CS 240 – Data Structures and Data Management

Module 5: Other Dictionary Implementations

A. Jamshidpey   N. Nasr Esfahani   M. Petrick
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Spring 2022

version 2022-06-06 15:09

Outline

5. Dictionaries with Lists revisited
   - Dictionary ADT: Implementations thus far
   - Skip Lists
   - Re-ordering Items

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Dictionary ADT: Implementations thus far

A *dictionary* is a collection of key-value pairs (KVPs), supporting operations *search*, *insert*, and *delete*.

Realizations we have seen so far:

- **Unordered array or linked list**: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- **Ordered array**: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- **Binary search trees**: $\Theta(\mathit{height})$ search, insert and delete
- **Balanced BST** (AVL trees):
  $\Theta(\log n)$ search, insert, and delete

Improvements/Simplifications?

- **Can show:** If the KVPs were inserted in random order, then the expected height of the binary search tree would be $O(\log n)$.
- How can we use randomization within the data structure to mirror what would happen on random input?

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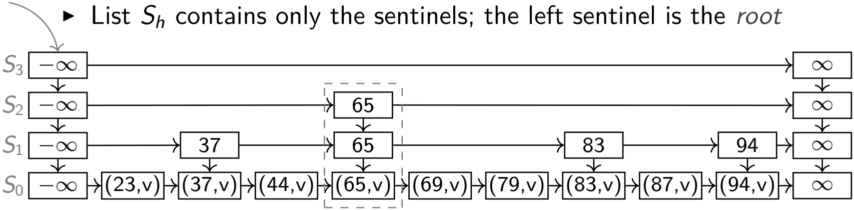
Skip Lists

- A hierarchy S of ordered linked lists (*levels*) $S_0, S_1, \cdots, S_h$:
  - Each list S_i contains the special keys $-\infty$ and $+\infty$ (sentinels)
  - List S_0 contains the KVPs of S in non-decreasing order. (The other lists store only keys, or links to nodes in S_0.)
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List S_h contains only the sentinels; the left sentinel is the *root*



- Each KVP belongs to a **tower** of nodes
- There are (usually) more *nodes* than *keys*
- The skip list consists of a reference to the topmost left node.
- Each node p has references *p.after* and *p.below*

Search in Skip Lists

For each level, find **predecessor** (node before where k would be).
This will also be useful for *insert*/*delete*.

getPredecessors (k)

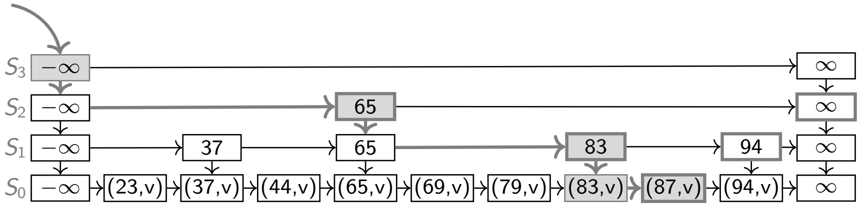
1. $p \leftarrow$ root
2. $P \leftarrow$ stack of nodes, initially containing p
3. **while** $p.below \neq$ `NIL` **do**
4. $p \leftarrow p.below$
5.   **while** $p.after.key < k$ **do** $p \leftarrow p.after$
6. $P.push(p)$
7. **return** P

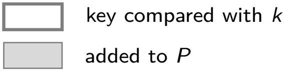
skipList::search (k)

1. $P \leftarrow$ *getPredecessors*(k)
2. $p_0 \leftarrow P.top()$ // predecessor of k in S_0
3. **if** $p_0.after.key = k$ **return** $p_0.after$
4. **else return** "not found, but would be after p_0"

Example: Search in Skip Lists

Example: *search*(87)





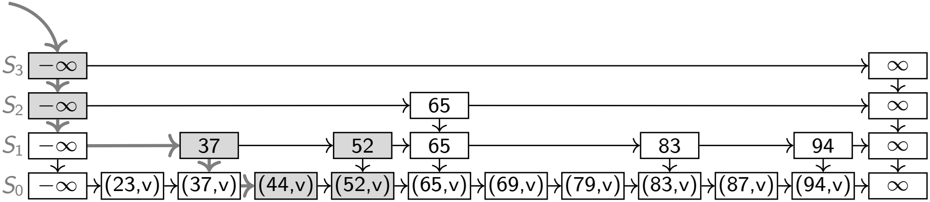
Insert in Skip Lists

skipList::insert(k, v)

- Randomly repeatedly toss a coin until you get tails
- Let i the number of times the coin came up heads
  - we want k to be in lists $S_0, \ldots, S_i$.
  - $i \rightarrow$ *height* of tower of k
  - $P(\text{tower of key } k \text{ has height} \geq i) = \left(\frac{1}{2}\right)^i$
- Increase height h of skip list, if needed, to have $h > i$ levels.
- Use *getPredecessors*(k) to get stack P.
  The top i items of P are the predecessors $p_0, p_1, \cdots, p_i$ of where k should be in each list $S_0, S_1, \cdots, S_i$
- Insert (k, v) after p_0 in S_0, and k after p_j in S_j for $1 \leq j \leq i$

Example: Insert in Skip Lists

Example: *skipList::insert*$(52, v)$
Coin tosses: H,T $\Rightarrow i = 1$
getPredecessors(52)



