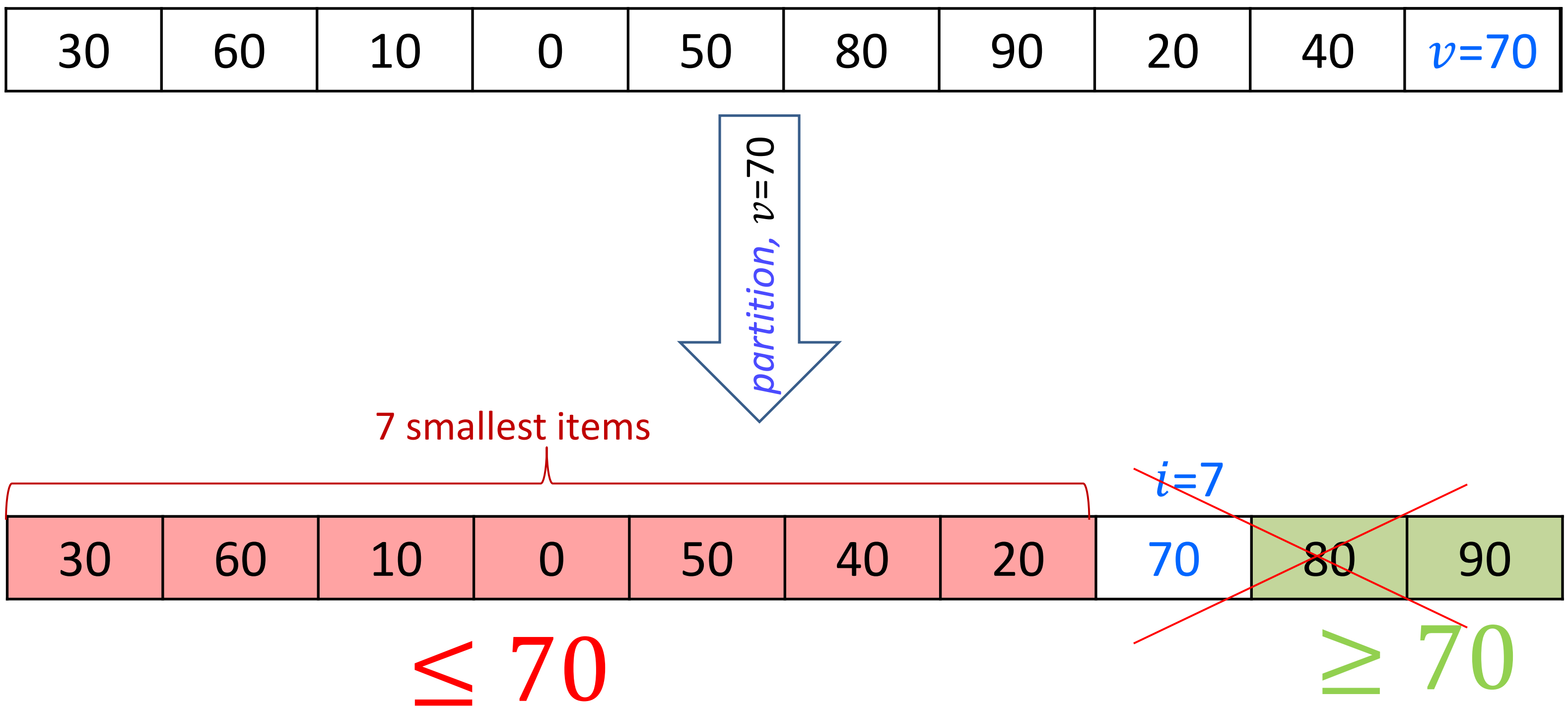
swap$(A[n-1],\ A[i])$

return i

- Running time is $\Theta(n)$

Quick Select Algorithm

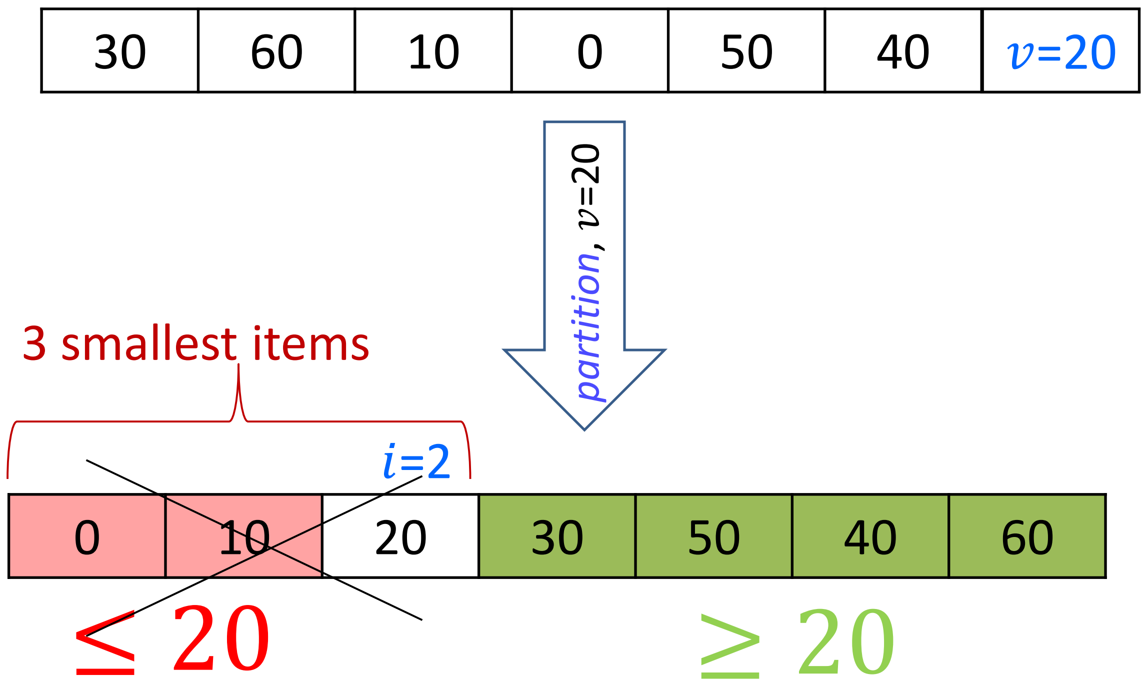
- Find item that would be in $A[k]$ if A was sorted
- Similar to quick-sort, but recurse only on one side ("quick-sort with pruning")
- Example: select($k = 4$)
  - [the correct answer is 40 in this case]



- $i > k$, search recursively in the left side to select k

Quick Select Algorithm

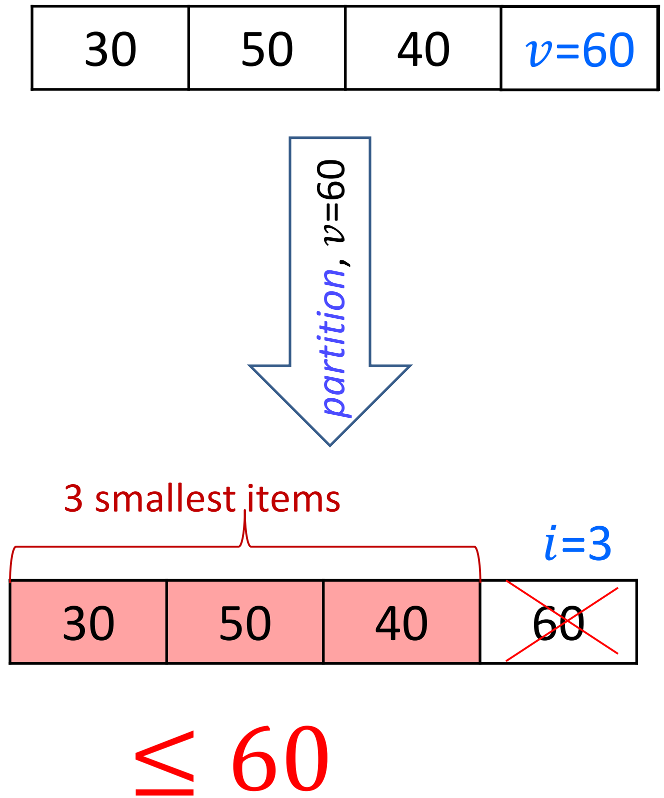
- Example continued: select($k = 4$)



- $i < k$, search recursively on the right, select $\boldsymbol{k - (i + 1)}$
  - $k = 1$ in our example

Quick Select Algorithm

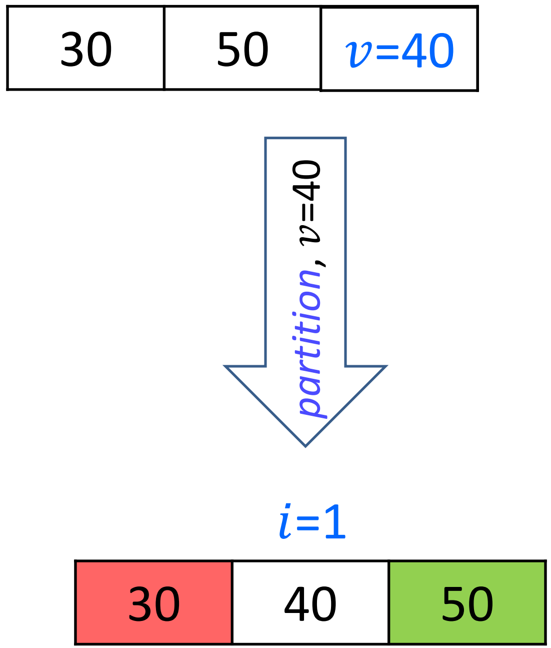
- Example continued: select($k = 1$)



- $i > k,$ search on the left to select k

Quick Select Algorithm

- Example continued: select($k = 1$)



- $i = k$, found our item, done!
- In our example, we got to subarray of size 3
- Often stop much sooner than that
  - running time?

QuickSelect Algorithm

quick-select1(A, k)
A: array of size n, k: integer s.t. $0 \le k < n$
  $p \leftarrow$ *choose-pivot1*(A)
  $i \leftarrow$ *partition*(A, p)
  if $i = k$ **then**
    return $A[i]$
  else if $i > k$ **then**
    return *quick-select1*$(A[0, 1, \ldots, i-1],\ k)$
  else if $i < k$ **then**
    return *quick-select1*$(A[i+1, \ldots, n-1],\ k-(i+1))$

- **Best case**
  - first chosen pivot could have pivot-index k
  - no recursive calls, total cost $\Theta(n)$
- **Worst case**: recurrence equation $T(n) = \begin{cases} cn + T(n-1) & n > 1 \\ c & n = 1 \end{cases}$

QuickSelect Algorithm

- **Worst case**: recurrence equation $T(n) = \begin{cases} cn + T(n-1) & n > 1 \\ c & n = 1 \end{cases}$
- Solution: repeatedly expand until we see a pattern forming

$$T(n) = cn + T(n-1)$$

$$T(n-1) = c(n-1) + T(n-2)$$

$$T(n) = cn + c(n-1) + T(n-2)$$ after 1 expansion

$$T(n-2) = c(n-2) + T(n-3)$$

$$T(n) = cn + c(n-1) + c(n-2) + T(n-3)$$ after 2 expansions

- After i expansions

$$T(n) = cn + c(n-1) + c(n-2) + \cdots + c(n-i) + T(n-(i+1))$$

- Stop expanding when get to base case $T(n-(i+1)) = T(1)$
- Happens when $n-(i+1) = 1$, or, rewriting, $i = n-2$
- Thus $T(n) = cn + c(n-1) + c(n-2) + \cdots + c \cdot 2 + T(1)$

$$= cn + c(n-1) + c(n-2) + \cdots + c \cdot 2 + c$$

$$= c(n + (n-1) + \cdots + 2 + 1) \in \Theta(n^2)$$

Average-Case Analysis of *quick-select1*

$$T^{avr}(n) = \frac{1}{\text{\# instances of size } n} \sum_{I:size(I)=n} T(I)$$

infinitely many

- Need to make some assumptions
- First assumption
  - all input numbers are distinct
  - this assumption is just for simpler analysis, can prove the same thing without this assumption

Average-Case Analysis of *quick-select1*

- QuickSelect is *comparison-based*
  - only cares if $A[i] < A[j]$ for i, j
  - does not care what the actual values of $A[i]$, $A[j]$ are

I_1

30	60	0	10

I_2

20	50	10	15

- QuickSelect makes exactly the same sequences of steps on I_1 and I_2
  - therefore $T(I_1) = T(I_2)$
- Any comparison based algorithm has exactly the same running time for arrays that have the same relative order of elements, regardless of actual array values
- Second assumption: we are sorting integers $0, \ldots, n-1$
  - now there are $n!$ possible input instances I
  - more formal proof uses *sorting permutations*
    - permutation π for which $A[\pi(0)] \leq A[\pi(1)] \leq \ldots \leq A[\pi(n-1)]$
    - for I_1 (and I_2) sorting permutation is $\pi = (2, 3, 0, 1)$
    - assume *each sorting permutation is equally likely*
    - $n!$ possible permutations

Average-Case Analysis of *quick-select1*

$$T^{avr}(n) = \frac{1}{\text{\# instances of size } n} \sum_{I:size(I)=n} T(I)$$

- Example for $n = 3$, using all the assumptions

$$T^{avr}(3) = \frac{1}{3!}(T(\{0,1,2\}) + T(\{0,2,1\}) + T(\{1,0,2\}) + T(\{1,2,0\}) + T(\{2,0,1\}) + T(\{2,1,0\}))$$

Average-Case Analysis of *quick-select1*

- Recall that pivot is last array element
- Pivot index is equal to pivot value due to assuming we sort $0, \ldots, n-1$

	0	1	2	3
A	2	3	0	v=1

for v=1, pivot index $i = 1$

- Partition sum over different pivot indexes

$$T^{avr}(n) = \frac{1}{n!} \sum_{I:Size(I)=n} T(I) \qquad = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\substack{I:size(I)=n, \\ pivot\ is\ i}} T(I)$$

- Example for $n = 3$

$$T^{avr}(3) = \frac{1}{3!}(T(\{0,1,2\}) + T(\{0,2,1\}) + T(\{1,0,2\}) + T(\{1,2,0\}) + T(\{2,0,1\}) + T(\{2,1,0\}))$$

$$\begin{aligned} T^{avr}(3) = \tfrac{1}{3!}\ &(T(\{1,2,\mathbf{0}\}) + T(\{2,1,\mathbf{0}\})) + \\ &(T(\{0,2,\mathbf{1}\}) + T(\{2,0,\mathbf{1}\})) + \\ &(T(\{0,1,\mathbf{2}\}) + T(\{1,0,\mathbf{2}\})) \end{aligned}$$

Average-Case Analysis of *quick-select1*

- Partition sum over different pivots $T^{avr}(n) = \frac{1}{n!}\sum_{i=0}^{n-1} \sum_{\substack{I:size(I)=n, \\ pivot\ is\ i}} T(I)$

- There are $(n-1)!$ input instances I with pivot index i

A					$\boldsymbol{i}$
	choice of $n-1$ items: anything but i	choice of $n-2$ items: anything but i and $A[0]$	…	'choice' of 1 items	no choice

- One can show (will only hint at the proof with example for $n = 4, i = 1$)

$$\sum_{\substack{I:size(I)=n, \\ pivot\ is\ \boldsymbol{i}}} T(I) \le (n-1)!\, cn + (n-1)!\, max\{T^{avr}(i)\,, T^{avr}(n-i-1)\,\}$$

- Therefore $T^{avr}(n) \le cn + \frac{1}{n}\sum_{i=0}^{n-1} max\{T^{avr}(i)\,, T^{avr}(n-i-1)\,\}$

Average-Case Analysis of *quick-select1*

- Let $n = 4, i = 1$

$$\sum_{\substack{I:size(I)=\mathbf{4},\\ pivot\ is\ \mathbf{1}}} T(I) = T(\{0,2,3,\mathbf{1}\}) + T(\{0,3,2,\mathbf{1}\}) + T(\{2,0,3,\mathbf{1}\}) + T(\{2,3,0,\mathbf{1}\}) + T(\{3,0,2,\mathbf{1}\}) + T(\{3,2,0,\mathbf{1}\})$$

- Total work is proportional to comparisons, will count comparisons

comparisons to partition:	3	3	3	3	3	3	Total: $3(3)!$
instances	$\{0,2,3,\mathbf{1}\}$	$\{0,3,2,\mathbf{1}\}$	$\{2,0,3,\mathbf{1}\}$	$\{2,3,0,\mathbf{1}\}$	$\{3,0,2,\mathbf{1}\}$	$\{3,2,0,\mathbf{1}\}$	
partitions (assume stable order)	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{3,2\}$	

Average-Case Analysis of *quick-select1*

- Let $n = 4, i = 1$

$$\sum_{\substack{I:size(I)=\mathbf{4},\\ pivot\ is\ \mathbf{1}}} T(I) = \begin{array}{l} T(\{0,2,3,\mathbf{1}\}) + T(\{0,3,2,\mathbf{1}\}) \\ + T(\{2,0,3,\mathbf{1}\}) + T(\{2,3,0,\mathbf{1}\}) \\ + T(\{3,0,2,\mathbf{1}\}) + T(\{3,2,0,\mathbf{1}\}) \end{array}$$

- Total work is proportional to comparisons, will count comparisons

comparisons to partition:	3	3	3	3	3	3	Total: $3(3)!$
instances	$\{0,2,3,\mathbf{1}\}$	$\{0,3,2,\mathbf{1}\}$	$\{2,0,3,\mathbf{1}\}$	$\{2,3,0,\mathbf{1}\}$	$\{3,0,2,\mathbf{1}\}$	$\{3,2,0,\mathbf{1}\}$	
partitions	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{3,2\}$	

Case 1: $k > i$

$$T(\{2,3\}) + T(\{3,2\}) + T(\{2,3\}) + T(\{2,3\}) + T(\{3,2\}) + T(\{3,2\})$$

$$= T(\{0,1\}) + T(\{1,0\}) + T(\{0,1\}) + T(\{0,1\}) + T(\{1,0\}) + T(\{1,0\})$$

since only relative order matters

swap

Average-Case Analysis of *quick-select1*

- Let $n = 4, i = 1$

$$\sum_{\substack{I:size(I)=4,\\ pivot\ is\ 1}} T(I) = \begin{aligned} &T(\{0,2,3,\mathbf{1}\}) + T(\{0,3,2,\mathbf{1}\}) \\ &+ T(\{2,0,3,\mathbf{1}\}) + T(\{2,3,0,\mathbf{1}\}) \\ &+ T(\{3,0,2,\mathbf{1}\}) + T(\{3,2,0,\mathbf{1}\}) \end{aligned}$$

- Total work is proportional to comparisons, will count comparisons

comparisons to partition:	3	3	3	3	3	3	Total: $3(3)!$
instances	$\{0,2,3,\mathbf{1}\}$	$\{0,3,2,\mathbf{1}\}$	$\{2,0,3,\mathbf{1}\}$	$\{2,3,0,\mathbf{1}\}$	$\{3,0,2,\mathbf{1}\}$	$\{3,2,0,\mathbf{1}\}$	
partitions	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{3,2\}$	

Case 1: $k > i$

$$T(\{2,3\}) + T(\{3,2\}) + T(\{2,3\}) + T(\{2,3\}) + T(\{3,2\}) + T(\{3,2\})$$

$$= \underbrace{T(\{0,1\}) + T(\{1,0\})}_{2!\,T^{avr}(2)} + \underbrace{T(\{0,1\}) + T(\{1,0\})}_{2!\,T^{avr}(2)} + \underbrace{T(\{0,1\}) + T(\{1,0\})}_{2!\,T^{avr}(2)}$$

since only relative order matters

Total recursive comparisons $\dfrac{3!}{2!}\, 2!\,T^{avr}(2) = 3!\,T^{avr}(2)$

Average-Case Analysis of *quick-select1*

- Let $n = 4, i = 1$

$$\sum_{\substack{I:size(I)=4,\\ pivot\ is\ 1}} T(I) = T(\{0,2,3,1\}) + T(\{0,3,2,1\}) + T(\{2,0,3,1\}) + T(\{2,3,0,1\}) + T(\{3,0,2,1\}) + T(\{3,2,0,1\})$$

