University of Waterloo
CS240 - Spring 2023
Assignment 1

Due Date: Wednesday May 24, 5pm

The integrity of the grade you receive in this course is very important to you and the University of Waterloo. You should read and sign the Academic Integrity Declaration before you start working on this assignment and submit it **before the deadline of May 17**; i.e., **you should have submitted AID01 by the due date of this assignment**. The agreement will indicate what you must do to ensure the integrity of your grade. If you are having difficulties with the assignment, course staff are there to help (provided it isn't last minute).

The Academic Integrity Declaration must be signed and submitted on time or the assessment will not be marked.

Please read `https://student.cs.uwaterloo.ca/~cs240/s23/assignments.phtml#guidelines` for guidelines on submission. **Each question must be submitted individually to MarkUs as a PDF** with the corresponding file names: a1q1.pdf, a1q2.pdf, ... , a1q5.pdf . It is a good idea to submit questions as you go so you aren't trying to create several PDF files at the last minute.

Late Policy: Assignments are due at 5:00pm, with the grace period until 11:59pm. Assignments submitted after 11:59pm on the due date will not be accepted but may be reviewed (by request) for feedback purposes only.

Notes:

- Logarithms are in base 2, if not mentioned otherwise.

Question 1 [3+3+3+4+4 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

1. $5n^3 + n^2 + 10n + 7 \in O(n^3)$
2. $n^2(\log n)^{1.00001} \in \Omega(n^2)$
3. $\frac{n^2}{n+\log n} \in \Theta(n)$
4. $n \log(\log(n)) \in o(n(\log(\log(n)))^2)$
5. $2^n \in \omega(n^{1/n})$

Question 2 [3+3+3 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Provide a brief justification of your answers. In your justification you may use any relationship or technique that is described in class.

1. $f(n) = n\sqrt{n}$, $g(n) = n \log n$
2. $f(n) = 10^n + 99n^{10}$, $g(n) = 75^n + 25n^{27}$
3. $f(n) = n^4$, $g(n) = n^4(\log(n))^3$

Question 3 [3+3+3+3 marks]

Analyze the following pieces of pseudocode and give a tight (Θ) bound on the running time as a function of n. Show your work. A formal proof is not required, but you should justify your answer (in all cases, n is assumed to be a positive integer).

1.
```
x = 0
for i = 1 to n + 12 do
    x = x * 4
   for j = 389 to 20100
      for k = 2i to 3i
          x = x * 77
```

2.
```
x = 0
for i = 1 to ceiling(log(n))
   for j = 1 to i
      for k = 1 to 10
          x = x + 1
```

3.
```
s = 0;
for i = 1 to n*n do
   for j = 1 to i*i do
       s = s + 1;
```

4.
```
s = n
while (s > 0)
  if (s is even)
     s = floor(s / 4)
  else
     s = 2 * s
```

Question 4 [2+6 marks]

We consider two algorithms, Algo1 and Algo2, that solve the same problem. For any input of size n, Algo1 takes time $T_1(n)$ and Algo2 takes time $T_2(n)$. Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, provide a counter example and explain it.

1. Suppose that $T_1(n) \in O(n^2(\log n)^5)$ and $T_2(n) \in O(n^3(\log n)^2)$. Does it imply that there exists n_0 such that for $n \geq n_0$, Algo1 runs faster than Algo2 on inputs of size n?
2. Same question, assuming that $T_1(n) \in \Theta(n^2)$ and $T_2(n) \in \Theta(n^3)$.

Question 5 [5+5 marks]

True or false? For each of the following assertions, indicate whether it is true or false. If true, prove it; if false, give a counter-example and briefly justify it. In both cases, f and g are functions that take positive values, with $\lim_{n\to\infty} f(n)$ and $\lim_{n\to\infty} g(n) = \infty$.

1. If $f(n)$ is $\Theta(g(n))$ and $h(n) \in \Theta(g(n))$, then $\frac{f(n)}{h(n)} \in \Theta(1)$.
2. If $f(n)$ is $\Theta(g(n))$, then $\log(f(n)) \in \Theta(\log(g(n)))$.