Example 2: Insert in Skip Lists

Example: *skipList::insert*$(100, v)$

Insert in Skip Lists

```
skipList::insert(k, v)
    P ← getPredecessors(k)
    for (i ← 0; random(2) = 1; i ← i+1) {}     // random tower height
    while i ≥ P.size()                          // increase skip-list height?
        root ← new sentinel-only list, linked in appropriately
        add left sentinel of root at bottom of stack P
    p ← P.pop()                                 // insert (k, v) in S_0
    z_below ← new node with (k, v), inserted after p
    while i > 0                                 // insert k in S_1, ..., S_i
        p ← P.pop()
        z ← new node with k added after p
        z.below ← z_below; z_below ← z
        i ← i − 1
```

Delete in Skip Lists

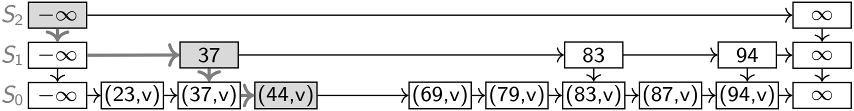
It is easy to remove a key since we can find all predecessors.
Then eliminate layers if there are multiple ones with only sentinels.

```
skipList::delete(k)
    P ← getPredecessors(k)
    while P is non-empty
        p ← P.pop()            // predecessor of k in some layer
        if p.after.key = k
            p.after ← p.after.after
        else break             // no more copies of k

    p ← left sentinel of the root-list
    while p.below.after is the ∞-sentinel
        // the two top lists are both only sentinels, remove one
        p.below ← p.below.below
        p.after.below ← p.after.below.below
```

Example: Delete in Skip Lists

Example: *skipList::delete*(65)

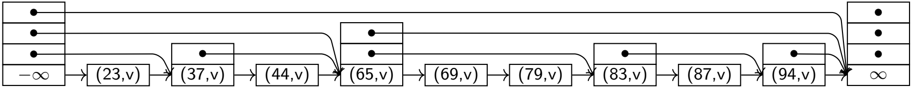


Analysis of Skip Lists

- Expected **space** usage: $O(n)$
- Expected **height**: $O(\log n)$
- Crucial for all operations:
  - How often do we *drop down* (execute $p \leftarrow p.below$)?
  - How often do we *step forward* (execute $p \leftarrow p.after$)?
- *skipList::search*: $O(\log n)$ expected time
  - # drop-downs = height
  - expected # forward-steps is ≤ 1 in each level
  - expected total # forward-steps is in $O(\log n)$
- *skipList::insert*: $O(\log n)$ expected time
- *skipList::delete*: $O(\log n)$ expected time

Summary of Skip Lists

- $O(n)$ expected space, all operations take $O(\log n)$ expected time.
- As described they are no faster than randomized binary search trees.
- Can show: A biased coin-flip to determine tower-height gives smaller expected run-times.
- Can save links (hence space) by implementing towers as array.



- Then skip lists are fast in practice and simple to implement.

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Re-ordering Items

- Recall: Unordered list/array implementation of ADT Dictionary
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Lists/arrays are a very simple and popular implementation. Can we do something to make search more effective in practice?

- No: if items are accessed equally likely
- Yes: otherwise (we have a probability distribution of the items)
  - Intuition: Frequently accessed items should be in the front.
  - Two cases: Do we know the access distribution beforehand or not?

  - For short lists or extremely unbalanced distributions this may be faster than AVL trees or Skip Lists, and much easier to implement.

Optimal Static Ordering

Example:

key	A	B	C	D	E
frequency of access	2	8	1	10	5
access-probability	$\frac{2}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{5}{26}$

- We count cost i for accessing the key in the ith position.
- Order A, B, C, D, E has expected access cost
  $\frac{2}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{1}{26} \cdot 3 + \frac{10}{26} \cdot 4 + \frac{5}{26} \cdot 5 = \frac{86}{26} \approx 3.31$
- Order D, B, E, A, C has expected access cost
  $\frac{10}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{5}{26} \cdot 3 + \frac{2}{26} \cdot 4 + \frac{1}{26} \cdot 5 = \frac{66}{26} \approx 2.54$

- **Claim:** Over all possible static orderings, the one that sorts items by non-increasing access-probability minimizes the expected access cost.
- **Proof Idea:** For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Dynamic Ordering: MTF

- What if we do *not know the access probabilities* ahead of time?
- Rule of thumb (**temporal locality**): A recently accessed item is likely to be used soon again.
- In list: Always insert at the front
- **Move-To-Front heuristic** (MTF): Upon a successful search, move the accessed item to the front of the list

$$A \to B \to C \to D \to E$$

$\downarrow$ search(D)

$$D \to A \to B \to C \to E$$

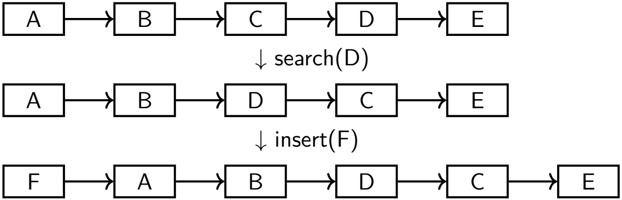
$\downarrow$ insert(F)

$$F \to D \to A \to B \to C \to E$$

- We can also do MTF on an array, but should then insert and search from the *back* so that we have room to grow.

Dynamic Ordering: Transpose

Transpose heuristic: Upon a successful search, swap the accessed item with the item immediately preceding it



Performance of dynamic ordering:

- Transpose does not adapt quickly to changing access patterns.
- MTF works well in practice.
- **Can show:** MTF is “2-competitive”: No more than twice as bad as the optimal static ordering.