- Total work is proportional to comparisons, will count comparisons

comparisons to partition:	3	3	3	3	3	3	Total: $3(3)!$
instances	$\{0,2,3,1\}$	$\{0,3,2,1\}$	$\{2,0,3,1\}$	$\{2,3,0,1\}$	$\{3,0,2,1\}$	$\{3,2,0,1\}$	
partitions	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{3,2\}$	

Case 2: $k < i$

$$\underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)}$$

Total recursive comparisons $\frac{3!}{1!}1!\,T^{avr}(1) = 3!\,T^{avr}(1)$

[Case 1, total recursive comparisons: $= 3!\,T^{avr}(2)$]

Combining both cases, total recursive comparisons : $\leq 3!\max\{T^{avr}(1), T^{avr}(2)\}$

Adding comparisons to partition: $\leq 3(3)! + 3!\max\{T^{avr}(1), T^{avr}(2)\}$

Average-Case Analysis of *quick-select1*

- Let $n = 4, i = 1$

$$\sum_{\substack{I:size(I)=4,\\ pivot\ is\ \mathbf{1}}} T(I) = T(\{0,2,3,\mathbf{1}\}) + T(\{0,3,2,\mathbf{1}\}) + T(\{2,0,3,\mathbf{1}\}) + T(\{2,3,0,\mathbf{1}\}) + T(\{3,0,2,\mathbf{1}\}) + T(\{3,2,0,\mathbf{1}\})$$

- Total work is proportional to comparisons, will count comparisons

comparisons to partition:	3	3	3	3	3	3	Total: $3(4-1)!$
instances	$\{0,2,3,\mathbf{1}\}$	$\{0,3,2,\mathbf{1}\}$	$\{2,0,3,\mathbf{1}\}$	$\{2,3,0,\mathbf{1}\}$	$\{3,0,2,\mathbf{1}\}$	$\{3,2,0,\mathbf{1}\}$	
partitions	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{2,3\}$	$\{0\}$ $\{3,2\}$	$\{0\}$ $\{3,2\}$	

Case 2: $k < i$

$$\underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)} + \underbrace{T(\{0\})}_{1!\,T^{avr}(1)}$$

Total recursive comparisons $\frac{3!}{}1!\,T^{avr}(1) = 3!\,T^{avr}(1)$

$$\sum_{\substack{I:size(I)=n,\\ pivot\ is\ \boldsymbol{i}}} T(I) \le (n-1)!\,cn + (n-1)!\,max\{T^{avr}(i)\,, T^{avr}(n-i-1)\,\}$$

Comb

Adding comparisons to partition: $\le 3(3)! + 3!\max\{T^{avr}(1), T^{avr}(2)\}$

Average-Case Analysis of *quick-select1*

$$T(n) \le c \cdot n + \frac{1}{n}\sum_{i=0}^{n-1} max\{T(i), T(n-i-1)\}$$

Theorem: $T(n) \in O(n)$

Proof:

- will prove $T(n) \le 4cn$ by induction on n
- base case, $n = 1$: $T(1) = c \le 4c \cdot 1$
- induction hypothesis: assume $T(m) \le 4cm$ for all $m < n$
- need to show $T(n) \le 4cn$

induction hypothesis applies to each one of these

$$T(n) \le c \cdot n + \frac{1}{n}\sum_{i=0}^{n-1} max\{T(i), T(n-i-1)\}$$

$$\le c \cdot n + \frac{1}{n}\sum_{i=0}^{n-1} max\{4ci, 4c(n-i-1)\}$$

$$\le c \cdot n + \frac{4c}{n}\sum_{i=0}^{n-1} max\{i, n-i-1\}$$

Average-Case Analysis of *quick-select1*

exactly what we need for the proof

Proof: (cont.) $T(n) \le c \cdot n + \frac{4c}{n} \sum_{i=0}^{n-1} max\{i, n-i-1\} \le c \cdot n + \frac{4c}{n} \cdot \frac{3}{4} n^2 = 4cn$

$$\sum_{i=0}^{n-1} max\{i, n-i-1\} = \sum_{i=0}^{\frac{n}{2}-1} max\{i, n-i-1\} + \sum_{i=\frac{n}{2}}^{n-1} max\{i, n-i-1\}$$

$$= max\{0, \underline{n-1}\} + max\{1, \underline{n-2}\} + max\{2, \underline{n-3}\} + \cdots + max\left\{\frac{n}{2}-1, \underline{\frac{n}{2}}\right\}$$

$$+ max\left\{\underline{\frac{n}{2}}, \frac{n}{2}-1\right\} + max\left\{\underline{\frac{n}{2}+1}, \frac{n}{2}-2\right\} + \cdots + max\{\underline{n-1}, 0\}$$

$$= \underbrace{(n-1) + (n-2) + \cdots + \frac{n}{2}}_{\left(\frac{3n}{2}-1\right)\frac{n}{4}} + \underbrace{\frac{n}{2} + \left(\frac{n}{2}+1\right) + \cdots (n-1)}_{\left(\frac{3n}{2}-1\right)\frac{n}{4}} = \left(\frac{3n}{2}-1\right)\frac{n}{2}$$

$$\le \frac{3}{4} n^2$$

Average-Case Analysis of *quick-select1*

- Proved average case time $T(n)$ is $\mathrm{O}(n)$
- Average case is also $\Omega(n)$ since have to perform *partition*(A, p)
- Therefore average case is $T(n)$ is $\Theta(n)$

Outline

- Sorting and Randomized Algorithms
  - QuickSelect
  - Randomized Algorithms
  - QuickSort
  - Lower Bound for Comparison-Based Sorting
  - Non-Comparison-Based Sorting

Randomized Algorithms

- A *randomized algorithm* is one which relies on some random numbers in addition to the input
- The cost will depend on both the input and the random numbers used
- **Goal**
  - shift the dependency of run-time from what we cannot control (the input), to what we can control (random numbers)
  - no more bad instances, just unlucky numbers
    - if running time is long on some instance, it's because we generated unlucky random numbers, not because of the instance itself
- Side note
  - computers cannot generate truly random numbers
  - we assume there is a pseudo-random number generator (PRNG), a deterministic program that uses an initial value or *seed* to generate a sequence of seemingly random numbers
  - quality of randomized algorithm depends on the quality of the PRNG

Expected Running Time

- How do we measure the running time of a randomized algorithm?
  - it depends on the input I and on R, the sequence of random numbers an algorithm choses during execution
- Define $T(I, R)$ to be running time of randomized algorithm for instance I and R
- The *expected running time* $T^{exp}(I)$ for instance I is expected value for $T(I, R)$

$$T^{exp}(I) = \boldsymbol{E}[T(I,R)] = \sum_{\substack{\text{all possible} \\ \text{sequences } R}} T(I,R) \cdot \Pr[R]$$

- *Worst-case expected running time*

$$T^{exp}(n) = \max_{\{I:size(I)=n\}} T^{exp}(I)$$

- *Average-case expected running time*

$$T^{exp}(n) = \frac{1}{|I: size(I) = n|} \sum_{I:Size(I)=n} T^{exp}(I)$$

- Usually design A so that all instances of size n have the same expected run time
- Thus the average and worst case expected run times are the same, and we just compute the worst case expected time

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- Usually design A so that all instances of size n have the same expected run time
- Thus average and worst case expected run times are usually the same
    - just compute the worst case expected time
- Sometimes we also want to know the running time if we got really unlucky with the random numbers R we generate during the execution, or, formally

$$\max_{R} \max_{\{I:size(I)=n\}} T(I,R)$$

Randomized QuickSelect: Shuffle

- **Goal**: create a randomized version of *QuickSelect* for which all input has the same expected run-time
- **First idea**: first randomly permute input using *shuffle* and then run selection algorithm

```
shuffle(A)
A : array of size n
    for i ⟵ 0 to n − 1 do
        swap(A[i], A[random(i + 1)])
```

- $random(n)$ returns an integer uniformly sampled from $\{0, 1, 2, \ldots, n-1\}$
- can show that expected running time is $\Theta(n)$, the same as average running time

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- $random(n)$ returns an integer uniformly sampled from $\{0, 1, 2, \dots, n-1\}$
- can show that expected running time is $\Theta(n)$, the same as average running time
- if we get very unlucky with random numbers, we could get a sorted or almost sorted array after shuffle, resulting in $O(n^2)$ performance for selection algorithm
  - probability of this happening is almost zero
- whereas the user is quite likely to give instance which is sorted or almost sorted to the selection algorithm
  - probability is far from zero, humans often produce almost sorted data

Randomized QuickSelect: Random Pivot

- **Second idea**: select a random pivot from $\{0,1,2,\ldots,n-1\}$

```
choose-pivot2(A)
    return random(A.size())
```

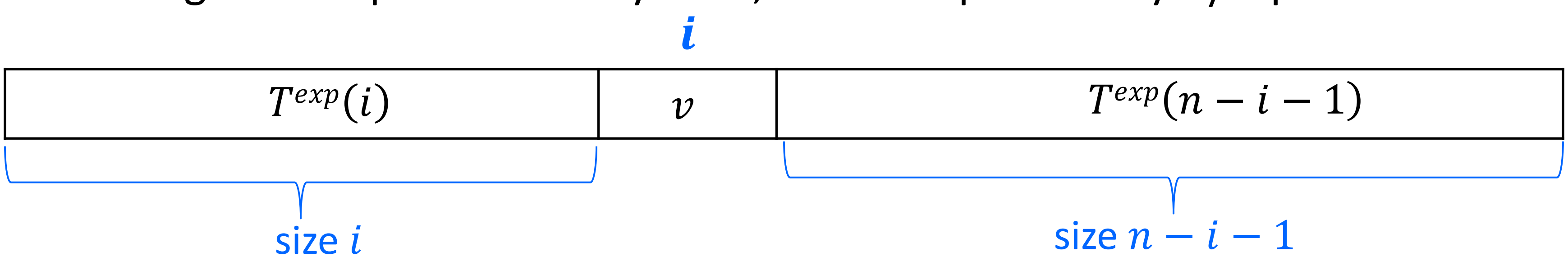
- Simpler and more efficient than shuffling the array
- Usually fastest in practice
- Expected running time is again $\Theta(n)$

Efficiency of Randomized QuickSelect

quick-select2(A, k)
  $p \longleftarrow$ *choose-pivot2*(A)
  "the rest"

choose-pivot2(A)
  return $random(A.size())$

- Assume all elements of A are distinct
- Select pivot with equal probability at each recursive call, and independently from other recursive calls
  - $P(\text{pivot has index } i) = \frac{1}{n}$ for any instance of size n
- $T^{exp}(I)$ depends only on the size of I, not the contents of I
- Let $T^{exp}(n)$ be expected time on an instance of size n
- Running time to partition array is cn, and with probability $1/n$ pivot-index is i



running time if pivot index is $i \leq c \cdot n + max\{T^{exp}(i), T^{exp}(n-i-1)\}$

Efficiency of Randomized QuickSelect

running time if pivot-index is $i \leq c \cdot n + max\{T^{exp}(i), T^{exp}(n-i-1)\}$

- Taking expectation over pivot index i

$$T^{exp}(n) = \sum_{i=0}^{n-1} (running\ time\ if\ pivot\ index\ is\ i) P(\text{index of pivot is } i)$$

$$\leq \sum_{i=0}^{n-1} (cn + max\{T^{exp}(i), Texp(n-i-1\}) \frac{1}{n}$$

$$\leq cn + \sum_{i=0}^{n-1} \frac{1}{n} max\{T^{exp}(i), T^{exp}(n-i-1)\}$$

- Same recurrence as for non-randomized average case
- Resolves to $\Theta(n)$ expected time on instance of size n
- Side note
    - there is selection algorithm “Median of Medians” (cs341) that has worst-case running time $O(n)$
        - uses double recursion
        - slower in practice

QuickSelect: Badly Designed Randomization

choose-random-pivot-badly(A)
    if $A.size \geq 3$ **return** $random(3)$
    else return 0

$$T^{exp}(n) = \max_{\{I: size(I)=n\}} T^{exp}(I)$$

- Worst instance is sorted array $I_n = \{0, 1, \ldots, n-1\}$
- $T^{exp}(I_n) = \begin{cases} cn + \frac{1}{3}T^{exp}(I_{n-1}) + \frac{1}{3}T^{exp}(I_{n-2}) + \frac{1}{3}T^{exp}(I_{n-3}) & \text{if } n \geq 3 \\ c & \text{if } n < 3 \end{cases}$
- $T^{exp}(I_n) \geq cn + T(I_{n-3})$ if $n \geq 3$
- Resolves to $\Theta(n^2)$
- Worst case expected time is $\Theta(n^2)$

Outline

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QuickSort

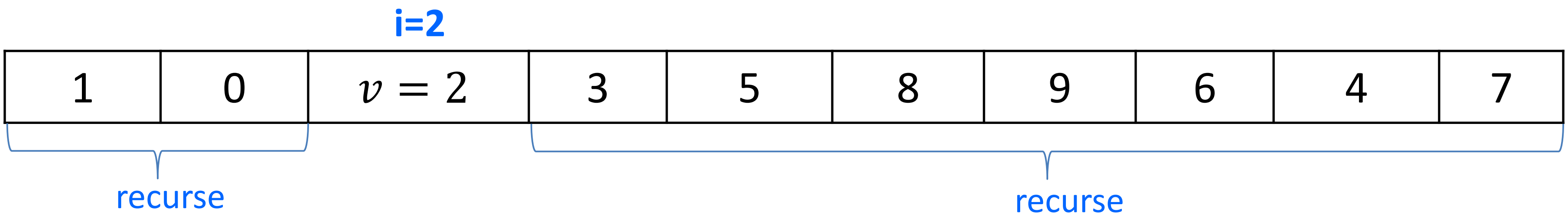
- Hoare developed *partition* and *quick-select* in 1960
- He also used them to *sort* based on partitioning

```
quick-sort1(A)
    Input: array A of size n
        if n ≤ 1 then return
        p ← choose-pivot1(A)
        i ← partition (A, p)
        quick-sort1(A[0, 1, …, i − 1])
        quick-sort1(A[i + 1, …, n − 1])
```

- Let $T(n)$ to be the runtime on size n array
- If we know pivot-index i, then $T(n) = cn + T(i) + T(n-i-1)$
- Worst case $T(n) = T(n-1) + cn$
  - recurrence solved in the same way as quick−select1, $\Theta(n^2)$
- Best case $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + cn$
  - solved in the same way as merge-sort, $\Theta(n \log n)$

Average-case analysis of quick-sort1

- Make the same assumptions as for quick-select1
- Deriving recurrence equation is similar to quick-select1, but recurse on both sides



- Using the same approach as for quick-select1, average running time is

$$T(n) = \frac{1}{n}\sum_{i=0}^{n-1}\big(cn + T(i) + T(n-i-1)\big), \qquad n \geq 2$$

- Running time is proportional to the number of comparisons
- Recurrence for counting comparisons

$$T(n) = \frac{1}{n}\sum_{i=0}^{n-1}\big(n + T(i) + T(n-i-1)\big), \qquad n \geq 2$$

Average-case analysis of quick-sort1

- First let us get a simpler recursive expression for $T(n)$

$$T(n) = \frac{1}{n}\sum_{i=0}^{n-1}\left(n + T(i) + T(n-i-1)\right)$$

$$= n + \frac{1}{n}\sum_{i=0}^{n-1} T(i) + \frac{1}{n}\sum_{i=0}^{n-1} T(n-i-1)$$

$T(0) + T(1) + \cdots + T(n-1)$ $\qquad$ $T(n-1) + T(n-2) + \cdots + T(0)$

$$= n + \frac{2}{n}\sum_{i=0}^{n-1} T(i)$$

- Thus $\quad T(n) = n + \frac{2}{n}\sum_{i=0}^{n-1} T(i)$

Average-case analysis of quick-sort1

$$T(n) = n + \frac{2}{n}\sum_{i=0}^{n-1} T(i) \text{ is } \Theta(n \log n)$$

Proof

Multiply by n: $\quad nT(n) = n^2 + 2\sum_{i=0}^{n-1} T(i)$

Plug in $n-1$: $\quad (n-1)T(n-1) = (n-1)^2 + 2\sum_{i=0}^{n-2} T(i)$

Subtract: $\quad nT(n) - (n-1)T(n-1) = 2n - 1 + 2T(n-1)$

Rearrange : $\quad nT(n) = (n+1)T(n-1) + 2n - 1$

Divide by $(n+1)n$: $\quad \frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2n-1}{n(n+1)}$

Let $A(n) = \frac{T(n)}{n+1}$: $\quad A(n) = A(n-1) + \frac{2n-1}{n(n+1)} = A(n-2) + \frac{2(n-1)-1}{(n-1)n} + \frac{2n-1}{n(n+1)}$

$$= \cdots = \sum_{i=1}^{n} \frac{2i-1}{i(i+1)} = \underbrace{\sum_{i=1}^{n} \frac{2}{i+1}}_{\Theta(\log n)} - \underbrace{\sum_{i=1}^{n} \frac{1}{i(i+1)}}_{\Theta(1)}$$

Therefore: $\quad A(n) = c \log n$

Finally: $\quad T(n) = (n+1)A(n) = c(n+1)\log n \in \Theta(n \log n)$

Improvement ideas for QuickSort

- Randomize by using *choose-pivot2*, giving $\Theta(n \log n)$ *expected time* for *quick-sort2*
- The auxiliary space is Ω(recursion depth)
  - $\Theta(n)$ in the worst-case
  - can be reduce to $\Theta(\log n)$ worst-case by
    - recurse in smaller sub-array first
    - replacing the other recursion by a while-loop (tail call elimination)
- Stop recursion when, say $n \leq 10$
  - array is not completely sorted, but almost sorted
  - at the end, run insertionSort, it sorts in just $O(n)$ time since all items are within 10 units of the required position
- Arrays with many duplicates sorted faster by changing *partition* to produce three subsets

$< v$	$= v$	$> v$

- Programming tricks
  - instead of passing full arrays, pass only the range of indices
  - avoid recursion altogether by keeping an explicit stack

QuickSort with Tricks

```
quick-sort3(A, n)
    initialize a stack S of index-pairs with {(0, n − 1)}
    while S is not empty
        (l, r) ← S.pop()                  // get the next subproblem
        while r − l + 1 > 10              // work on it if it's larger than 10
            p ← choose-pivot2(A, l, r)
            i ← partition (A, l, r, p)
            if i − l > r − i do           // is left side larger than right?
                S.push((l, i − 1))        // store larger  problem in S for later
                l ← i + 1                 // next work on the right side
            else
                S.push((i + 1, r))        // store larger problem in S for later
                r ← i − 1                 // next work on the left side
    InsertionSort(A)
```

- This is often the most efficient sorting algorithm in practice

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Lower bounds for sorting

- We have seen many sorting algorithms

Sort	Running Time	Analysis
Selection Sort	$\Theta(n^2)$	worst-case
Insertion Sort	$\Theta(n^2)$	worst-case
Merge Sort	$\Theta(n \log n)$	worst-case
Heap Sort	$\Theta(n \log n)$	worst-case
quick-sort1 quick-sort2	$\Theta(n \log n)$ $\Theta(n \log n)$	average-case expected

- **Question**: Can one do better than $\Theta(n \log n)$ running time?
- **Answer**: *It depends on what we allow*
  - No: comparison-based sorting lower bound is $\Omega(n \log n)$
    - no restriction on input, just must be able to compare
  - Yes: non-comparison-based sorting can achieve $O(n)$
    - restrictions on input

The Comparison Model

- All sorting algorithms seen so far are in the comparison model
- In the *comparison model* data can only be accessed in two ways
  - comparing two elements
    - $A[i] \leq A[j]$
  - moving elements around (e.g. copying, swapping)
- This makes very few assumptions on the things we are sorting
  - just count the number of above operations
- Under comparison model, will show that any sorting algorithm requires $\Omega(n \log n)$ comparisons
- This lower bound is not for an algorithm, it is for the sorting problem
- How can we talk about problem without algorithm?
  - count number of comparisons any sorting algorithm has to perform

