CS 240 – Data Structures and Data Management

Module 7: Dictionaries via Hashing

Simon Abélard   Ahmad Biniaz   Éric Schost
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Spring 2019

References: Sedgewick 12.2, 14.1-4
Goodrich & Tamassia 6.4

version 2019–06–12 08:56

Outline

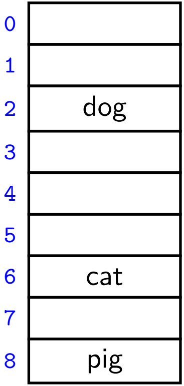
1. Dictionaries via Hashing
   - Hashing Introduction
   - Separate Chaining
   - Probe Sequences
   - Cuckoo hashing
   - Hash Function Strategies

Outline

Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.



- *search*(k): Check whether $A[k]$ is `NIL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NIL`

Each operation is $\Theta(1)$.
Total space is $\Theta(M)$.

What sorting algorithm does this remind you of?

Hashing

Two disadvantages of direct addressing:

- It cannot be used if the keys are not integers.
- It wastes space if M is unknown or $n \ll M$.

Hashing idea: Map (arbitrary) keys to integers in range $\{0, \ldots, M-1\}$ and then use direct addressing.

Details:

- **Assumption:** We know that all keys come from some **universe** U. (Typically $U = \mathbb{N}$.)
- We design a **hash function** $h : U \to \{0, 1, \ldots, M-1\}$. (Commonly used: $h(k) = k \bmod M$. We will see other choices later.)
- Store dictionary in **hash table**, i.e., an array T of size M.
- An item with key k should ideally be stored in **slot** $h(k)$, i.e., at $T[h(k)]$.

Hashing example

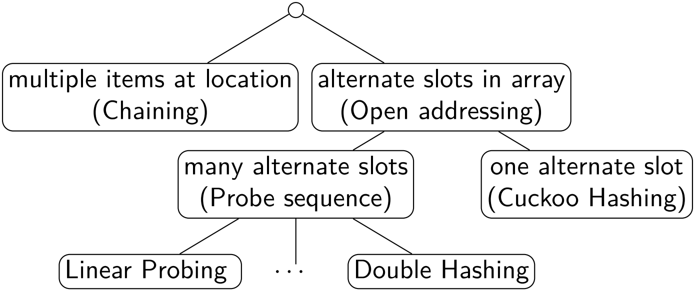
$U = \mathbb{N}$, $M = 11$, $\quad h(k) = k \bmod 11$.

The hash table stores keys 7, 13, 43, 45, 49, 92. (Values are not shown).

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Collisions

- Generally hash function h is not injective, so many keys can map to the same integer.
  - For example, $h(46) = 2 = h(13)$ if $h(k) = k \bmod 11$.
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.
- There are many strategies to resolve collisions:



Outline

Separate Chaining

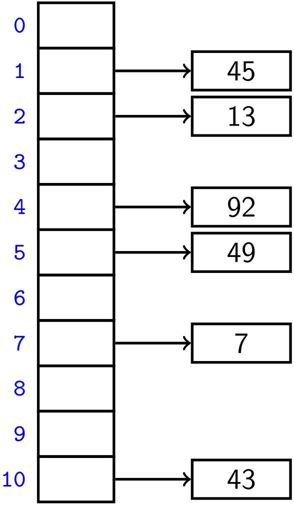
Simplest collision-resolution strategy: Each slot stores a **bucket** containing 0 or more KVPs.

- A bucket could be implemented by any dictionary realization (even another hash table!).
- The simplest approach is to use unsorted linked lists for buckets. This is called collision resolution by **separate chaining**.

- *search*(k): Look for key k in the list at $T[h(k)]$. Apply MTF-heuristic!
- *insert*(k, v): Add (k, v) to the front of the list at $T[h(k)]$.
- *delete*(k): Perform a search, then delete from the linked list.

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$



Complexity of chaining

Run-times: *insert* takes time $O(1)$.
search and *delete* have run-time $O(1 + \text{size of bucket } T(h(k)))$.

- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)
- However, this does not imply that the *average-case* cost of *search* and *delete* is $O(1 + \alpha)$.
  (If all keys hash to the same slot, then the average bucket-size is still α, but the operations take time $\Theta(n)$ on average.)
- **Uniform Hashing Assumption**: Each hash function value is equally likely.
  (This depends on the input and how we choose the function $\rightsquigarrow$ later.)
- Under this assumption, each key collides is expected to collide with $\frac{n-1}{M}$ other keys and the average-case cost of *search* and *delete* is hence $O(1 + \alpha)$.

Load factor and re-hashing

- For all collision resolution strategies, the run-time evaluation is done in terms of the *load factor* $\alpha = n/M$.
- We keep the load factor small by **rehashing** when needed:
  - Keep track of n and M throughout operations
  - If α gets too large, create new (twice as big) hash-table, new hash-function(s) and re-insert all items in the new table.
- Rehashing costs $\Theta(M + n)$ but happens rarely enough that we can ignore this term when amortizing over all operations.
- We should also re-hash when α gets too small, so that $M \in \Theta(n)$ throughout, and the space is always $\Theta(n)$.

Summary: If we maintain $\alpha \in \Theta(1)$, then (under the uniform hashing assumption) the average cost for hashing with chaining is $O(1)$ and the space is $\Theta(n)$.

Outline

1. Dictionaries via Hashing
   - Hashing Introduction
   - Separate Chaining
   - Probe Sequences
   - Cuckoo hashing
   - Hash Function Strategies

Open addressing

Main idea: Avoid the links needed for chaining by permitting only one item per slot, but allowing a key k to be in multiple slots.

search and *insert* follow a **probe sequence** of possible locations for key k: $\langle h(k,0), h(k,1), h(k,2), \ldots \rangle$ until an empty spot is found.

delete becomes problematic:

- Cannot leave an empty spot behind; the next search might otherwise not go far enough.
- Idea 1: Move later items in the probe sequence forward.
- Idea 2: **lazy deletion**: Mark spot as *deleted* (rather than `NIL`) and continue searching past deleted spots.

Simplest method for open addressing: *linear probing*
$h(k,i) = (h(k) + i) \bmod M$, for some hash function h.

Linear probing example

$M = 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Probe sequence operations

```
probe-sequence::insert(T, (k, v))
    for (j = 0; j < M; j++)
        if T[h(k, j)] is NIL or "deleted"
            T[h(k, j)] = (k, v)
            return "success"
    return "failure to insert"        // need to re-hash
```

```
probe-sequence-search(T, k)
    for (j = 0; j < M; j++)
        if T[h(k, j)] is NIL
            return "item not found"
        else if T[h(k, j)] has key k
            return T[h(k, j)]
        // ignore "deleted" and keep searching
    return "item not found"
```

Independent hash functions

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
- Using two modular hash-functions may often lead to dependencies.
- Better idea: Use *multiplicative method* for second hash function: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
  - A is some floating-point number
  - $kA - \lfloor kA \rfloor$ computes fractional part of kA, which is in $[0, 1)$
  - Multiply with M to get floating-point number in $[0, M)$
  - Round down to get integer in $\{0, \ldots, M-1\}$

  Knuth suggests $A = \varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$.

Double Hashing

- Assume we have two hash independent functions h_0, h_1.
- Assume further that $h_1(k) \neq 0$ and that $h_1(k)$ is relative prime with the table-size M for all keys k.
  - Choose M prime.
  - Modify standard hash-functions to ensure $h_1(k) \neq 0$
    E.g. modified multiplication method: $h(k) = 1 + \lfloor (M-1)(kA - \lfloor kA \rfloor) \rfloor$
- **Double hashing**: open addressing with probe sequence

$$h(k, i) = h_0(k) + i \cdot h_1(k) \bmod M$$

- *search*, *insert*, *delete* work just like for linear probing, but with this different probe sequence.

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Outline

Cuckoo hashing

We use two independent hash functions h_0, h_1 and two tables T_0, T_1.

Main idea: An item with key k can *only* be at $T_0[h_0(k)]$ or $T_1[h_1(k)]$.

- *search* and *delete* then take constant time.
- *insert* *always* initially puts a new item into $T_0[h_0(k)]$

  If $T_0[h_0(k)]$ is occupied: "kick out" the other item, which we then attempt to re-insert into its alternate position $T_1[h_1(k)]$

  This may lead to a loop of "kicking out". We detect this by aborting after too many attempts.

  In case of failure: rehash with a larger M and new hash functions.

insert may be slow, but is expected to be constant time if the load factor is small enough.

Cuckoo hashing insertion

```
cuckoo::insert(k, v)
    i ← 0
    do at most 2n times:
        if T_i[h_i(k)] is NIL
            T_i[h_i(k)] ← (k, v)
            return "success"
        swap((k, v), T_i[h_i(k)])
        i ← 1 − i
    return "failure to insert"        // need to re-hash
```

After $2n$ iterations, there definitely was a loop in the "kicking out" sequence (why?)

In practice, one would stop the iterations much earlier already.