Decision Tree

- Decision tree succinctly describes all the decisions that are taken during the execution of an algorithm and the resulting outcome
- For each sorting algorithm we can construct a corresponding decision tree
- Given decision tree, we can deduce the algorithm
- Decision tree can be constructed for any algorithm, not just sorting

Decision Tree Example

- Decision tree for a concrete comparison based sorting algorithm, with 3 non-repeating elements $[x_0, x_1, x_2]$

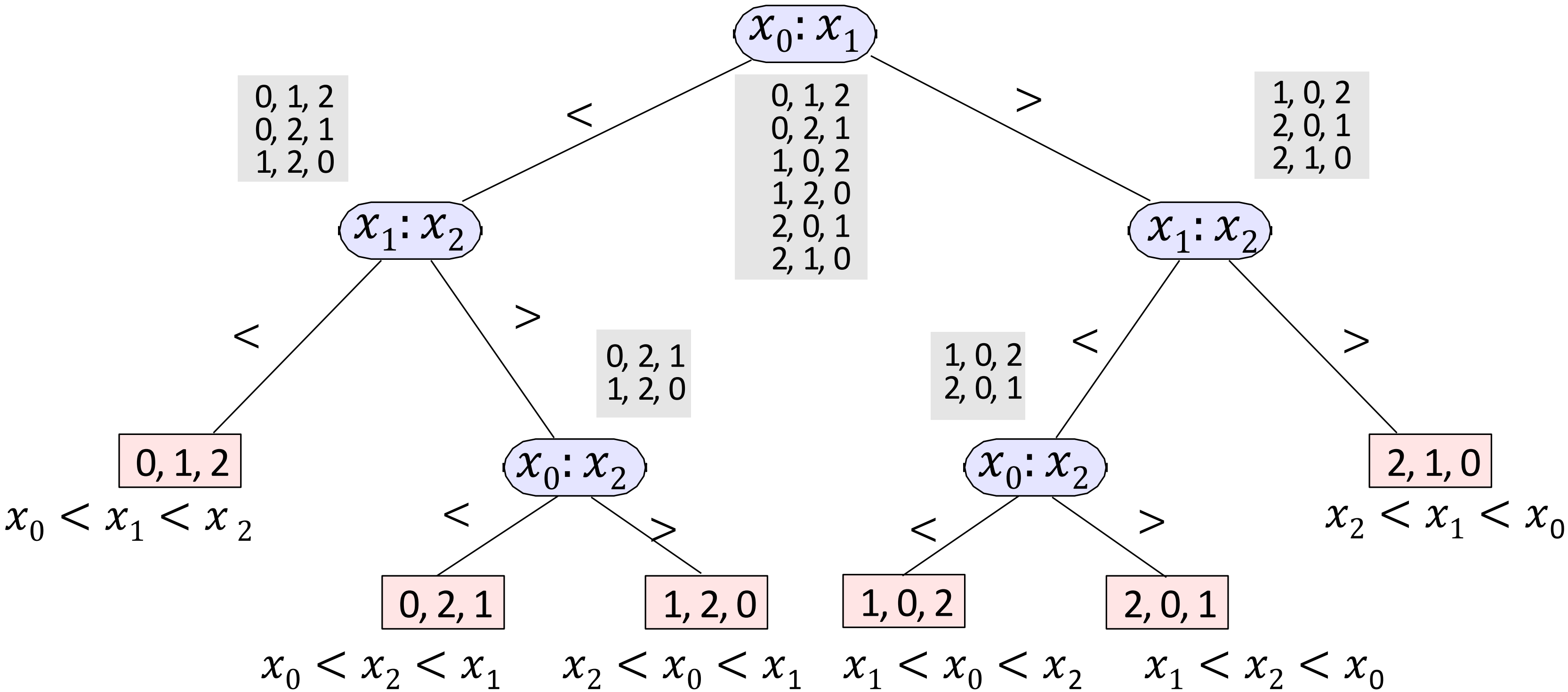
Set of all possible inputs

0, 1, 2	$x_0 < x_1 < x_2$	output $[x_0, x_1, x_2]$
0, 2, 1	$x_0 < x_2 < x_1$	output $[x_0, x_2, x_1]$
1, 0, 2	$x_1 < x_0 < x_2$	output $[x_1, x_0, x_2]$
1, 2, 0	$x_2 < x_0 < x_1$	output $[x_2, x_0, x_1]$
2, 0, 1	$x_1 < x_2 < x_0$	output $[x_1, x_2, x_0]$
2, 1, 0	$x_2 < x_1 < x_0$	output $[x_2, x_1, x_0]$

- Have to determine which of the 6 inputs we are given before can give output
  - unique output for each distinct input

Decision Tree

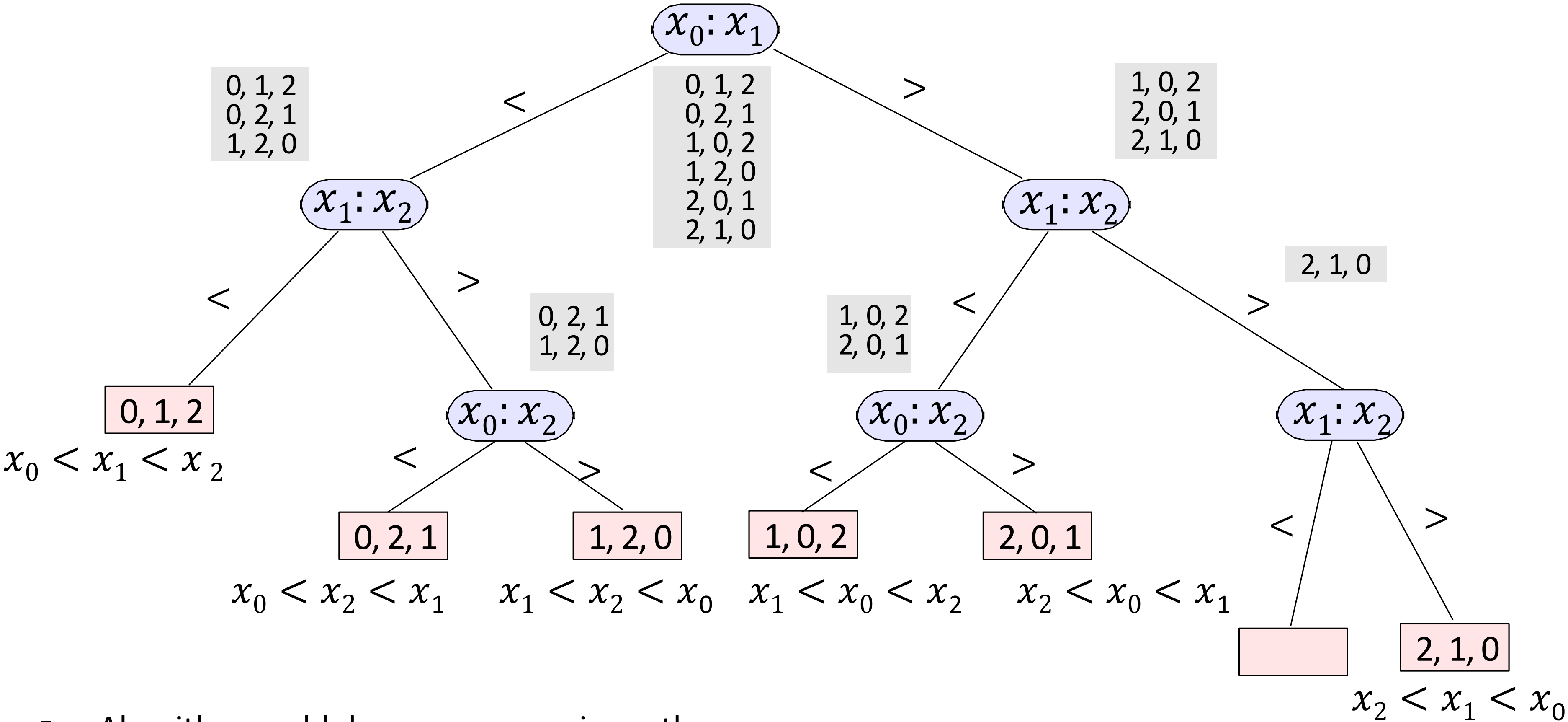
- Decision tree for a concrete comparison based sorting algorithm, with 3 non-repeating elements



- Root corresponds to the set of all possible inputs
- Interior nodes are comparisons: each comparison splits the set of possible inputs into two
- Know correct sorting order only when the set of possible inputs shrinks to size one
  - nodes where possible input shrunk to size one are leaves, when reach them, can output sorting result
- Sorting algorithm will traverse a path starting at root and ending at a leaf
  - length of the path is the number of comparisons to be made
- Tree height is the number of comparisons required for sorting in the worst case

Decision Tree

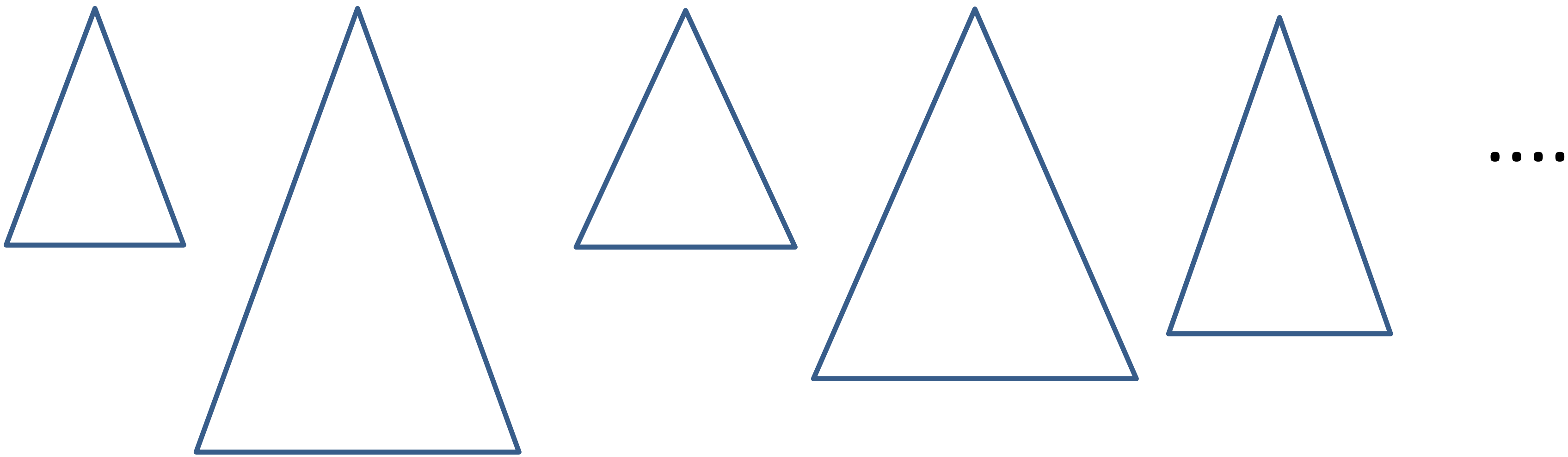
- Decision tree for a concrete comparison based sorting algorithm, with 3 non-repeating elements



- Algorithm could do more comparisons than necessary
- Thus can have more leafs than possible inputs
- But the number of leaves must be *at least* the number of possible inputs

Decision Tree

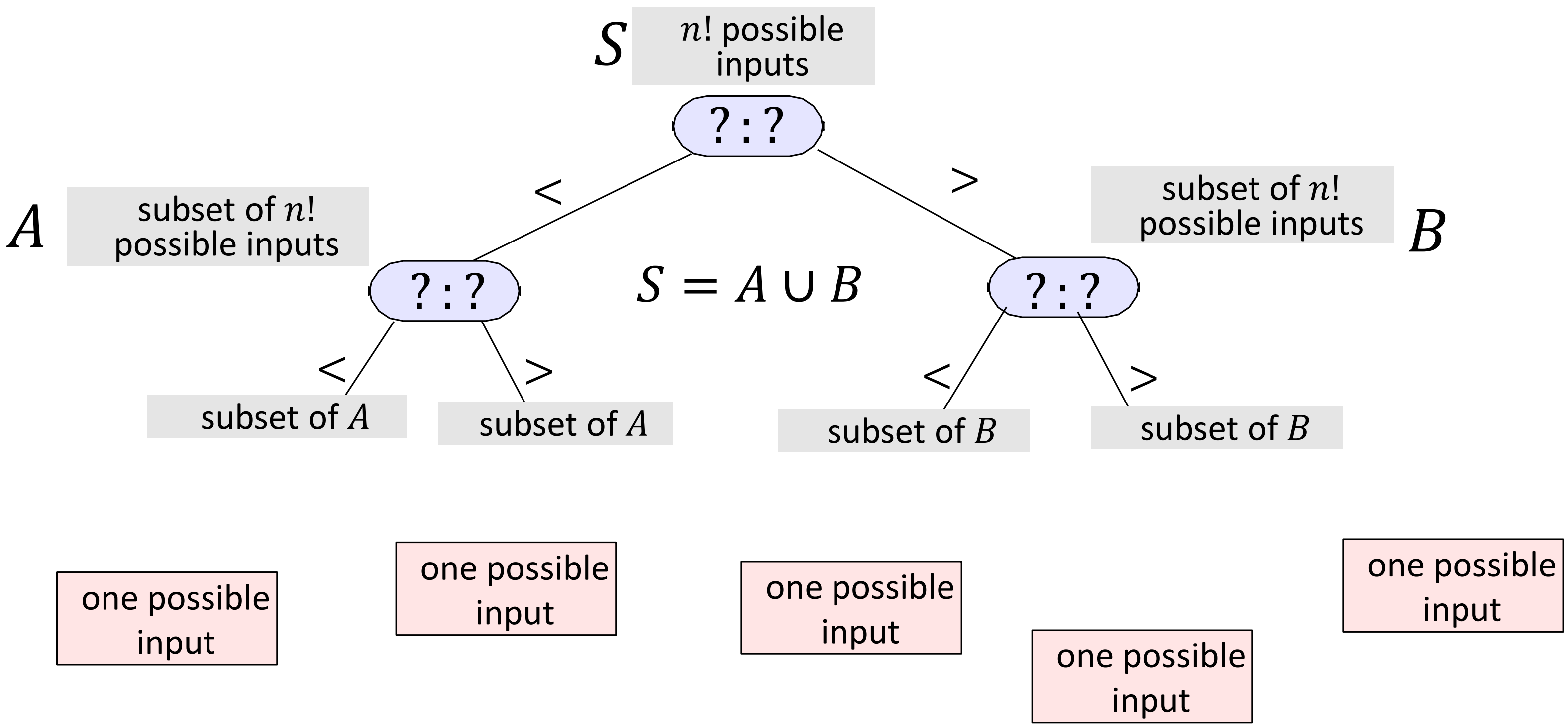
- General case: n non-repeating elements
- Many sorting algorithms, for each one we have its own decision tree
  - decision trees will have various heights



- Smallest height gives us the lower bound on the sorting problem
- Can we reason about the best (smallest) possible height any decision tree must have?

Decision Tree

- Can reason about decision tree for *any* comparison-based sorting algorithm with n non-repeating elements



- Tree must have at least $n!$ leaves
- Binary tree with height h has at most 2^h leaves
- Height h must be at least such that $2^h \geq n!$
- Tree height is the number of comparisons required in the worst case

Lower bound for sorting in the comparison model

Theorem: Any correct comparison-based sorting algorithm requires at least $\Omega(n \log n)$ comparisons

Proof:

- There exists a set of $n!$ possible inputs s.t. each leads to a different output
- Decision tree must have at least $n!$ leaves
- Binary tree with height h has at most 2^h leaves
- Height h must be at least such that $2^h \geq n!$
- Taking logs of both sides

$$h \geq \log(n!) = \log(n(n-1)\ldots\cdot 1) = \underbrace{\log n + \cdots + \log(\tfrac{n}{2}+1)}_{\geq \log\frac{n}{2}} + \log\frac{n}{2} + \cdots + \log 1$$

$$\geq \underbrace{\log\frac{n}{2} + \cdots + \log\frac{n}{2}}_{\frac{n}{2}\text{ of them}} = \frac{n}{2}\log\frac{n}{2} = \frac{n}{2}\log n - \frac{n}{2} \in \Omega(n \log n)$$

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Non-Comparison-Based Sorting

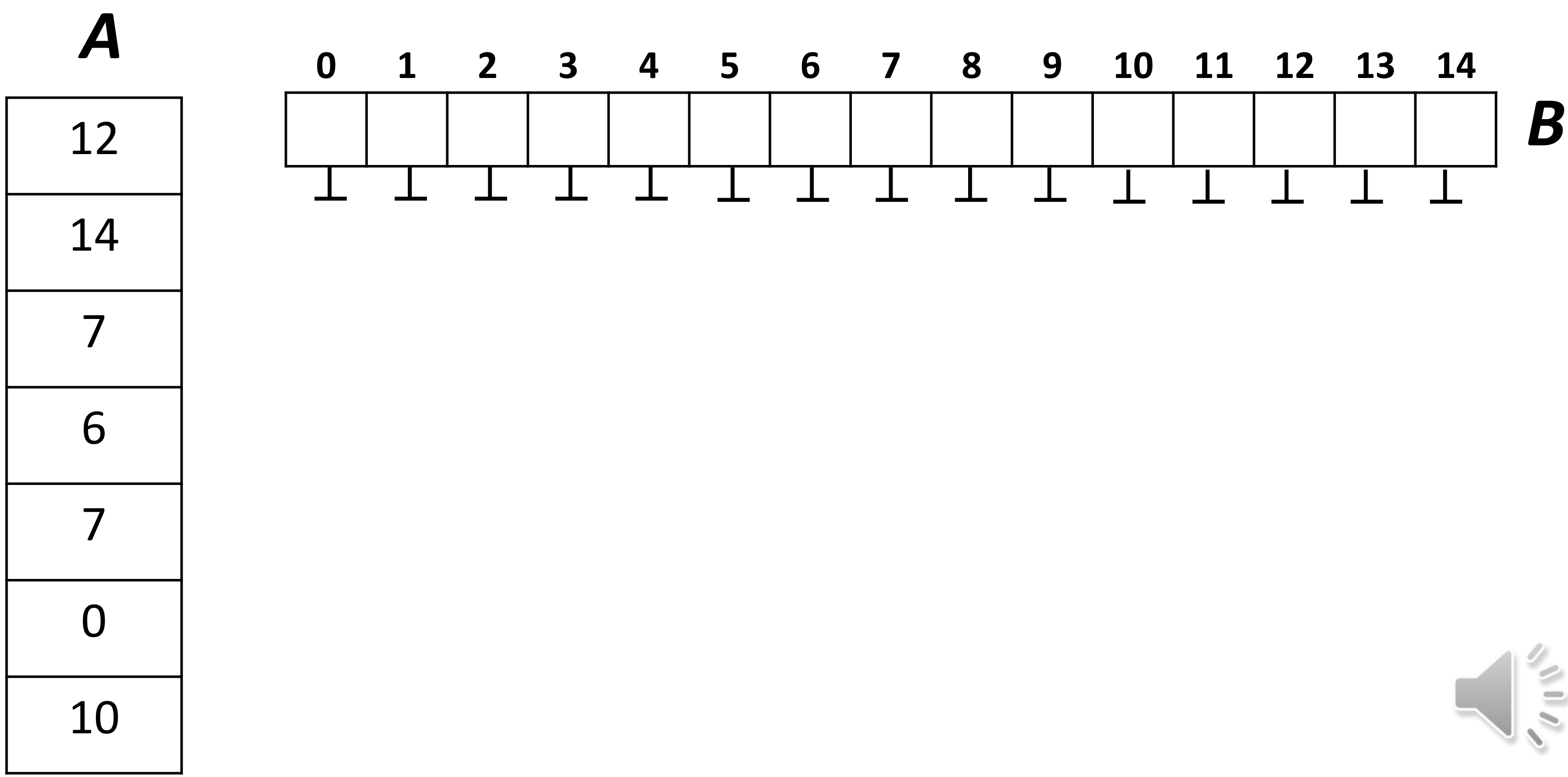
- Sort without comparing items to each other
- Non-comparison based sorting is less general than comparison based
- In particular, we need to make assumptions about items we sort
    - unlike in comparison based sorting, which sorts any data, as long as it can be compared
- Will assume we are sorting non-negative integers
    - can adapt to negative integers
    - also to some other data types, such as strings
    - but cannot sort arbitrary data

Non-Comparison-Based Sorting

- Simplest example
  - suppose all keys in A are integers in range $[0, \dots, L-1]$
- For non-comparison sorting, running time depends on both
  - array size n
  - L

Bucket Sort

- Suppose all keys in A are integers in range $[0, \ldots, L-1]$
- Use an axillary *bucket array* $B[0, \ldots, L-1]$ to sort
    - i.e. array of initially empty linked lists, initialization is $\Theta(L)$
- Example with $L = 15$



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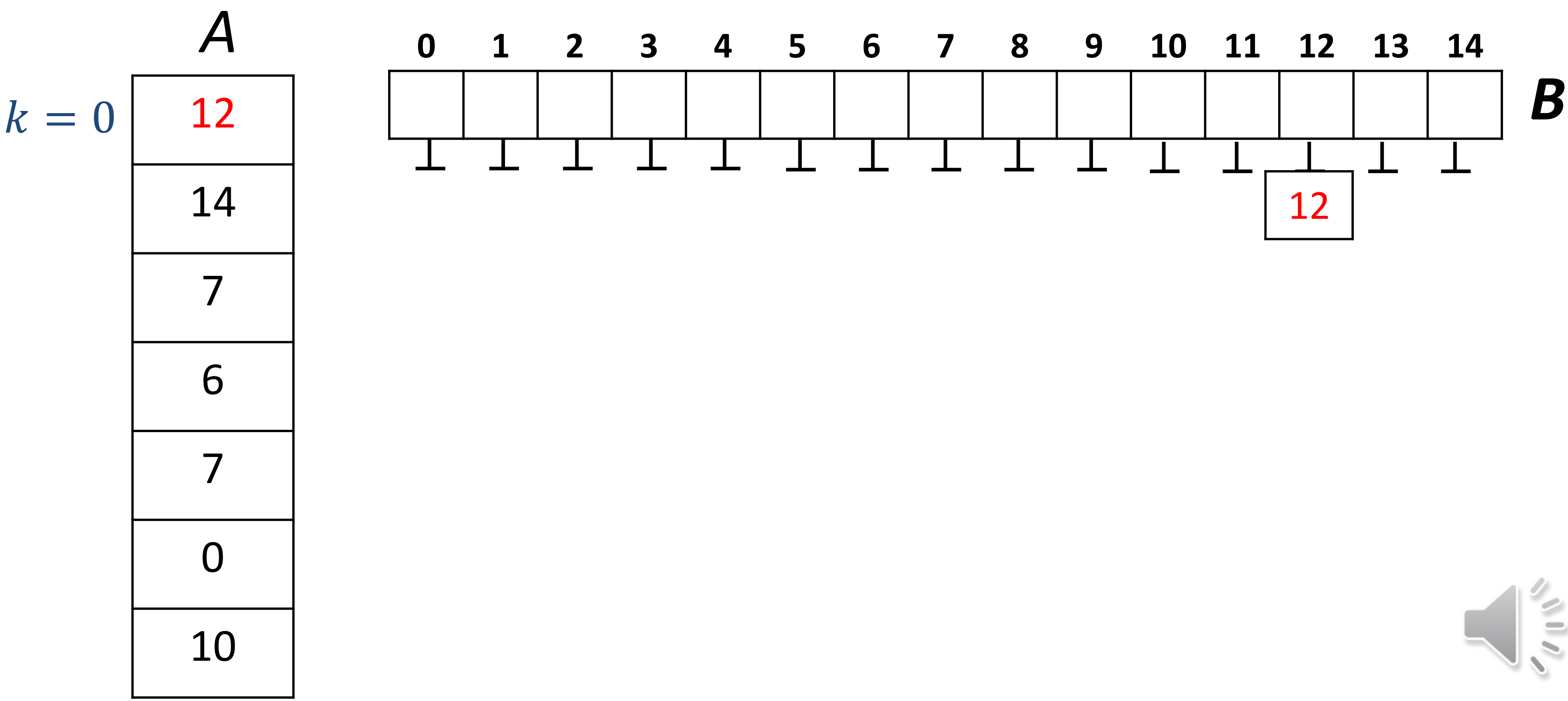

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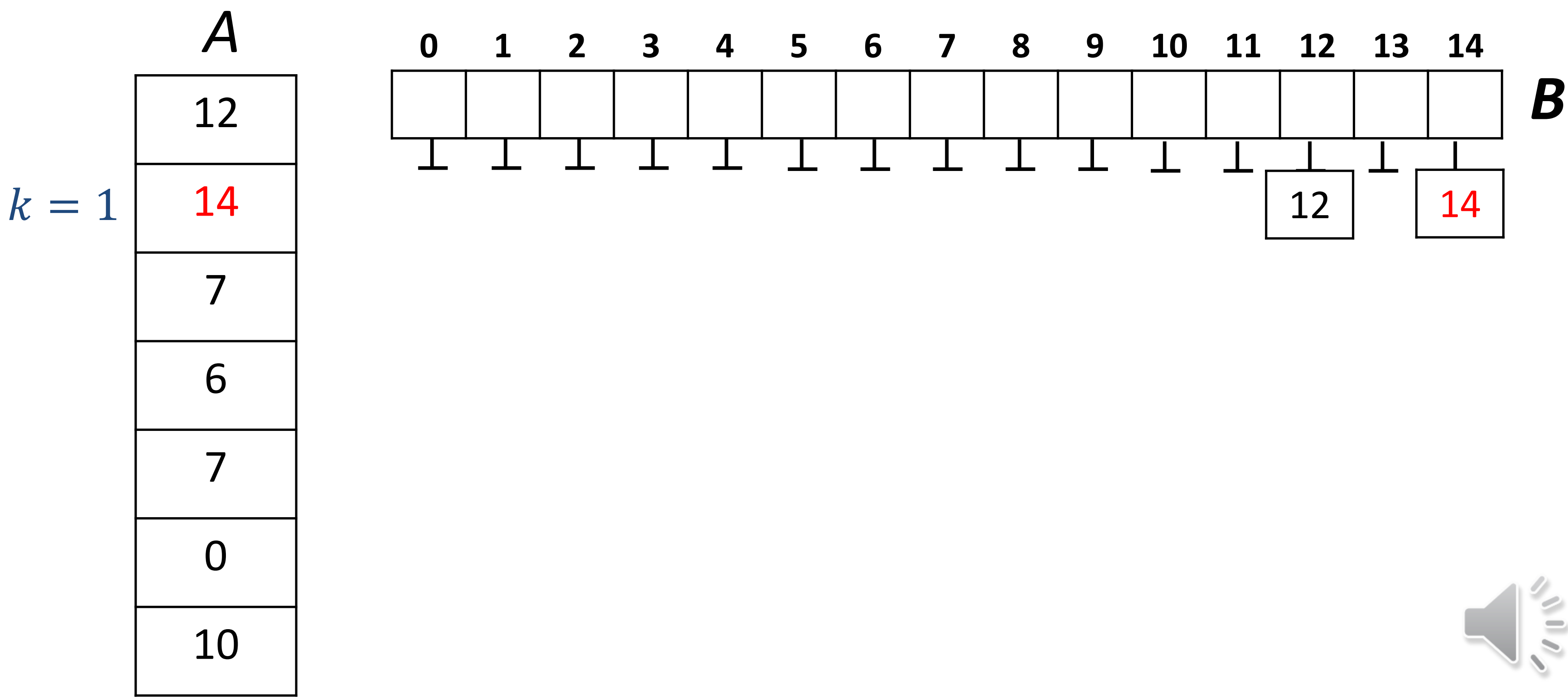

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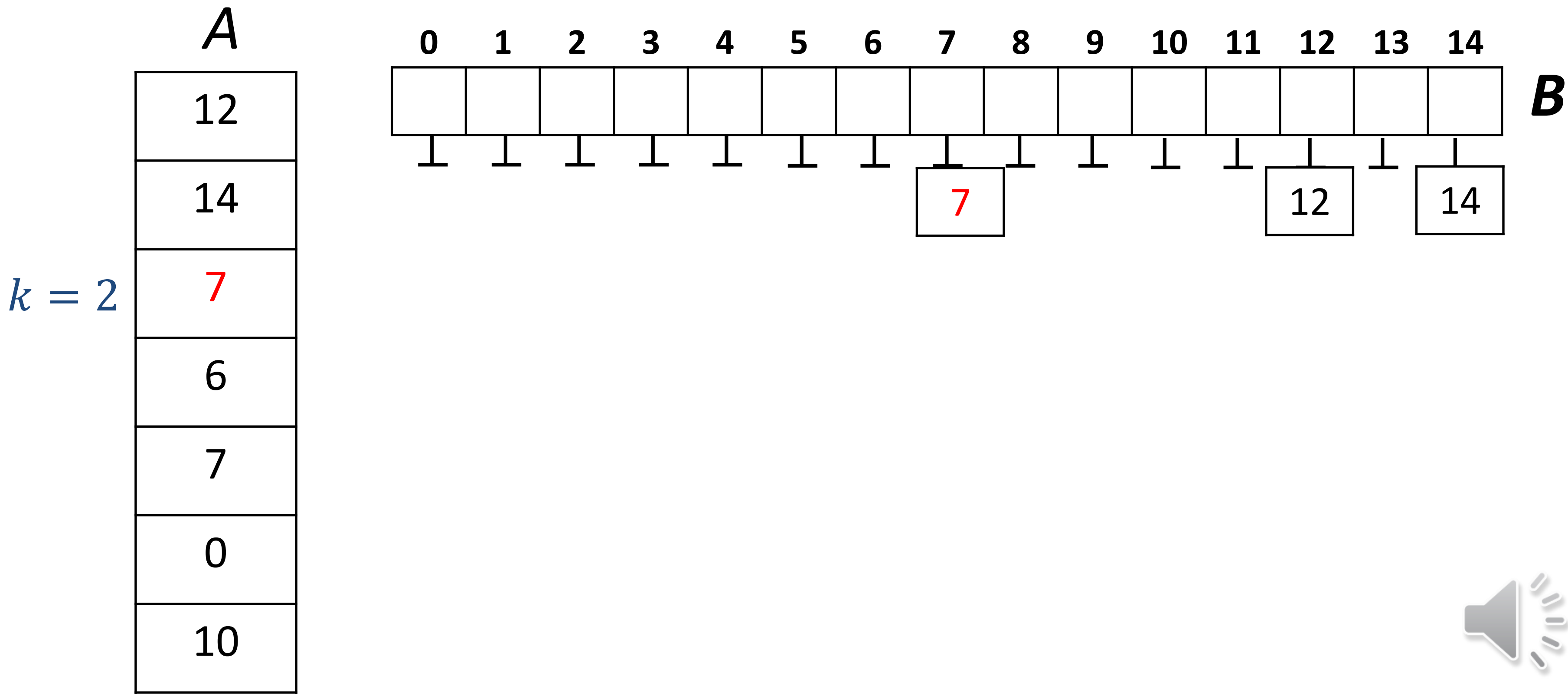

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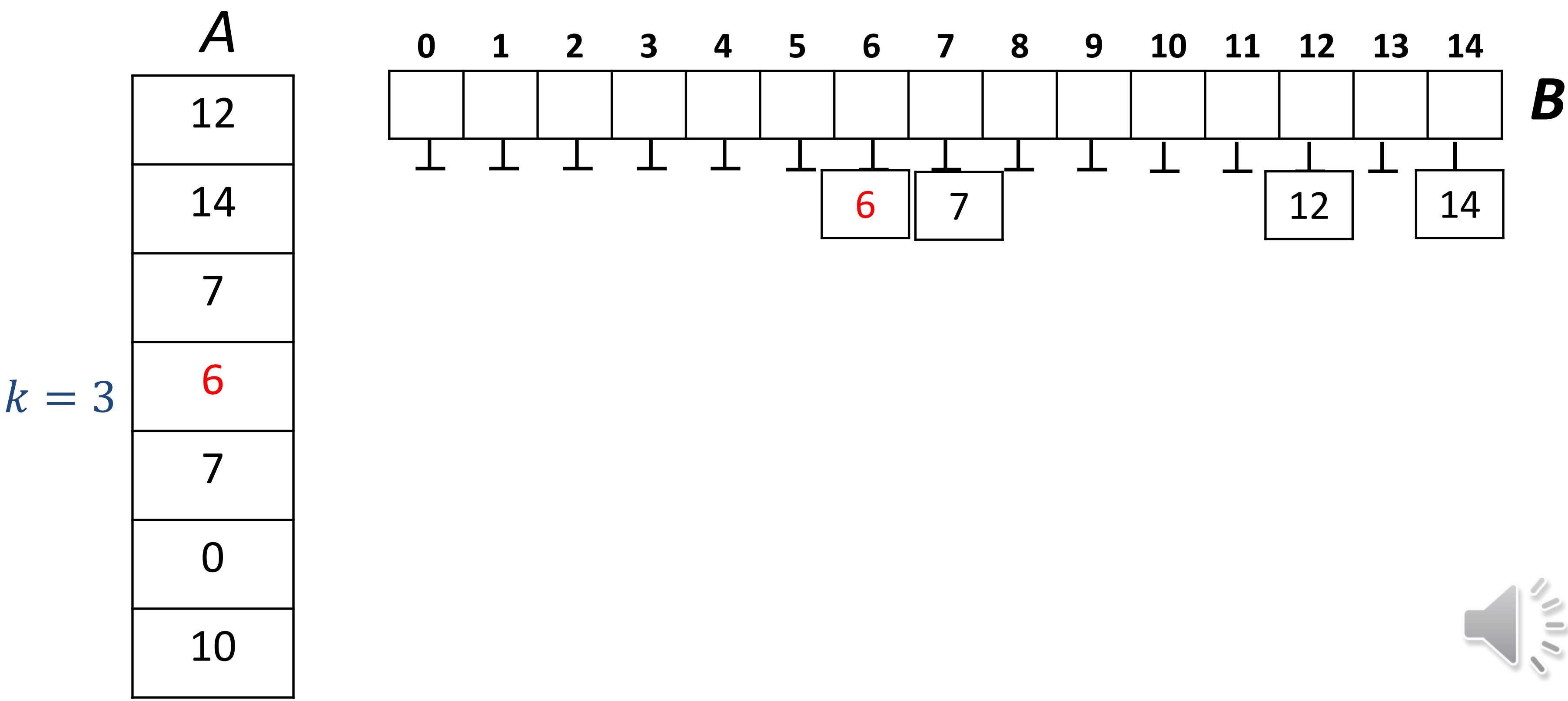

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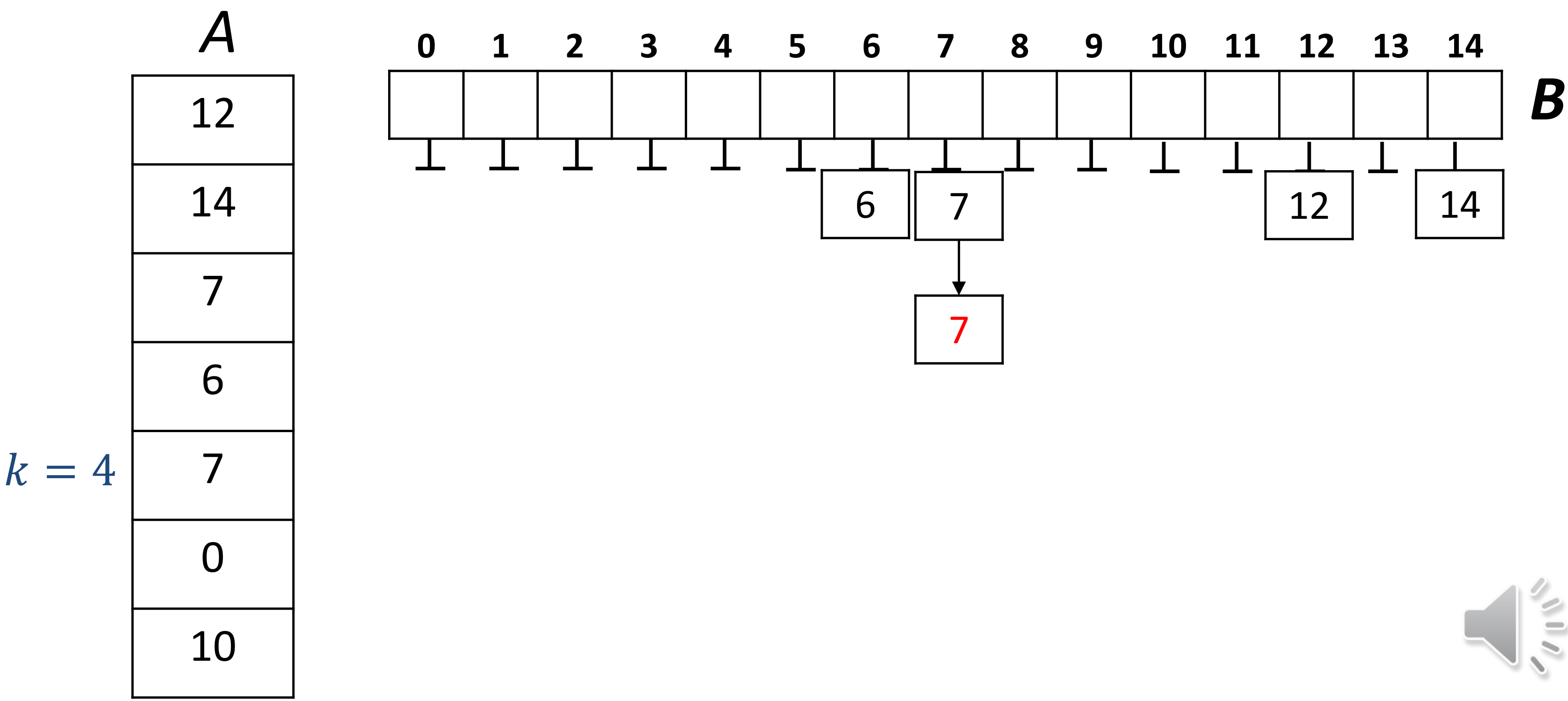

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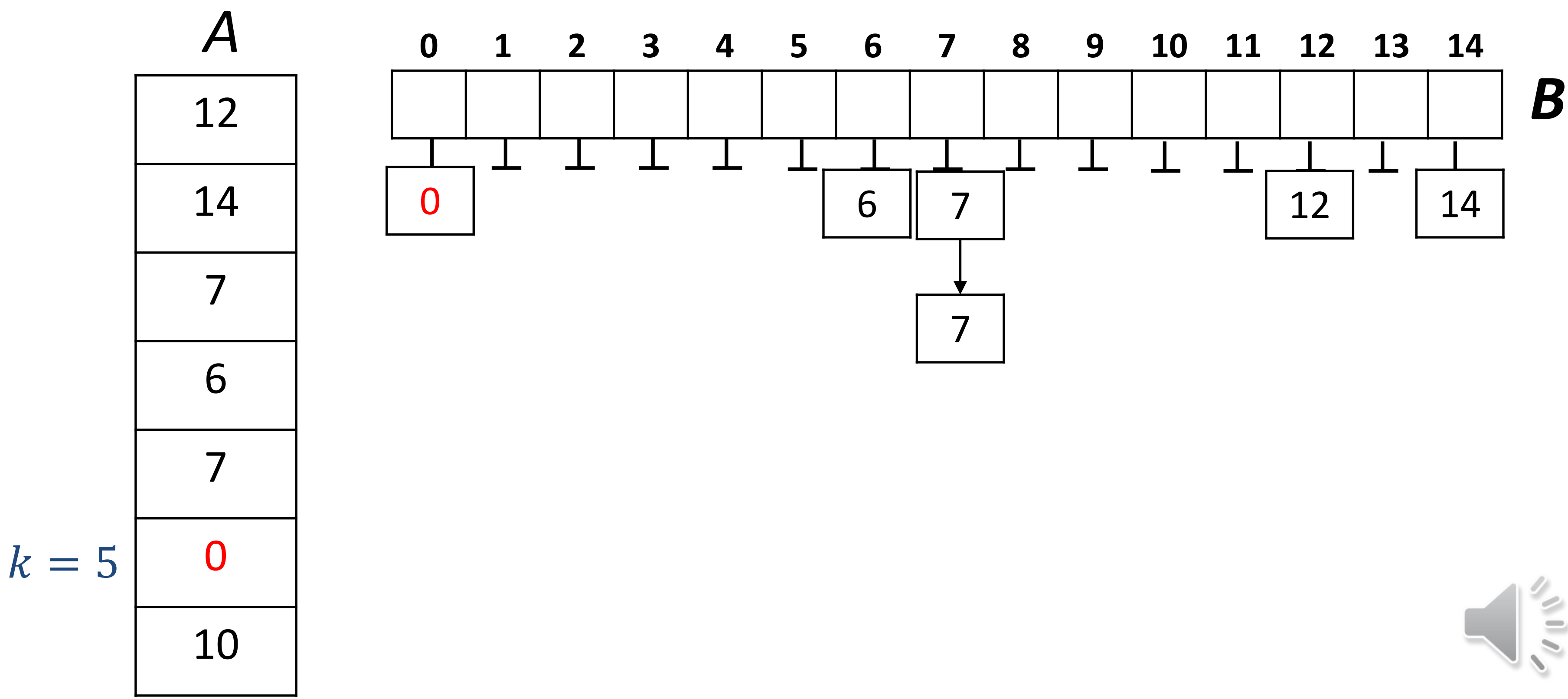

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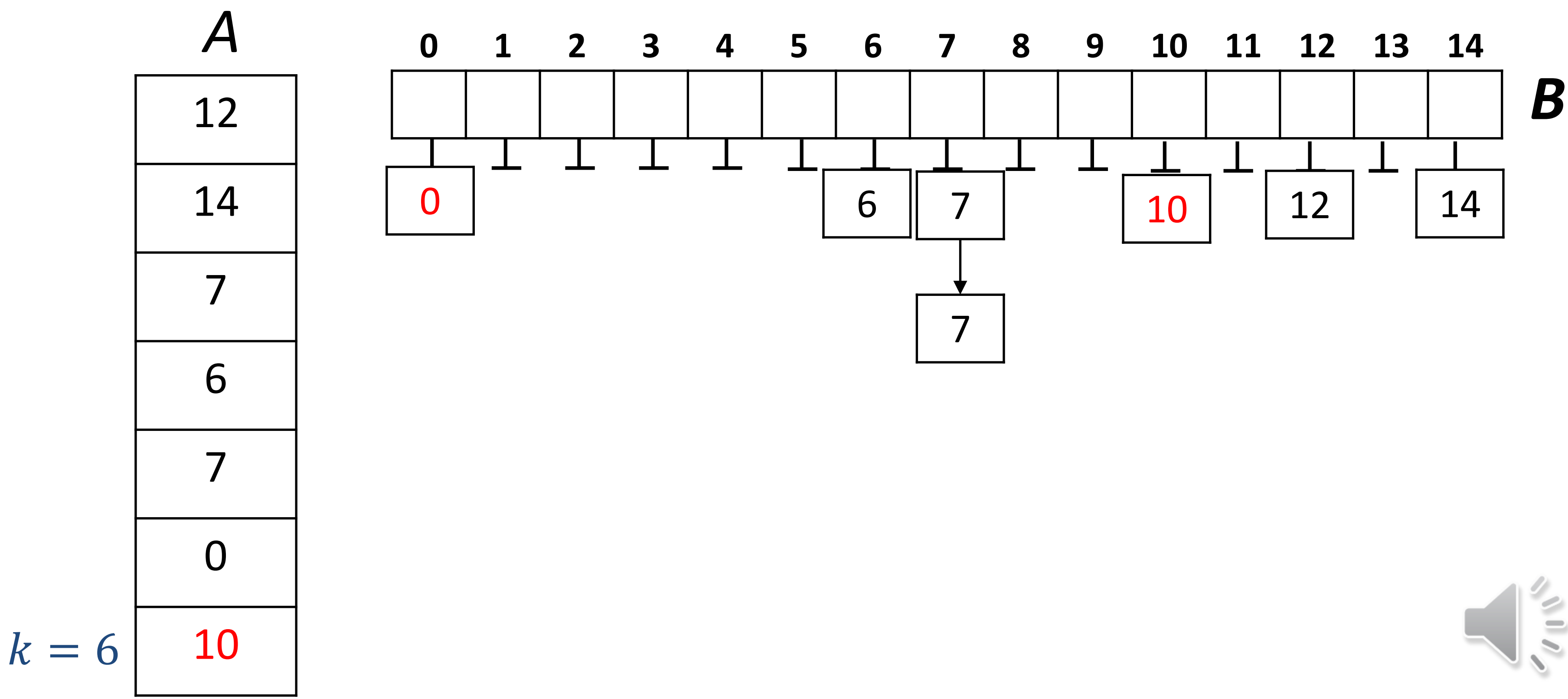
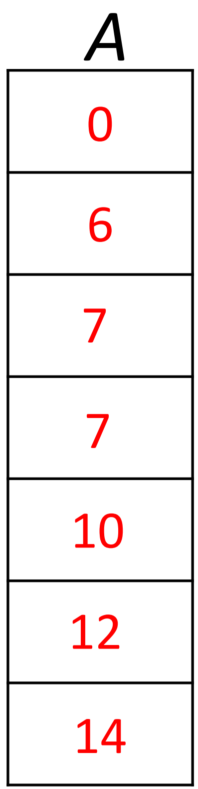
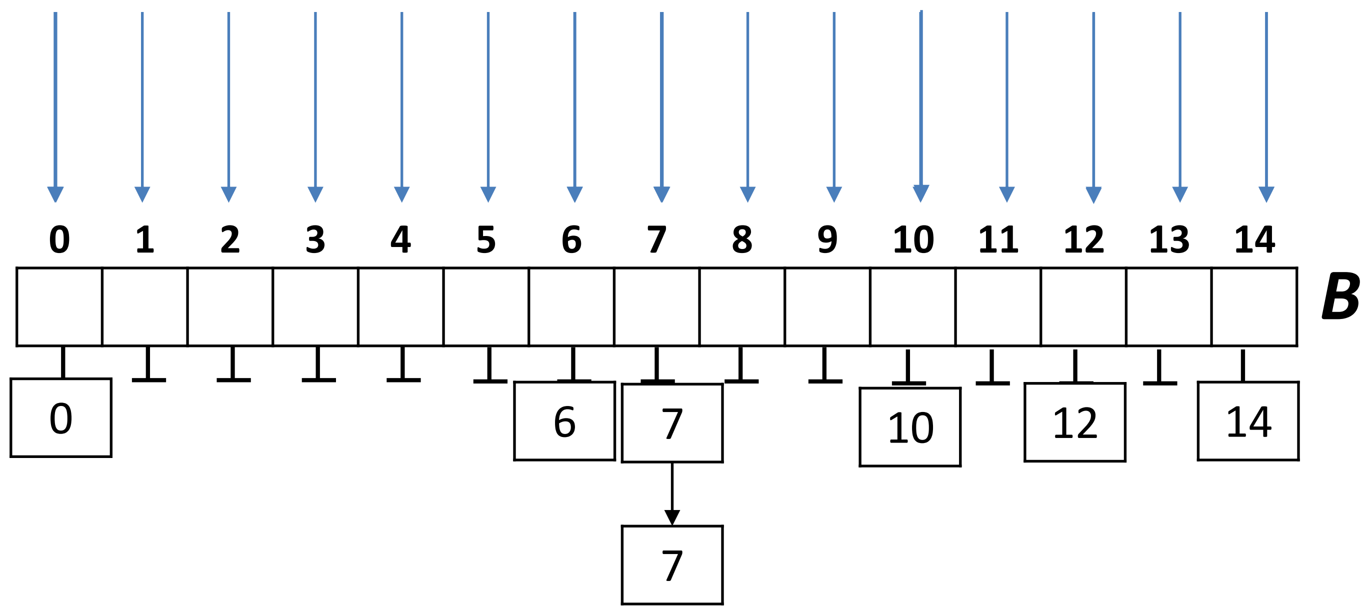


Bucket Sort

- Suppose all keys in A are integers in range $[0, \dots, L-1]$
- Use an axillary *bucket array* $B[0, \dots, L-1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$
- Now iterate through B and copy non-empty buckets to A





- Time complexity is $\Theta(L+n)$
  - n is size of A

Digit Based Non-Comparison-Based Sorting

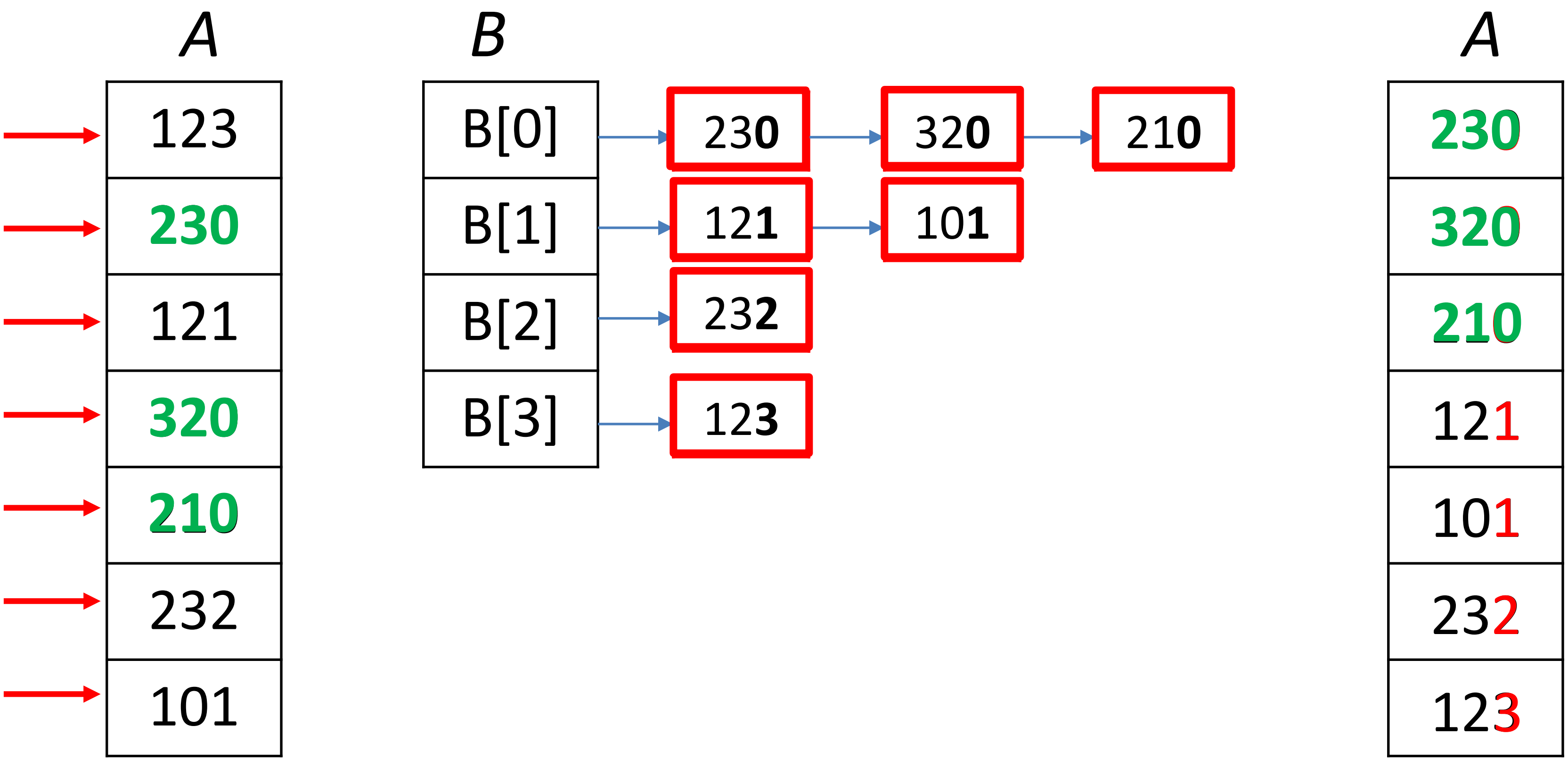
- Running time of bucket sort is $\Theta(L+n)$
  - n is size of A
  - L is range $[0, L)$ of integers in A
- What if L is much larger than n?
  - i.e. A has size 100, range of integers in A is $[0, \dots, 99999]$
- Assume at most m digits in any key
  - pad with leading 0s

123	230	021	320	210	232	101

- Can sort 'digit by digit', can go
  - forward, from digit $1 \rightarrow m$ (more obvious)
  - backward, from from digit $m \rightarrow 1$ (less obvious)
  - bucketsort is perfect for sorting 'by digit'
- Example: A has size 100, range of integers in A is [0,…,99999]
  - integers have at most 5 digits, need only 5 iterations of bucketsort

Bucket Sort on Last Digit

- Equivalent to normal bucket sort if we redefine comparison
  - $a \leq b$ if the last digit of a is smaller than (or equal) to the last digit of b



- Bucket sort is stable: equal items stay in original order
  - crucial for developing LSD radix sort later

Base R number representation

- Number of distinct digits gives the number of buckets R
- Useful to control number of buckets
  - larger R means less digits (less iterations), but more work per iteration (larger bucket array)
  - may want exactly 2, or 4, or even 128 buckets
- Can do so with base R representation
  - digits go from 0 to $R - 1$
  - R buckets
  - numbers are in the range $\{0, 1, \dots, R^m - 1\}$
- From now on, assume keys are numbers in base R (R: radix)
  - $R = 2, 10, 128, 256$ are common
- Example ($R = 4$)

123	230	21	320	210	232	101

Single Digit Bucket Sort

Bucket-sort(A, d)

A : array of size n, contains numbers with digits in $\{0, \dots, R-1\}$

d: index of digit by which we wish to sort

```
initialize array B[0, ..., R − 1] of empty lists (buckets)
for i ← 0 to n − 1 do
    next ← A[i]
    append next at end of B[dth digit of next]
i ← 0
for j ← 0 to R − 1 do
    while B[j] is non-empty do
        move first element of B[j] to A[i++]
```

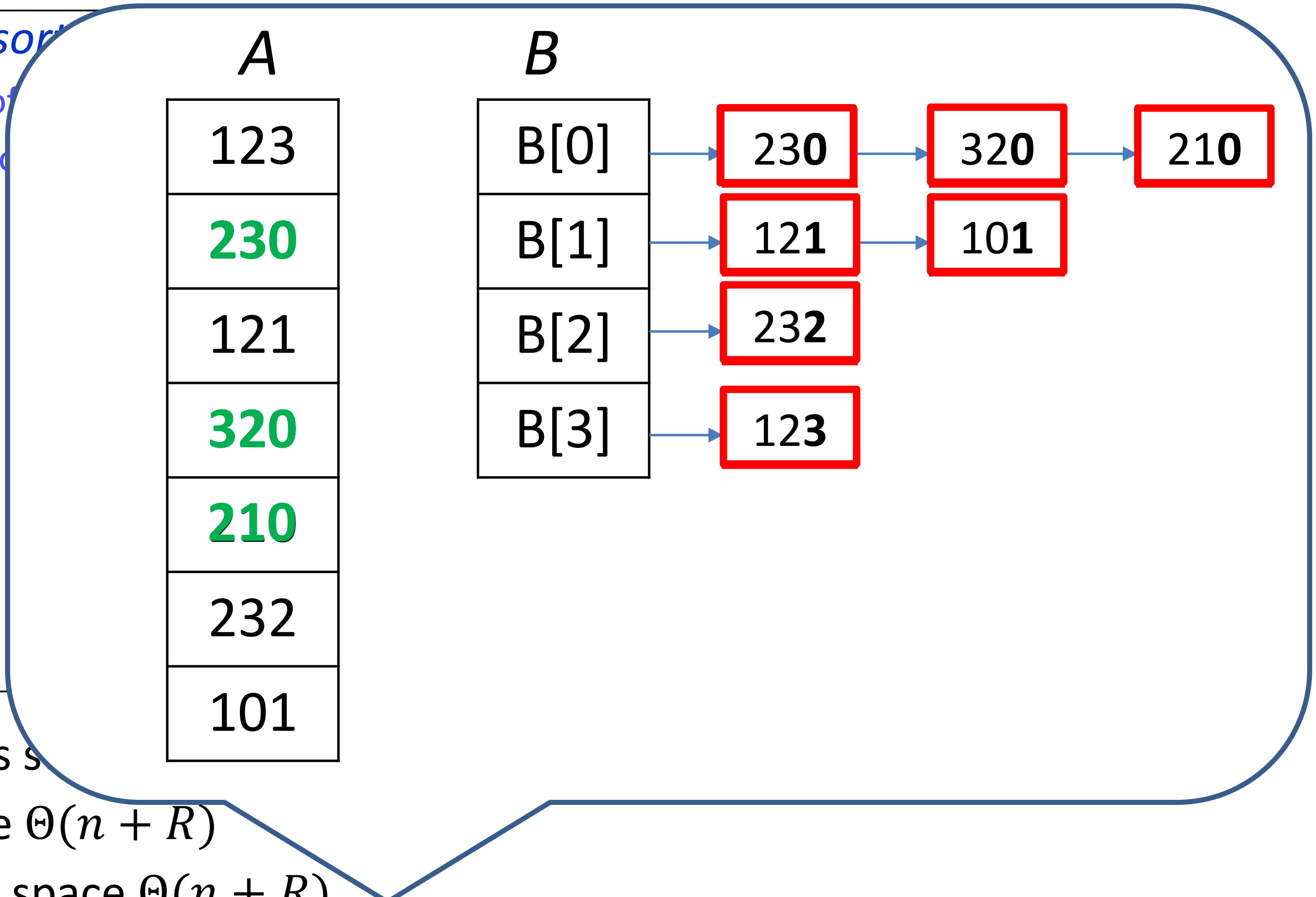
- Sorting is stable: equal items stay in original order
- Run-time $\Theta(n + R)$
- Auxiliary space $\Theta(n + R)$
    - $\Theta(R)$ for array B, and linked lists are $\Theta(n)$

Single Digit Bucket Sort

Bucket-sor

A : array of

d: index



- Sorting is s
- Run-time $\Theta(n + R)$
- Auxiliary space $\Theta(n + R)$
  - $\Theta(R)$ for array B, and linked lists are $\Theta(n)$
- Can replace lists by two auxiliary arrays of size R and n, resulting in *count-sort*
  - no details

MSD-Radix-Sort

- Sorts multi-digit numbers from the most significant to the least significant
- Start by sorting the whole array by the first digit

123
232
021
320
210
230
101

MSD-Radix-Sort

- Sorts multi-digit numbers from the most significant to the least significant
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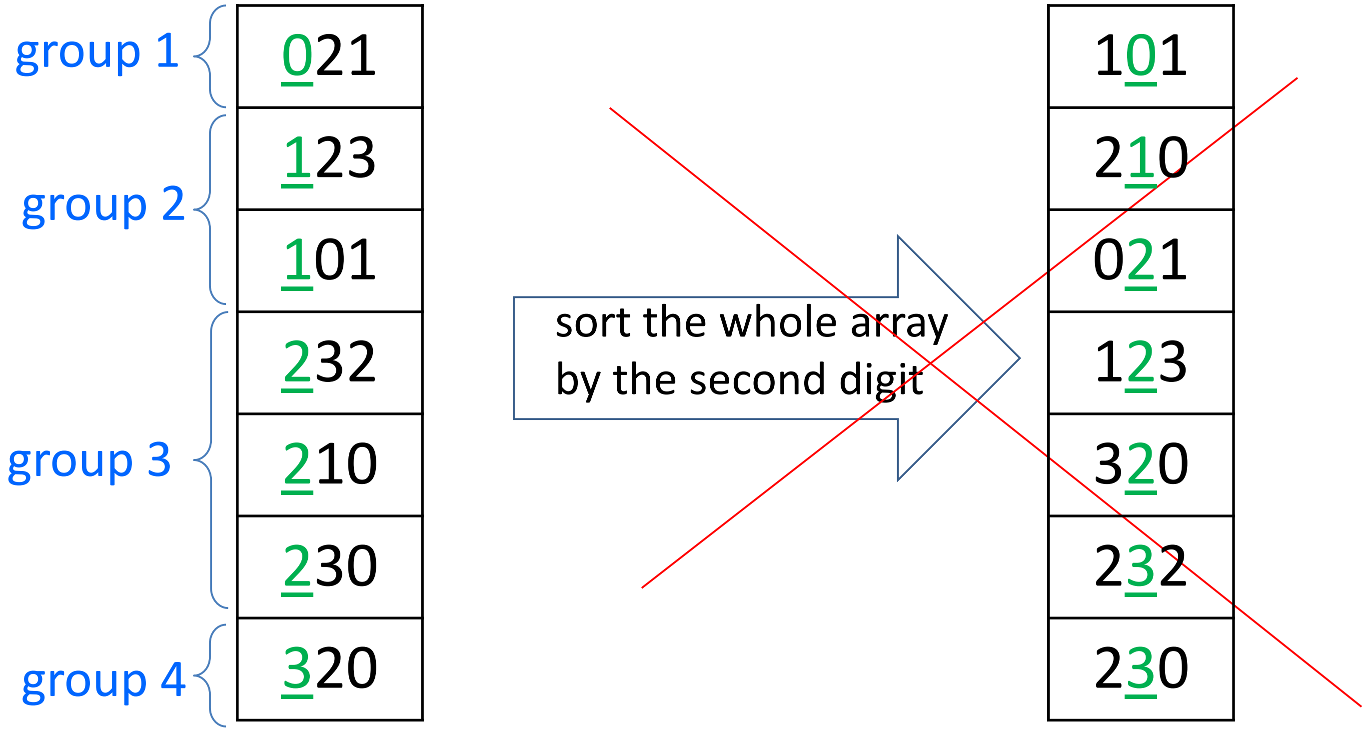
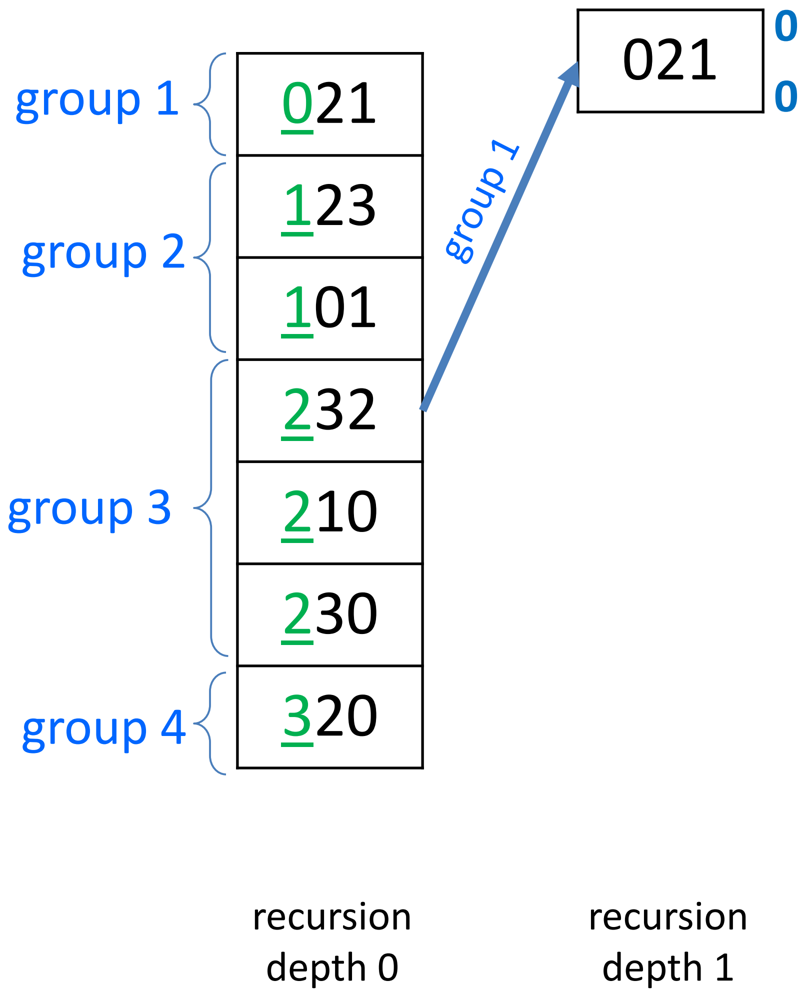


- Cannot sort the whole array by the second digit, will mess up the order
- Have to break down in groups by the first digit
  - each group can be safely sorted by the second digit
  - call sort recursively on each group, with appropriate array bounds

MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group



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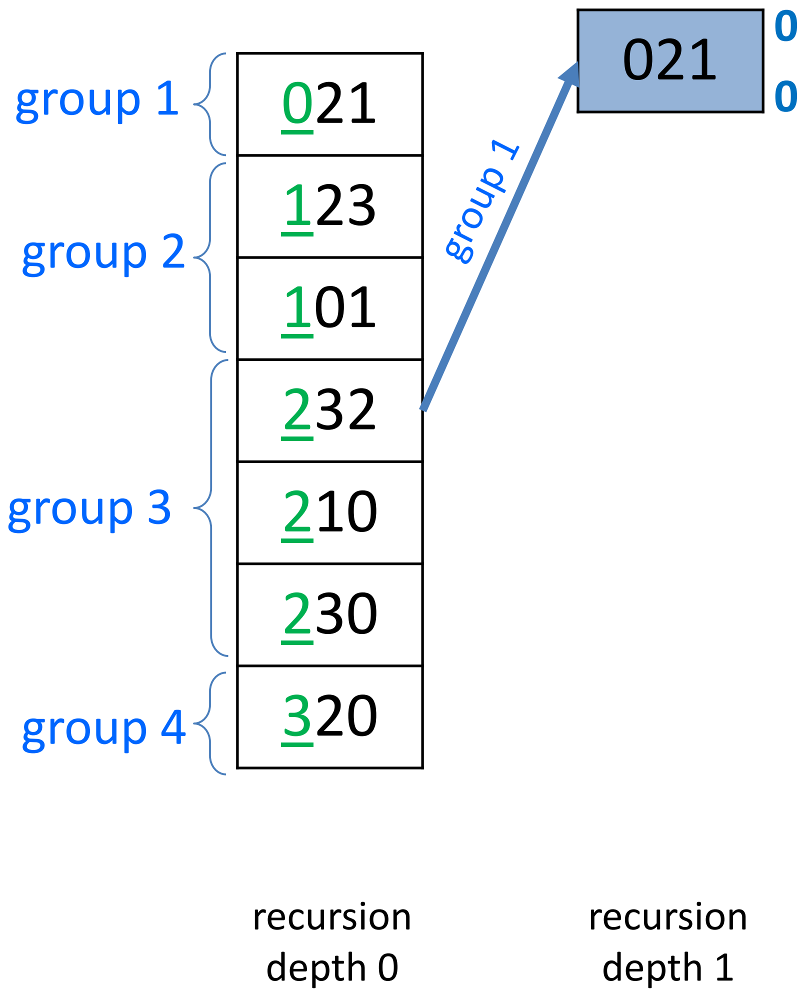

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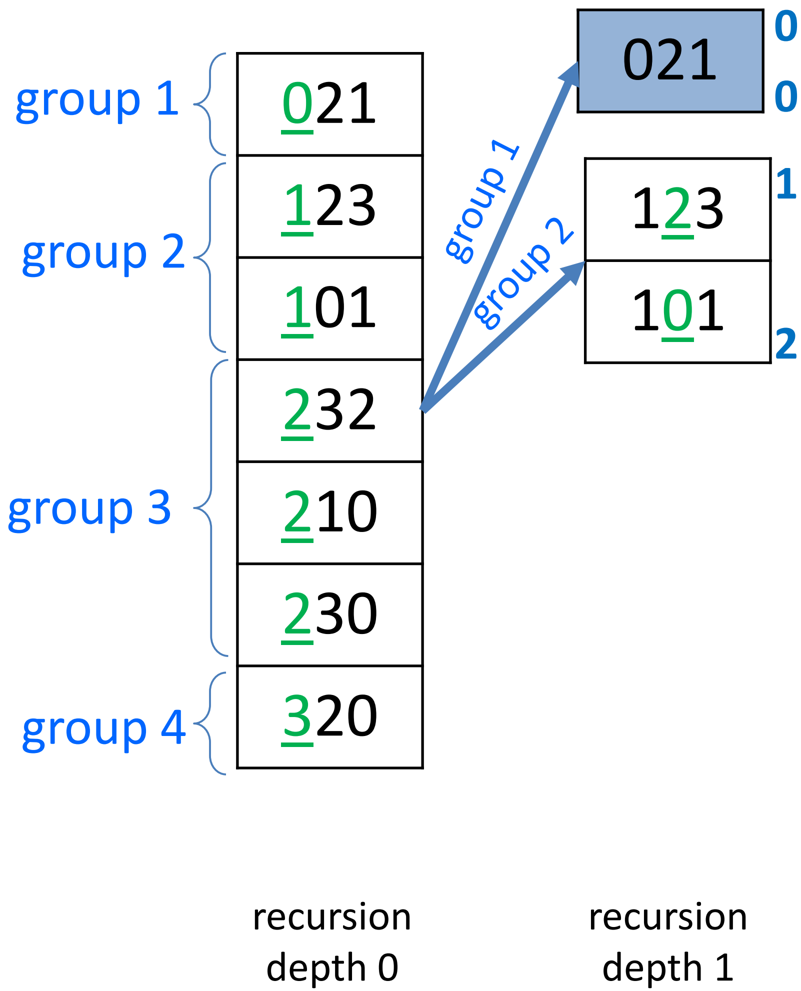

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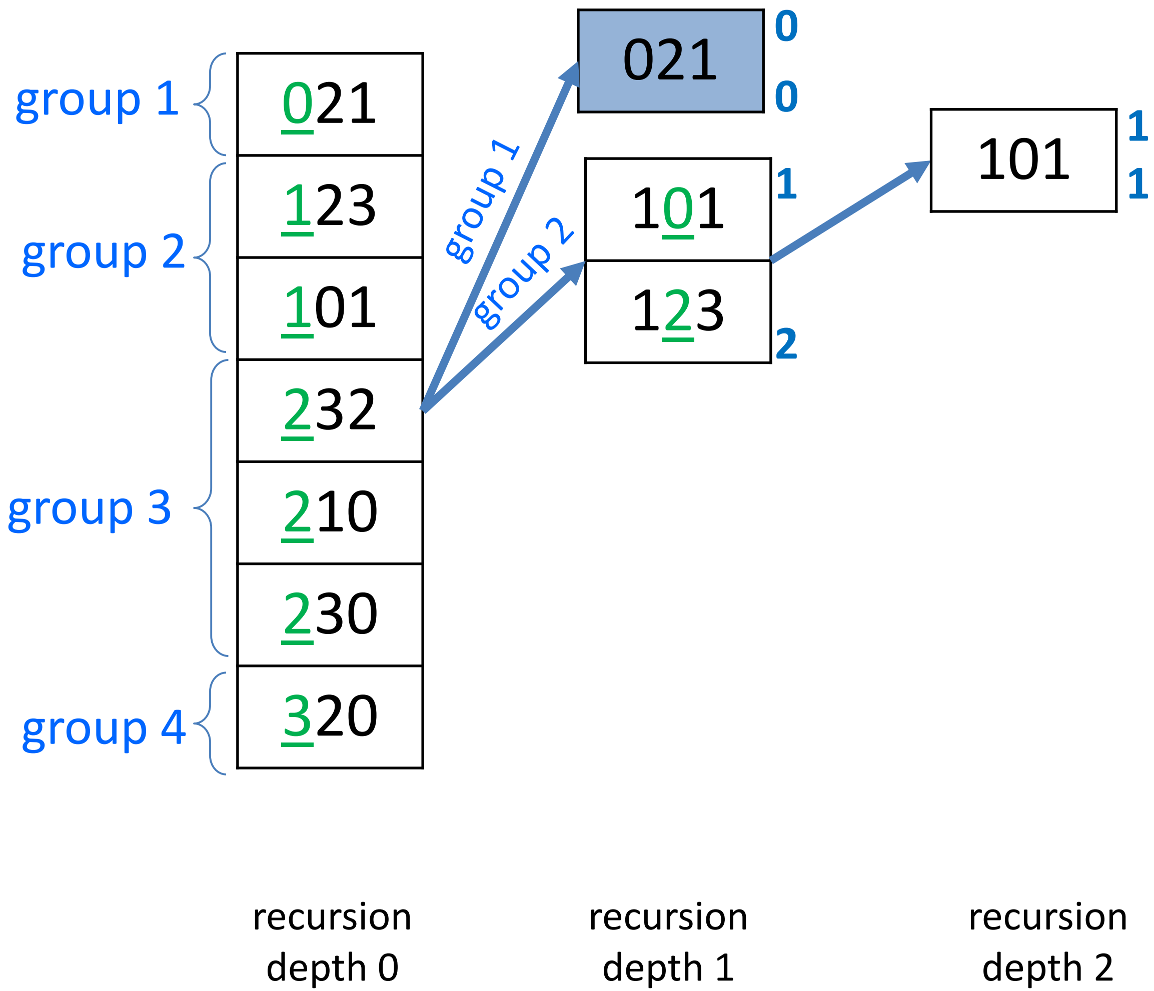

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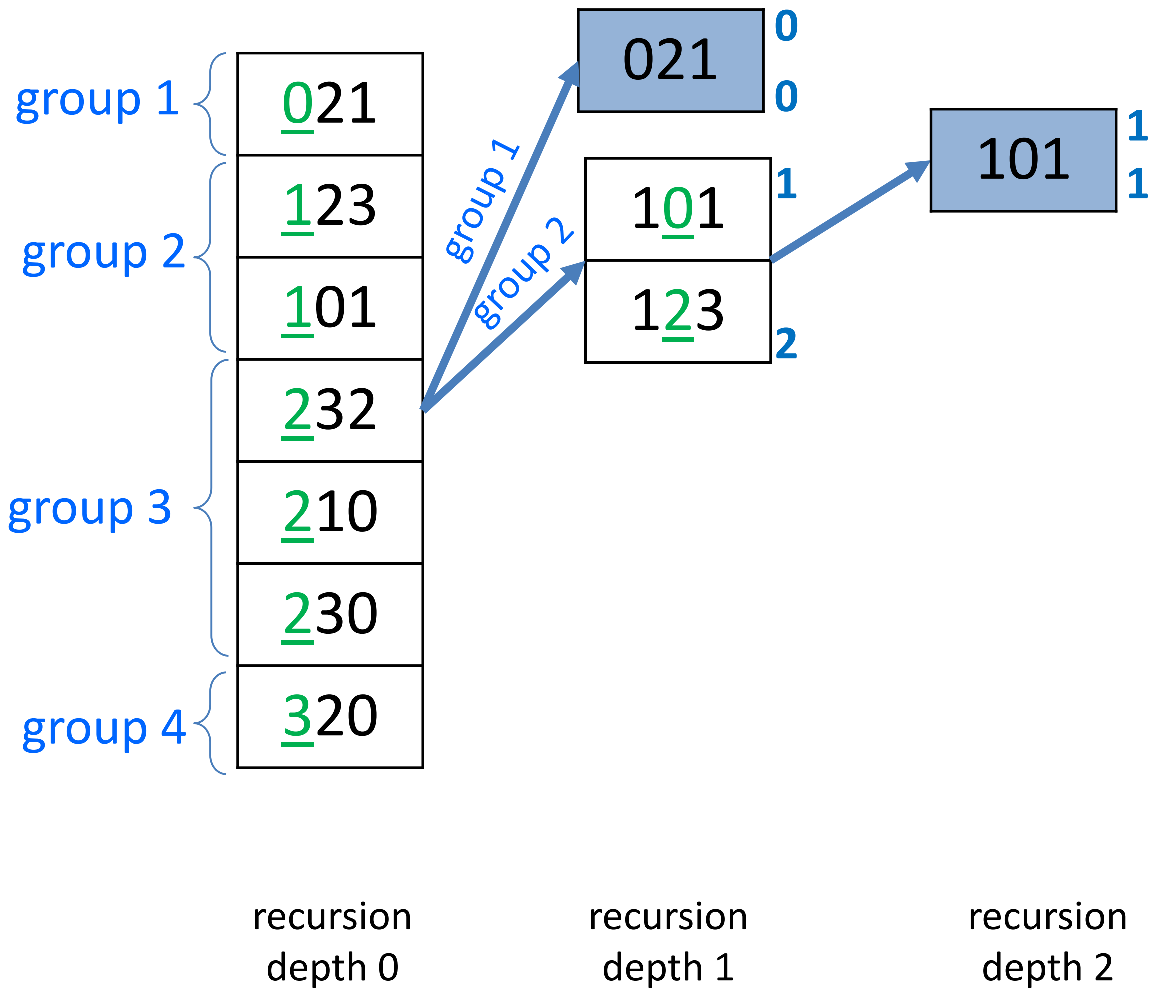

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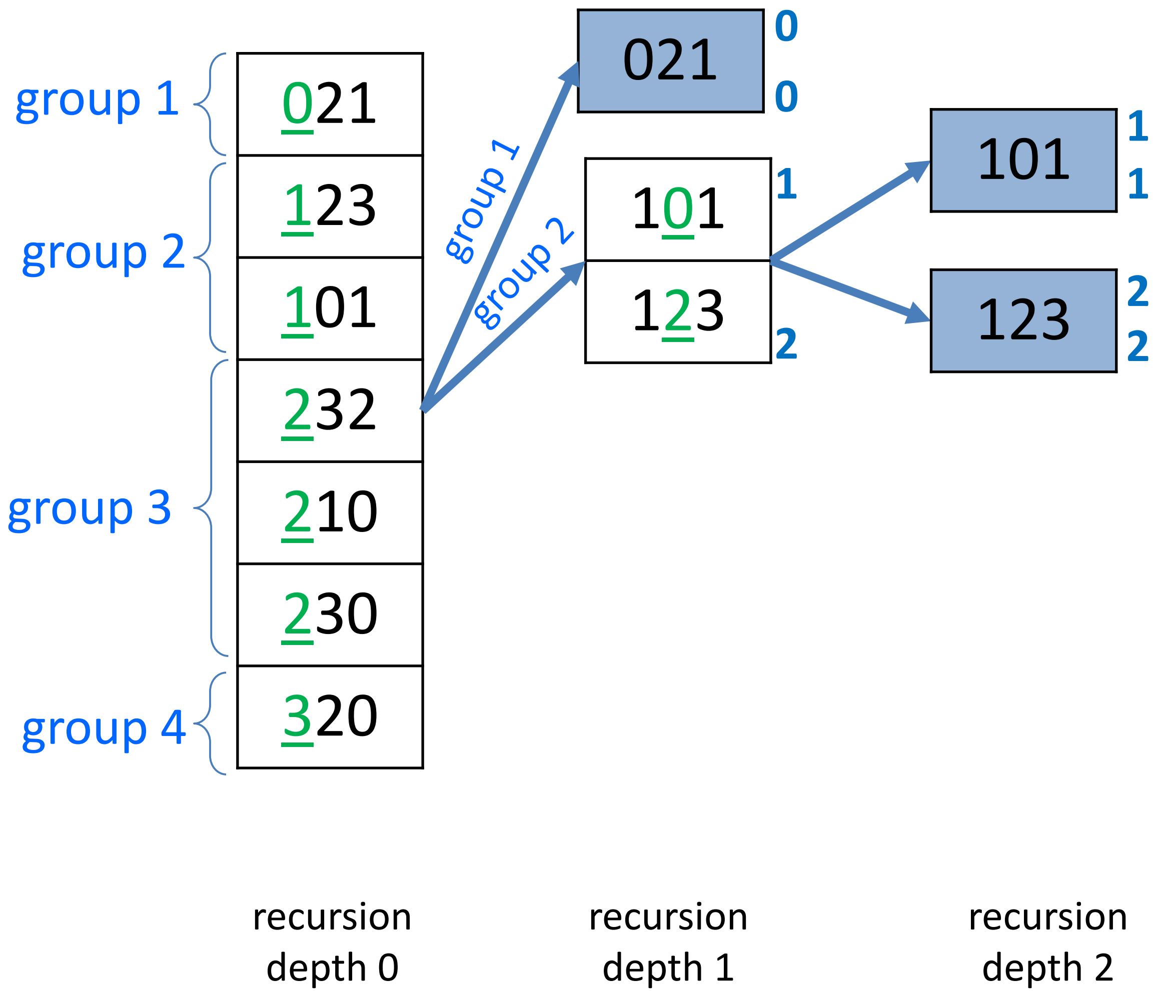

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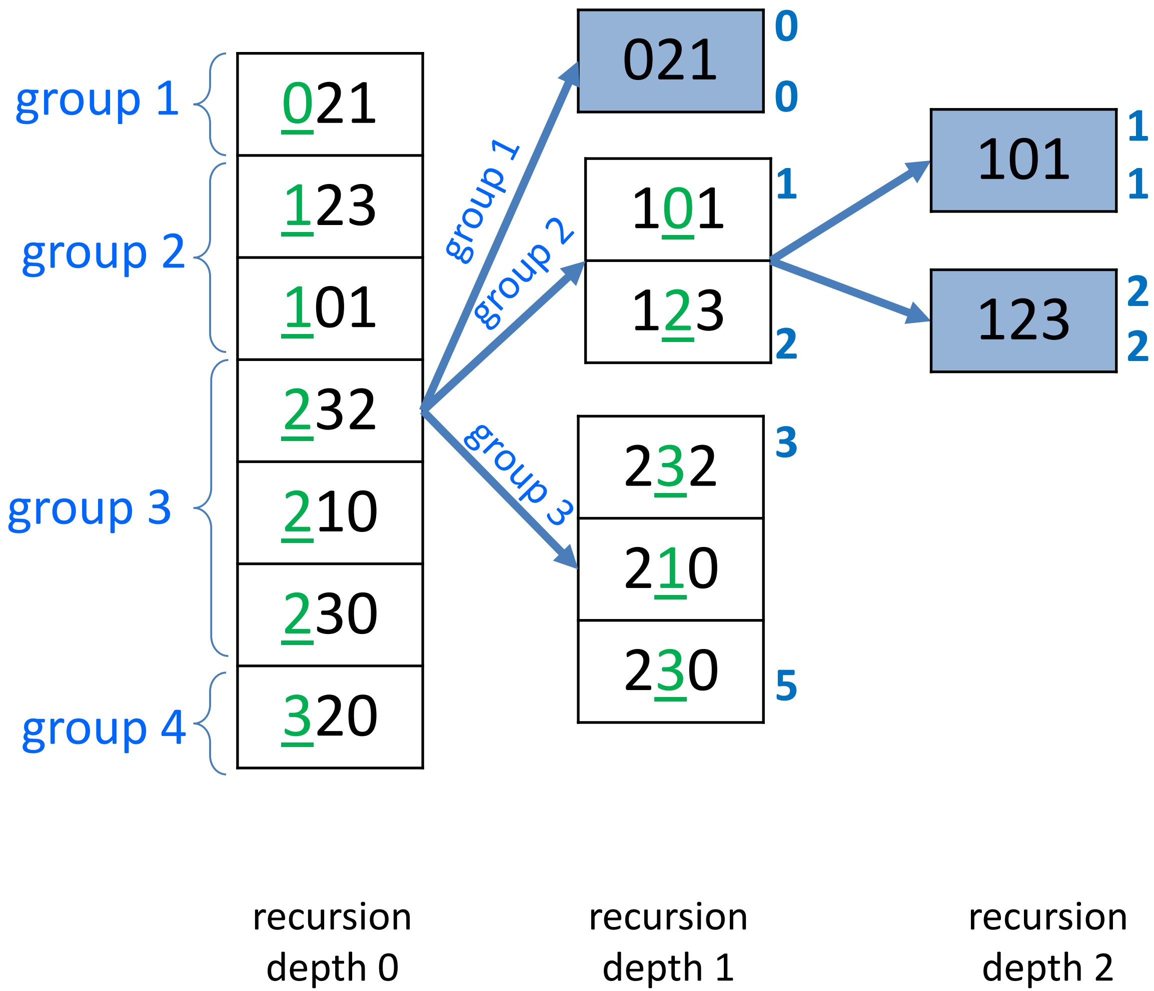

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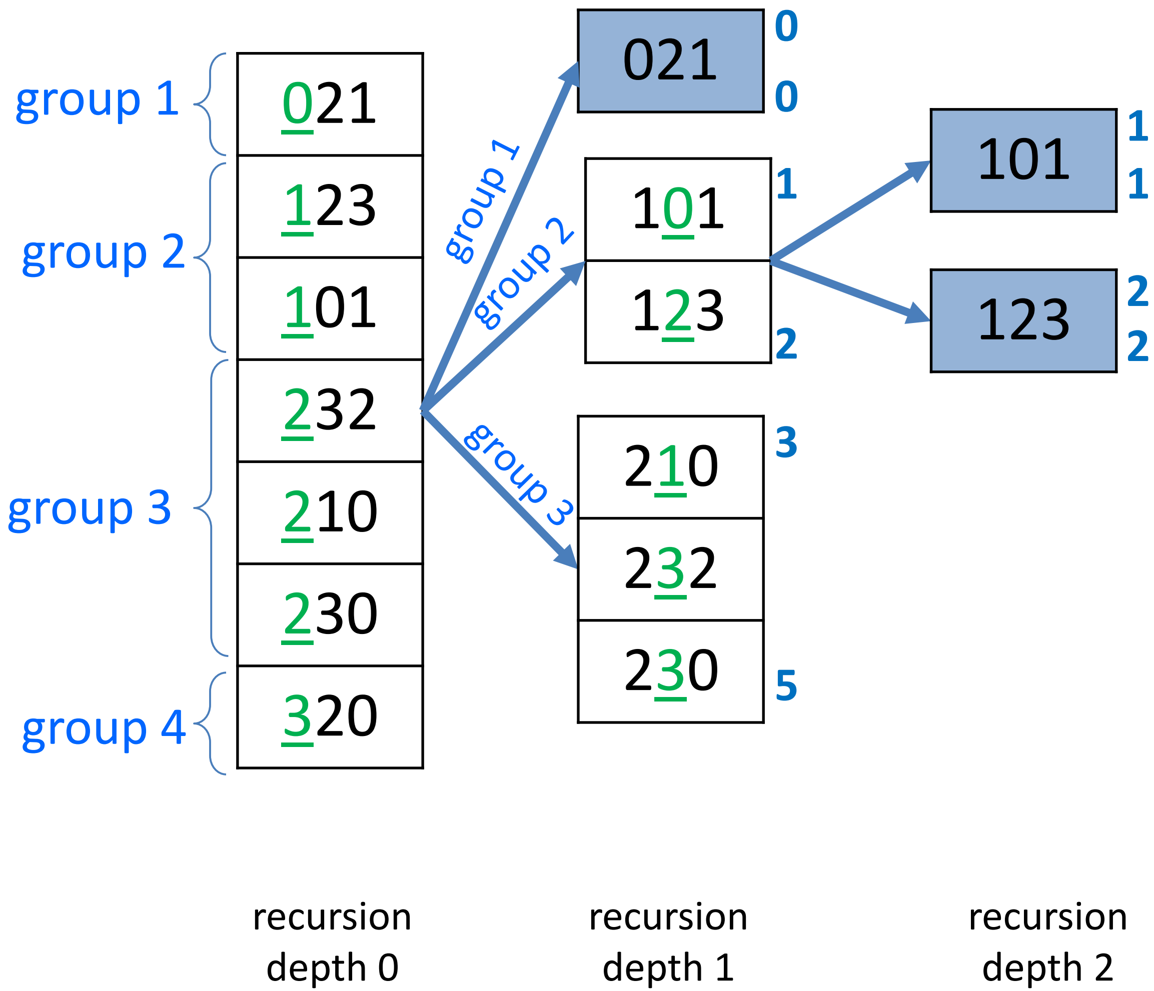

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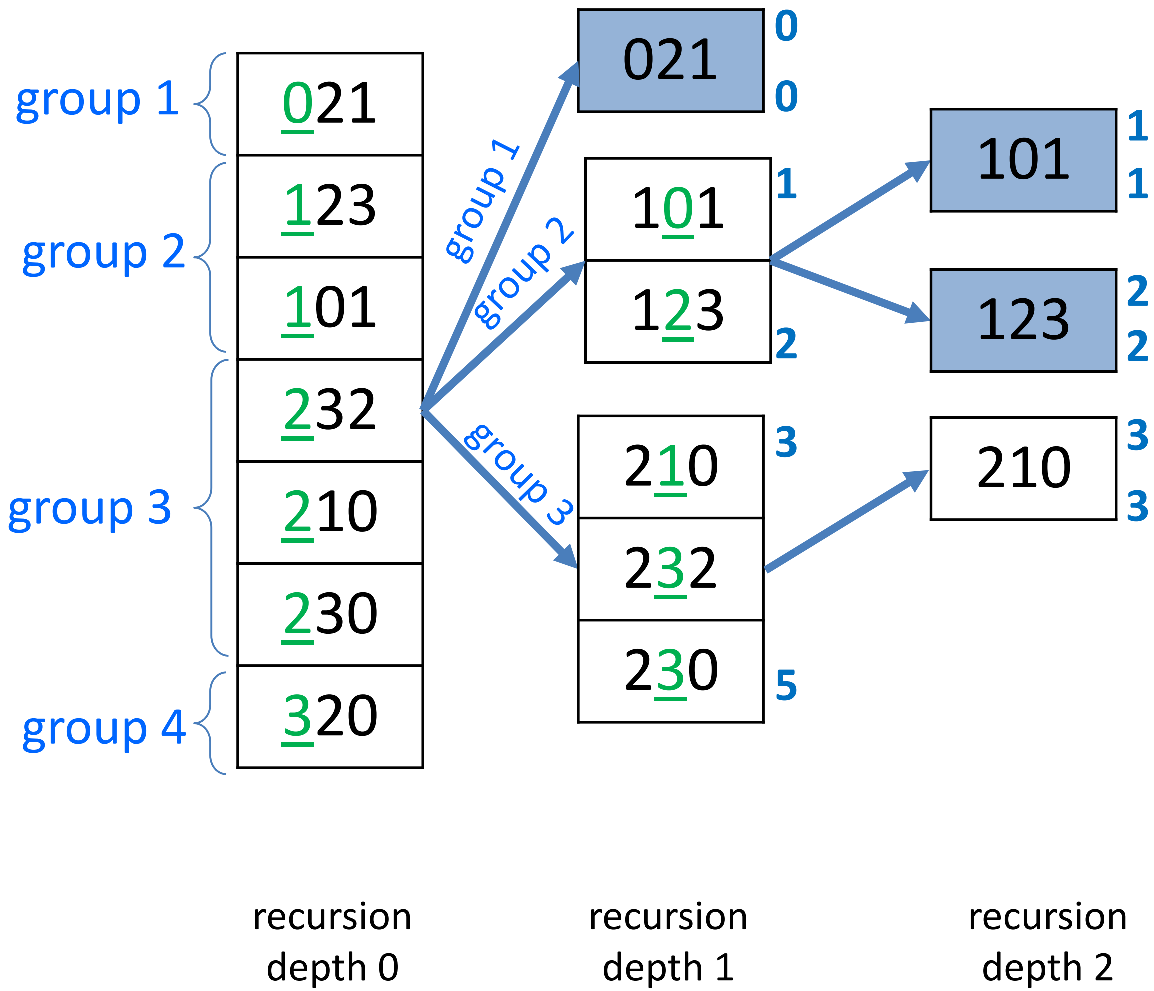

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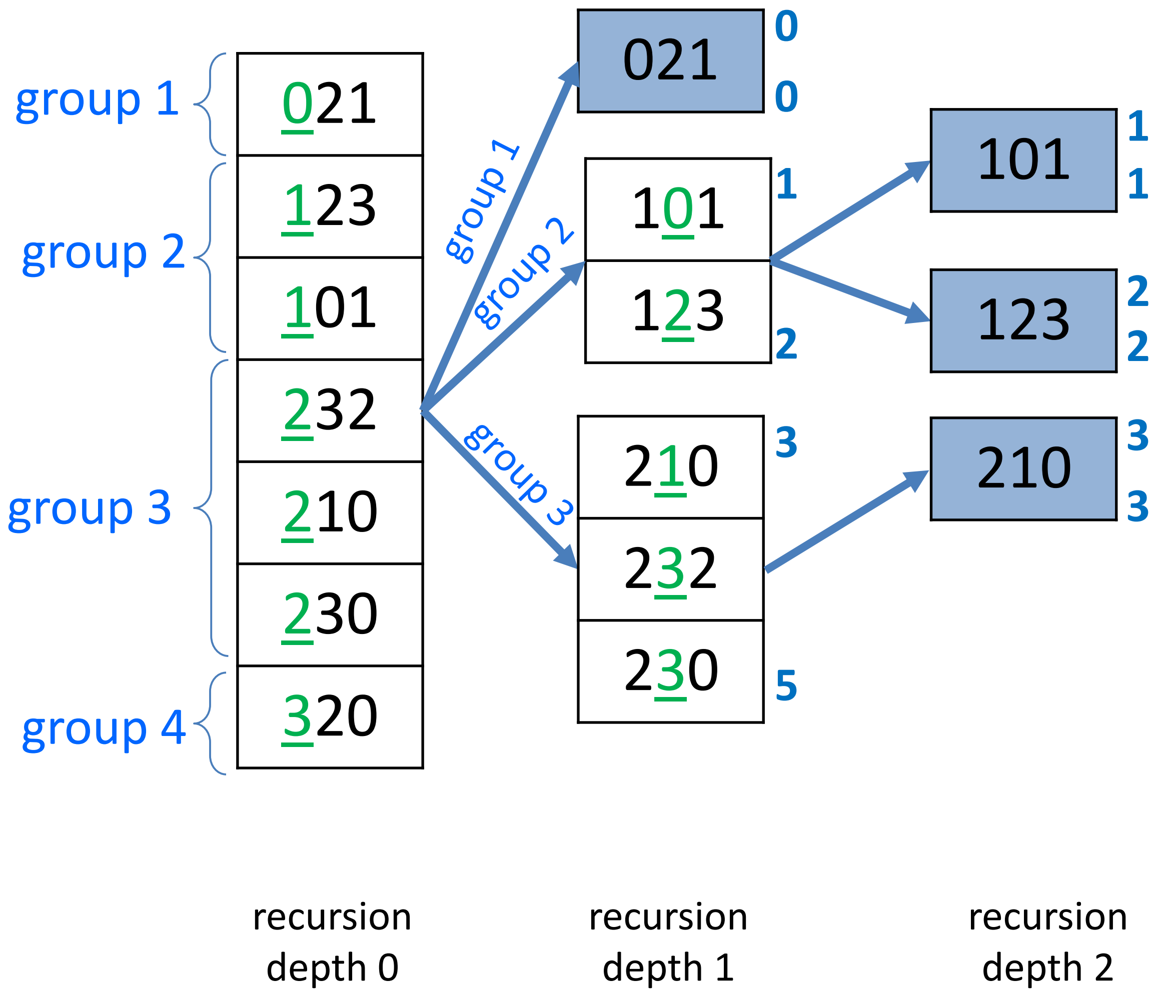

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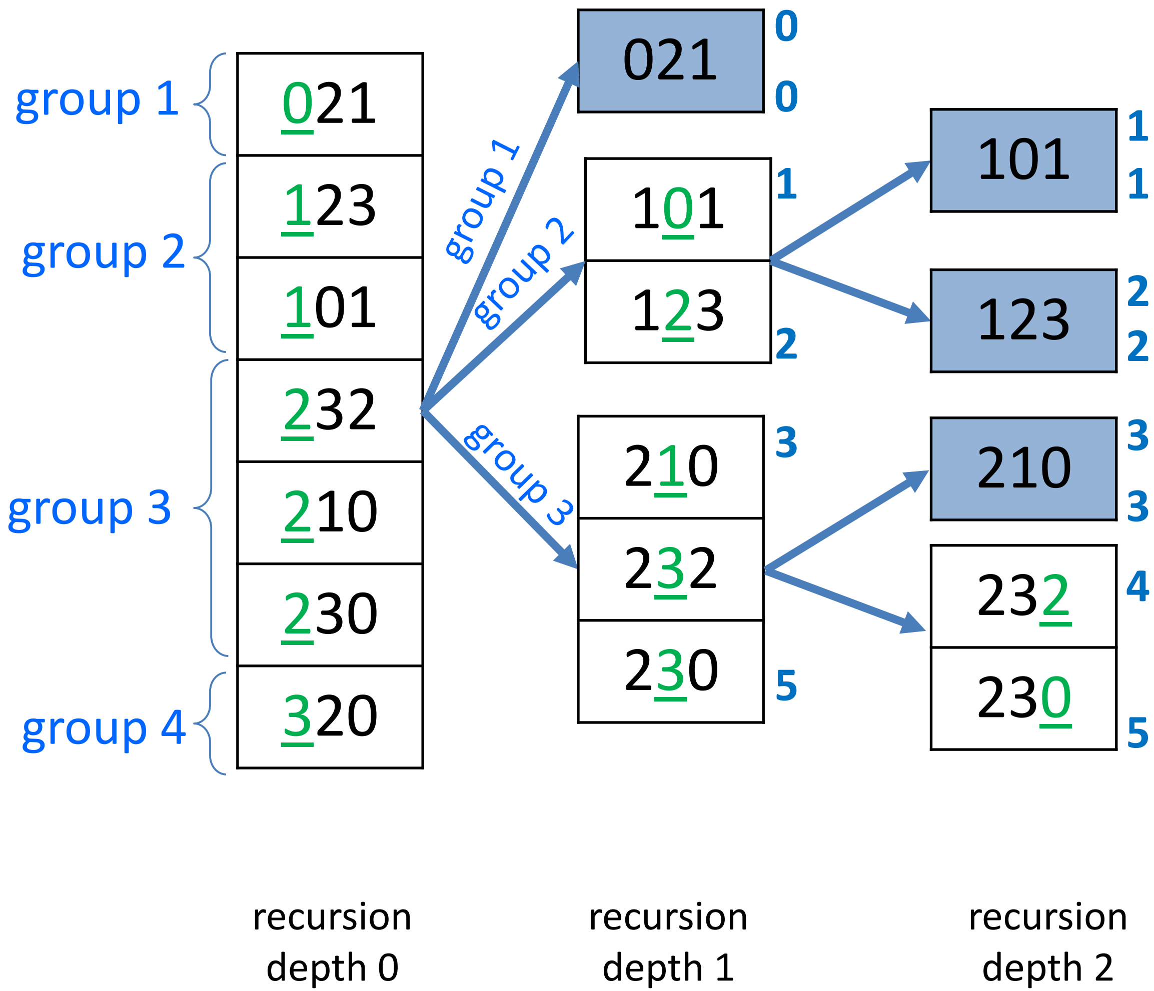

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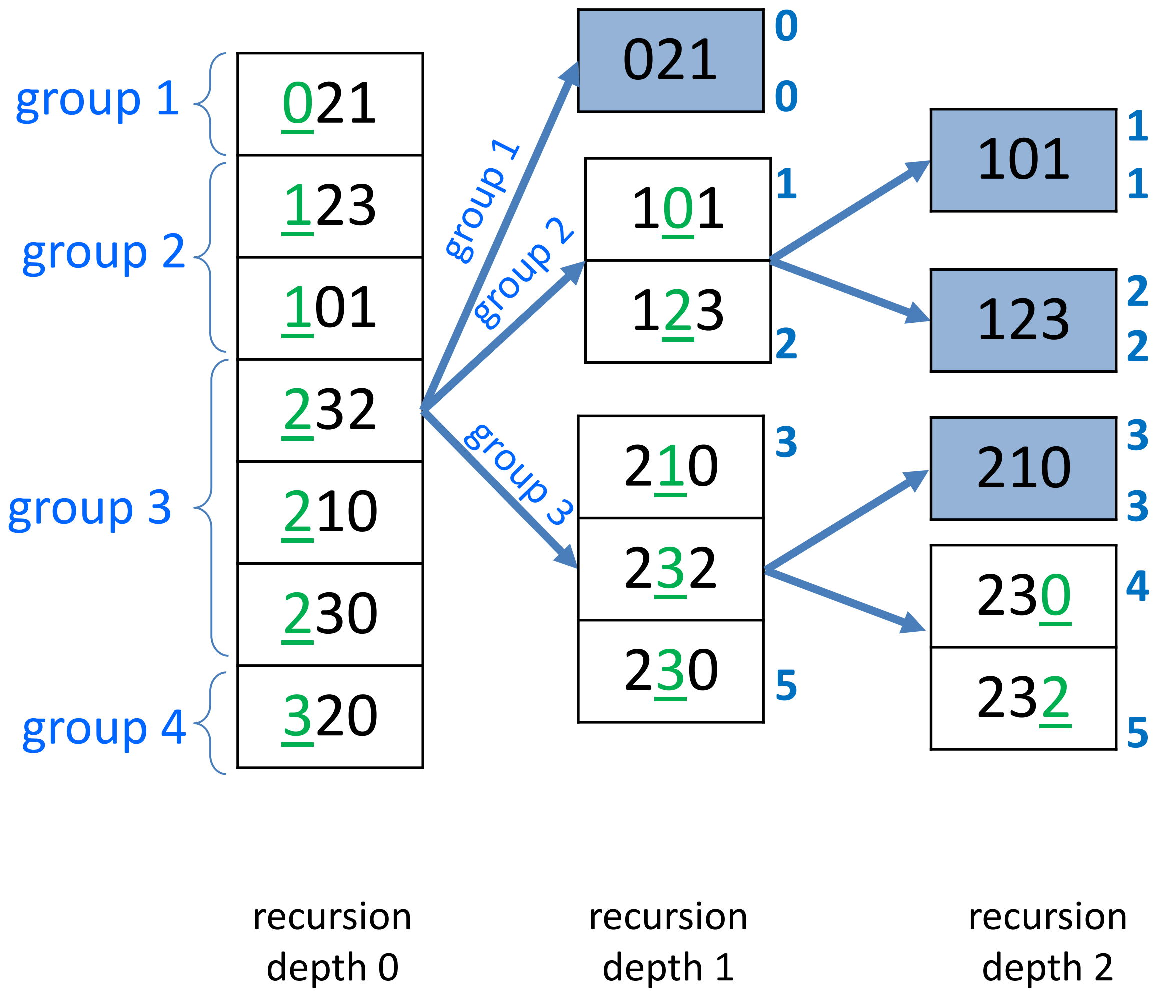

MSD-Radix-Sort Pseudocode

- Sorts array of m-digit radix-R numbers recursively
- Sort by leading digit, then each group by next digit, etc.

MSD-Radix-sort$(A,\ l \leftarrow 0, r \leftarrow n-1, d \leftarrow leading\ digit\ index)$

l, r : indexes between which to sort, $0 \leq l, r \leq n-1$

    if $l < r$
        bucket-sort$(A\ [l \ldots r],\ d)$
        if there are digits left
            $l' \leftarrow l$
            while $(l' \leq r)$ **do**
                let $r' \geq l'$ be the maximal s.t $A\ [l' \ldots r']$ have the same dth digit
                MSD-Radix-sort$(A, l', r', d+1)$
                $l' \leftarrow r' + 1$

- Run-time $O(mnR)$
- Auxiliary space is $\Theta(m+n+R)$ for bucket sort and recursion stack
- Drawback of *MSD-Radix-sort* is many recursions

MSD-Radix-Sort

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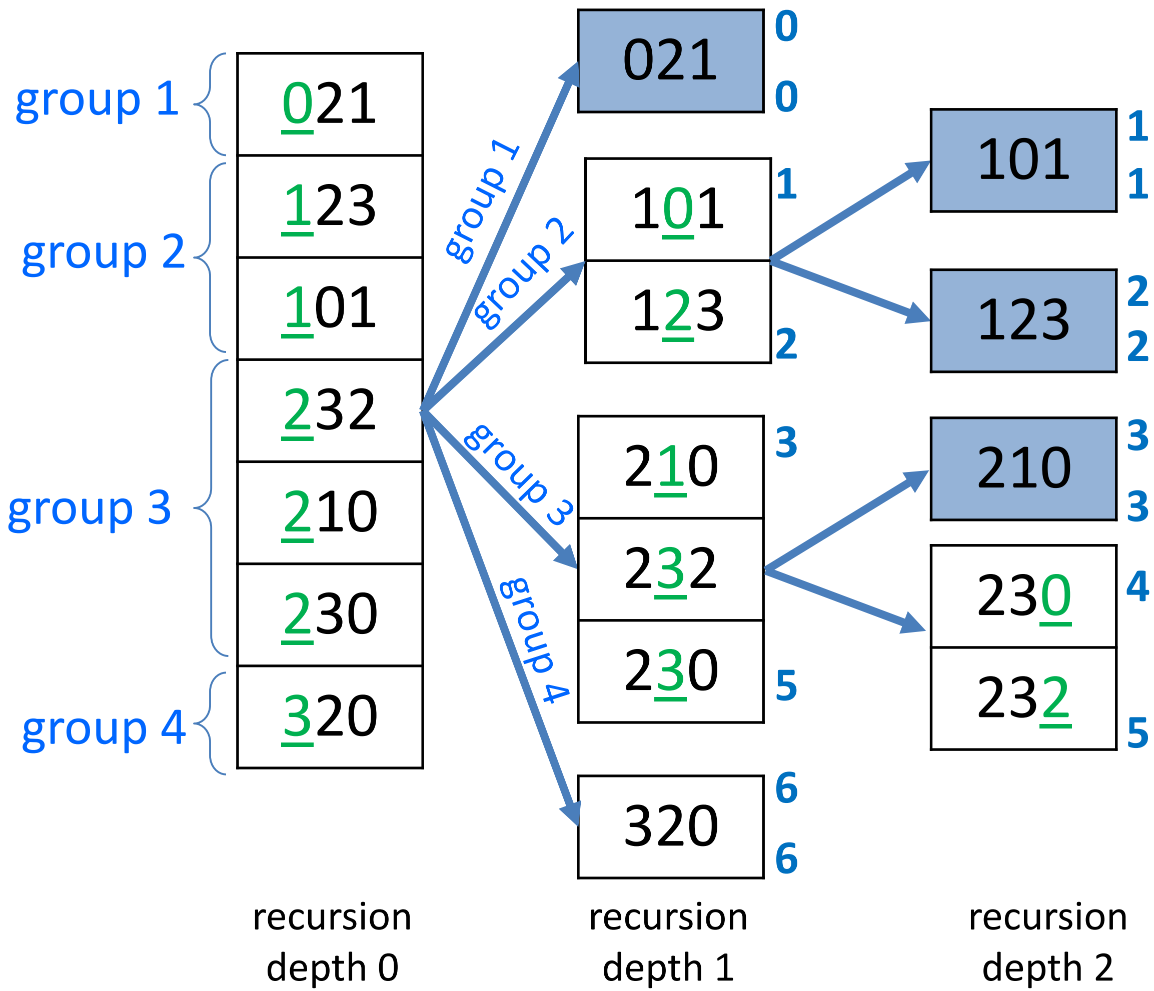

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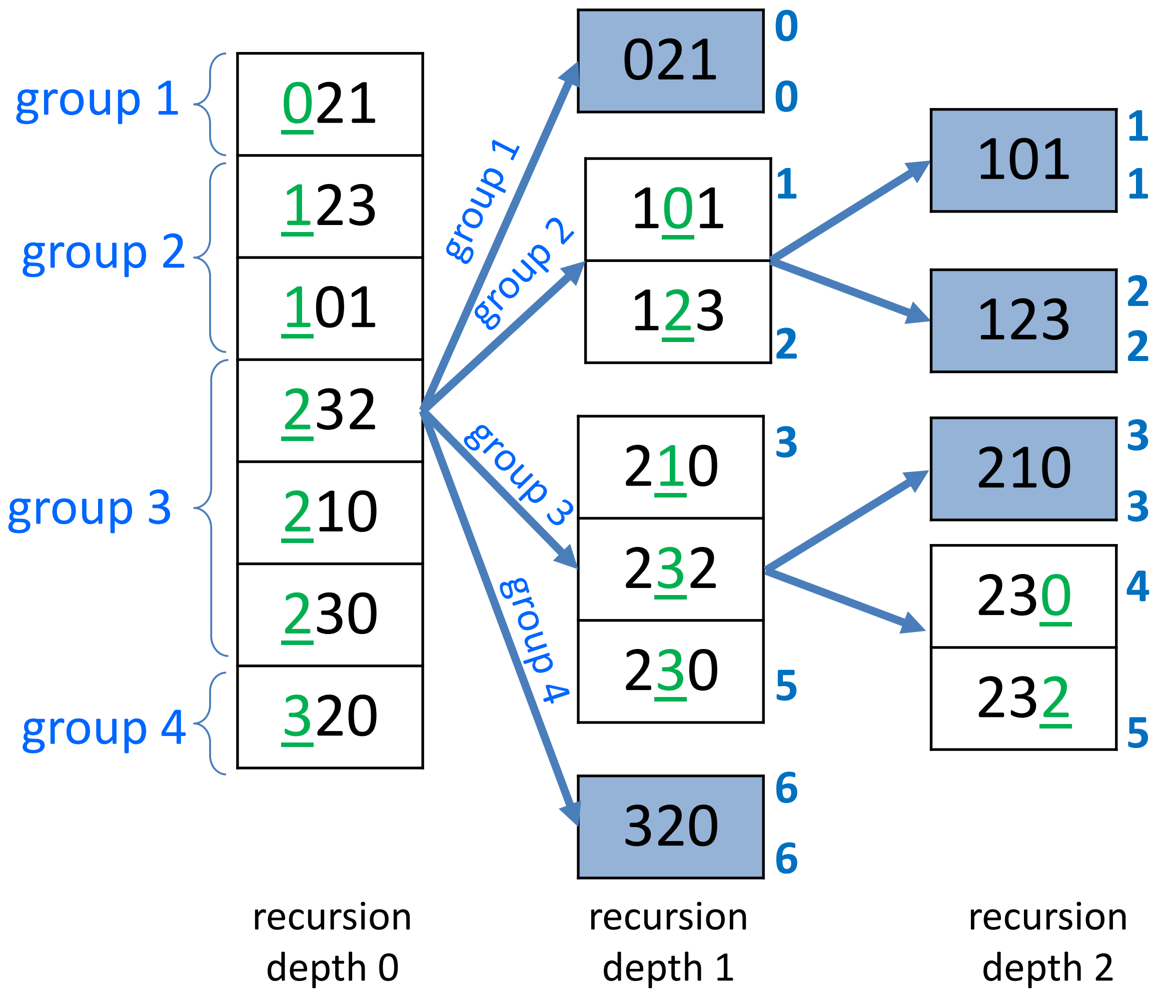
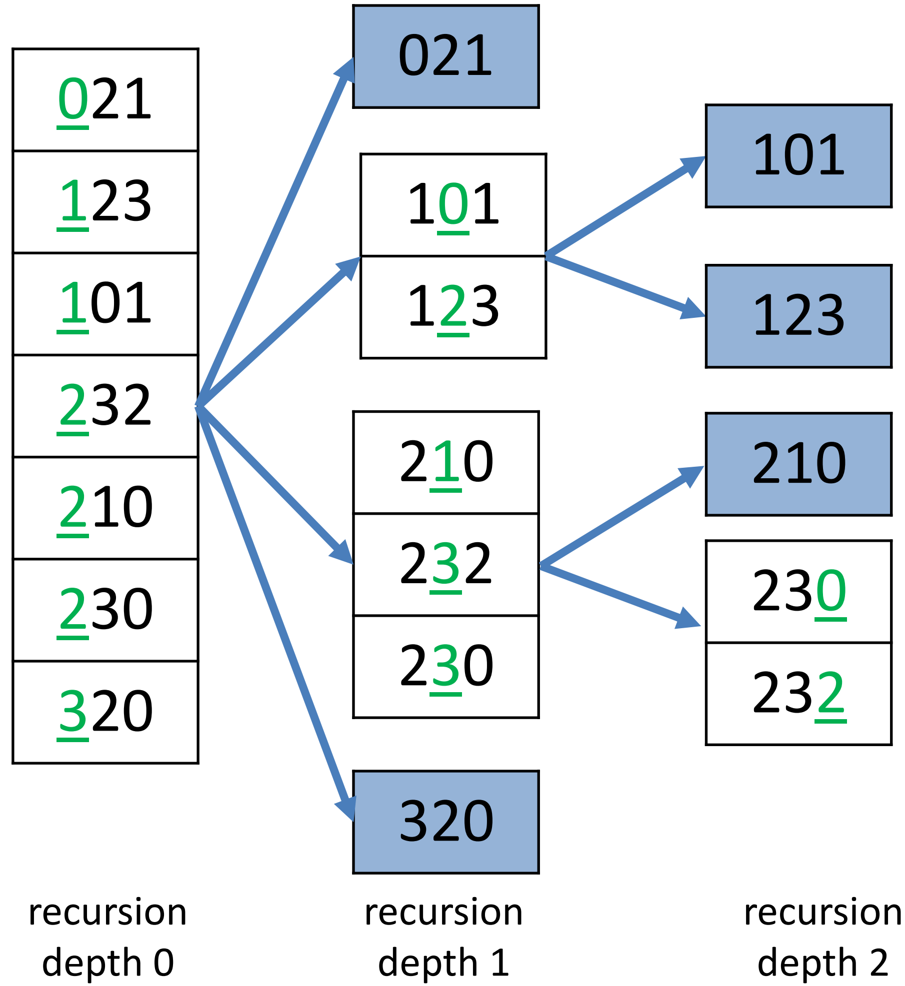


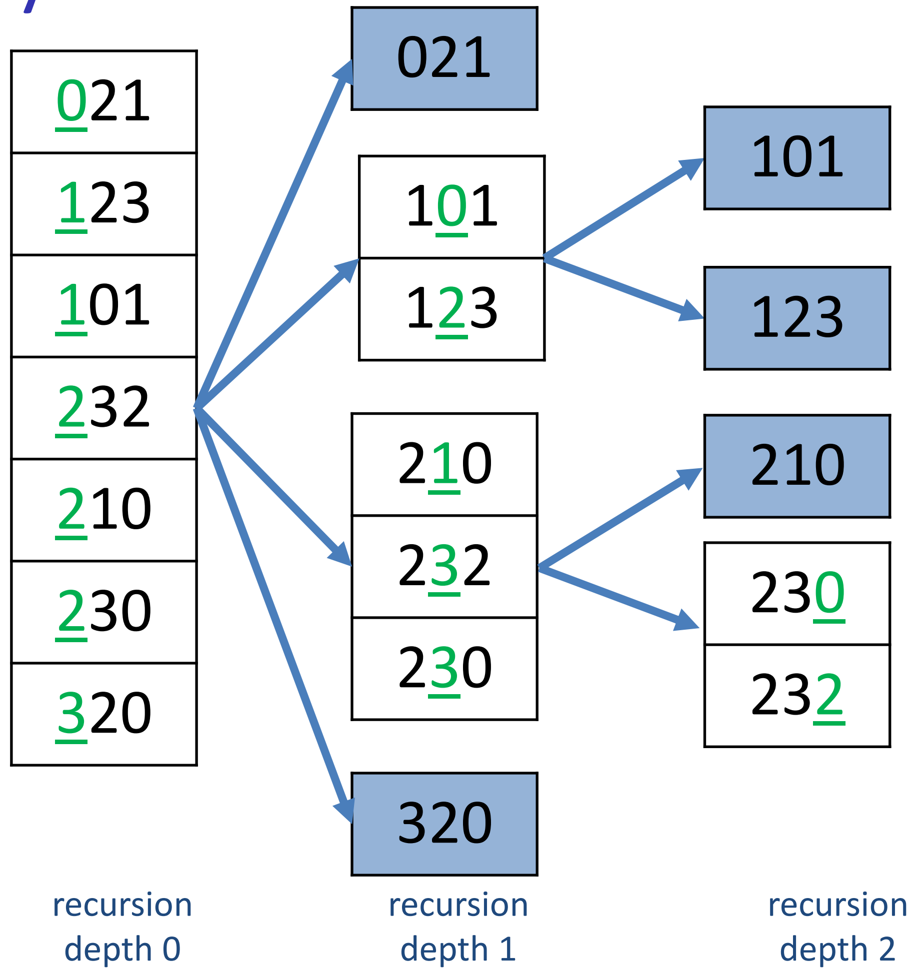
MSD-Radix-Sort Space Analysis

- Bucket-sort
  - auxiliary space $\Theta(n+R)$
- Recursion depth is $m-1$
  - auxiliary space $\Theta(m)$
- Total auxiliary space $\Theta(n+R+m)$



MSD-Radix-Sort Time Analysis

- Time spent for each recursion depth
  - Depth 0
    - one bucket sort on n items
    - $\Theta(n + R)$
  - All other depths
    - lets k be the number of bucket sorts at each depth
    - $k \leq n$
      - cannot have more bucket sorts than the array size
    - each bucket sort is on n_i items
    - $\sum_{i=0}^{k} n_i = n$
    - each bucket sort is $n_i + R$
    - $\sum_{i=0}^{k}(n_i + R) = n + \sum_{i=0}^{k} R \leq n + nR$
    - total time at any depth is $O(nR)$
- Number of depths is at most $m - 1$
- Total time $O(mnR)$



MSD-Radix-Sort Time Analysis

- Total time $O(mnR)$
- This is $O(n)$ if sort items in limited range
  - suppose $R = 2$, and we sort are n integers in the range $[0, 2^{10})$
  - then $m = 10$, $R = 2$, and sorting is $O(n)$
    - note that n, the number of items to sort, can be arbitrarily large

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  - then $m = 10$, $R = 2$, and sorting is $O(n)$
    - note that n, the number of items to sort, can be arbitrarily large
- This does not contradict $\Omega(n \log n)$ bound on the sorting problem, since the bound applies to comparison-based sorting

LSD-Radix-Sort

- **Idea**: apply single digit bucket sort from least significant digit to the most significant digit
- Observe that digit bucket sort is stable
    - equal elements stay in the original order
    - therefore, we can apply single digit bucket sort to the **whole array**, and the output will be sorted after iterations over all digits

LSD-Radix-Sort

123	230	230	101	101	101
230	320	320	210	210	121
121	210	210	320	320	123
320	121	121	121	121	210
210	101	101	123	123	230
232	232	232	230	230	232
101	123	123	232	232	320
prepare to sort by last digit	last digit sorted	prepare to sort by middle digit	last two digits sorted	prepare to sort by first digit	last three digits sorted

- m bucket sorts, on n items each, one bucket sort is $\Theta(n + R)$
- Total time cost $\Theta(m(n + R))$

LSD-Radix-Sort

LSD-radix-sort(A)
A: array of size *n*, contains *m*-digit radix-*R* numbers
    for $d \leftarrow$ least significant **down to** most significant digit **do**
        bucket-sort(A, d)

- Loop invariant: after iteration i, A is sorted w.r.t. the last i digits of each entry
- Time cost $\Theta(m(n+R))$
- Auxiliary space $\Theta(n+R)$

Summary

- Sorting is an important and *very* well-studied problem
- Can be done in $\Theta(n \log n)$ time
  - faster is not possible for general input
- HeapSort is the only $\Theta(n \log n)$ time algorithm we have seen with $O(1)$ auxiliary space
- MergeSort is also $\Theta(n \log n)$ time
- Selection and insertion sorts are $\Theta(n^2)$
- QuickSort is worst-case $\Theta(n^2)$, but often the fastest in practice
- BucketSort and RadixSort can achieve $o(n \log n)$ if the input is special
- Best-case, worst-case, average-case can all differ
- Randomized algorithms can eliminate "bad cases", resulting in the same expected time for all cases