Cuckoo hashing example

$M = 11$, $\quad h_0(k) = k \bmod 11$, $\quad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

	T_0		T_1
0	44	0	
1		1	
2		2	
3		3	
4	59	4	
5		5	
6		6	
7		7	
8		8	
9	92	9	
10		10	

Cuckoo hashing discussions

- The two hash-tables need not be of the same size.
- *Load factor* $\alpha = n/(\text{size of } T_0 + \text{size of } T_1)$
- One can argue: If the load factor α is small enough then insertion has $O(1)$ expected run-time.
- This crucially requires $\alpha < \frac{1}{2}$.

There are many possible variations:

- The two hash-tables could be combined into one.
- Be more flexible when inserting: Always consider both possible positions.
- Use $k > 2$ allowed locations (i.e., k hash-functions).

Complexity of open addressing strategies

For any open addressing scheme, we *must* have $\alpha < 1$ (why?).
Cuckoo hashing requires $\alpha < 1/2$.

Avg.-case costs:	*search (unsuccessful)*	*insert*	*search (successful)*
Linear Probing	$\dfrac{1}{(1-\alpha)^2}$	$\dfrac{1}{(1-\alpha)^2}$	$\dfrac{1}{1-\alpha}$
Double Hashing	$\dfrac{1}{1-\alpha}$	$\dfrac{1}{1-\alpha}$	$\dfrac{1}{\alpha}\log\left(\dfrac{1}{1-\alpha}\right)$
Cuckoo Hashing	1 (worst-case)	$\dfrac{\alpha}{(1-2\alpha)^2}$	1 (worst-case)

Summary: All operations have $O(1)$ average-case run-time if the hash-function is uniform and α is kept sufficiently small.
But worst-case run-time is (usually) $\Theta(n)$.

Outline

1. Dictionaries via Hashing
   - Hashing Introduction
   - Separate Chaining
   - Probe Sequences
   - Cuckoo hashing
   - Hash Function Strategies

Choosing a good hash function

- **Goal:** Satisfy uniform hashing assumption (each hash-index is equally likely)
- Proving this is usually impossible, as it requires knowledge of the input distribution and the hash function distribution.
- We can get good performance by choosing a hash-function that is
  - unrelated to any possible patterns in the data, and
  - depends on all parts of the key.
- We saw two basic methods for integer keys:
  - **Modular method**: $h(k) = k \bmod M$.
    We should choose M to be a prime.
  - **Multiplicative method**: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
    for some constant floating-point number A with $0 < A < 1$.

Universal Hashing

Every hash function *must* do badly for some sequences of inputs:

- If the universe contains at least $M \cdot n$ keys, then there are n keys that all hash to the same value.
- For this set of keys, we have the worst case.

Idea: Randomization!

- When initializing or re-hashing, use as hash function

$$h(k) = \big((ak + b) \bmod p\big) \bmod M$$

  where $p > M$ is a prime number, and a, b are *random* numbers in $\{0, \ldots p-1\}$, $a \neq 0$.
- Can prove: For any (fixed) numbers $x \neq y$, the probability of a collision using this random function h is at most $\frac{1}{M}$.
- Therefore the expected run-time is $O(1)$ if α is kept small enough.

We have again shifted the performance from "bad input" to "bad luck".

Multi-dimensional Data

What if the keys are multi-dimensional, such as strings in Σ^*?

Standard approach is to *flatten* string w to integer $f(w) \in \mathbb{N}$, e.g.

$$\begin{aligned} A \cdot P \cdot P \cdot L \cdot E \quad &\rightarrow \quad (65, 80, 80, 76, 69) \quad \text{(ASCII)} \\ &\rightarrow \quad 65R^4 + 80R^3 + 80R^2 + 76R^1 + 68R^0 \\ &\qquad \text{(for some radix } R\text{, e.g. } R = 255) \end{aligned}$$

We combine this with a modular hash function: $h(w) = f(w) \bmod M$

To compute this in $O(|w|)$ time without overflow, use Horner's rule and apply mod early. For exampe, $h(APPLE)$ is

$$\left(\left(\left(\left(\left(\left((65R+80) \bmod M\right)R+80\right) \bmod M\right)R+76\right) \bmod M\right)R+69\right) \bmod M$$

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- Predictable space usage (exactly n nodes)
- Never need to rebuild the entire structure
- Supports ordered dictionary operations (rank, select etc.)

Advantages of Hash Tables

- $O(1)$ operations (if hashes well-spread and load factor small)
- We can choose space-time tradeoff via load factor
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete