CS 240 – Data Structures and Data Management

Module 7: Dictionaries via Hashing

Leili Rafiee Sevyeri   Éric Schost
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Outline

7 Dictionaries via Hashing

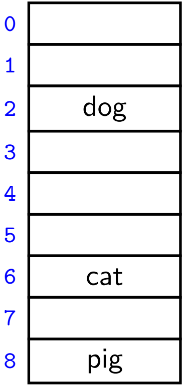
- Hashing Introduction
- Hashing with Chaining
- Probe Sequences
- Cuckoo hashing
- Hash Function Strategies

Outline

Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.



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- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NIL`

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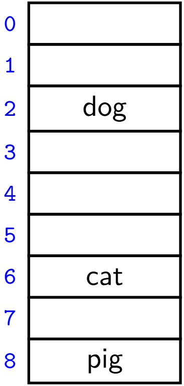

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Total space is $\Theta(M)$.

What sorting algorithm does this remind you of?

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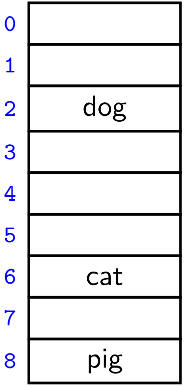

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Bucket Sort

Hashing

Two disadvantages of direct addressing:

- It cannot be used if the keys are not integers.
- It wastes space if M is unknown or $n \ll M$.

Hashing idea: Map (arbitrary) keys to integers in range $\{0, \ldots, M-1\}$ and then use direct addressing.

Details:

- **Assumption:** We know that all keys come from some **universe** U. (Typically $U =$ non-negative integers, sometimes $|U|$ finite.)
- We design a **hash function** $h : U \to \{0, 1, \ldots, M-1\}$. (Commonly used: $h(k) = k \bmod M$. We will see other choices later.)
- Store dictionary in **hash table**, i.e., an array T of size M.
- An item with key k should ideally be stored in **slot** $h(k)$, i.e., at $T[h(k)]$.

Hashing example

$U = \mathbb{N}$, $M = 11$, $\qquad h(k) = k \bmod 11$.

The hash table stores keys 7, 13, 43, 45, 49, 92. (Values are not shown).

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Collisions

- Generally hash function h is not injective, so many keys can map to the same integer.
  - For example, $h(46) = 2 = h(13)$ if $h(k) = k \bmod 11$.
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.

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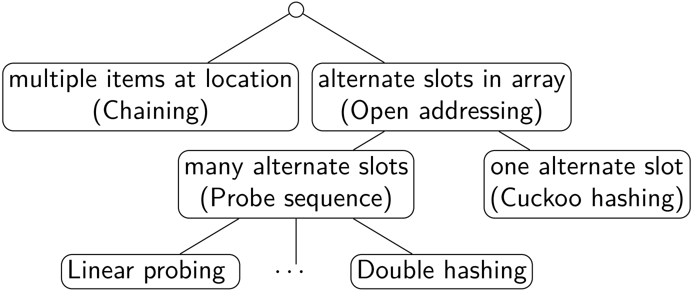
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.
- There are many strategies to resolve collisions:



Outline

Hashing with Chaining

Simplest collision-resolution strategy: Each slot stores a **bucket** containing 0 or more KVPs.

- A bucket could be implemented by any dictionary realization (even another hash table!).
- The simplest approach is to use unsorted linked lists for buckets. This is called collision resolution by **chaining**.

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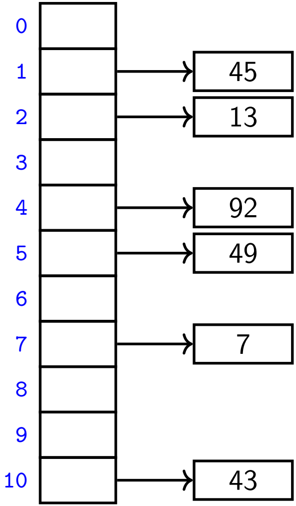
- The simplest approach is to use unsorted linked lists for buckets. This is called collision resolution by **chaining**.

- *search*(k): Look for key k in the list at $T[h(k)]$. Apply MTF-heuristic!
- *insert*(k, v): Add (k, v) to the front of the list at $T[h(k)]$.
- *delete*(k): Perform a search, then delete from the linked list.

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

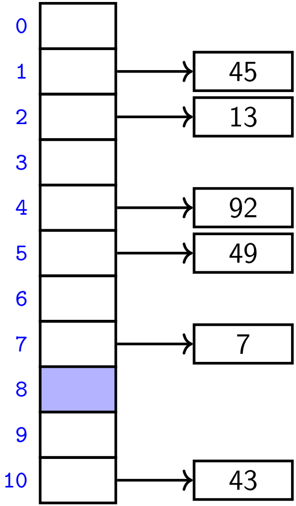


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41) = 8$



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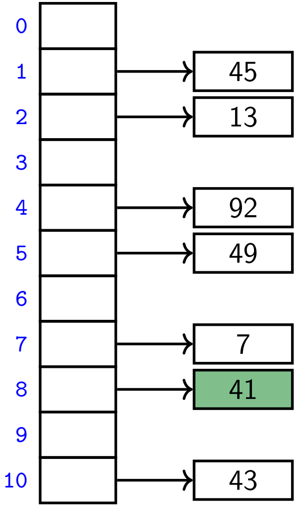

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insert(46)

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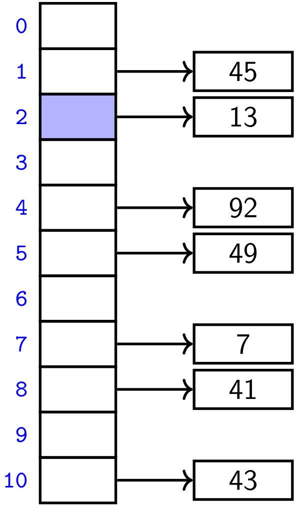

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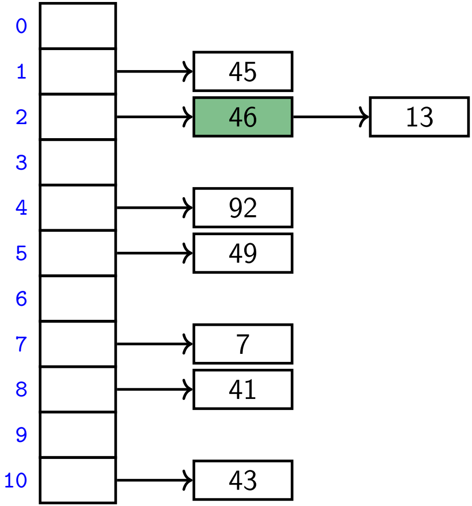
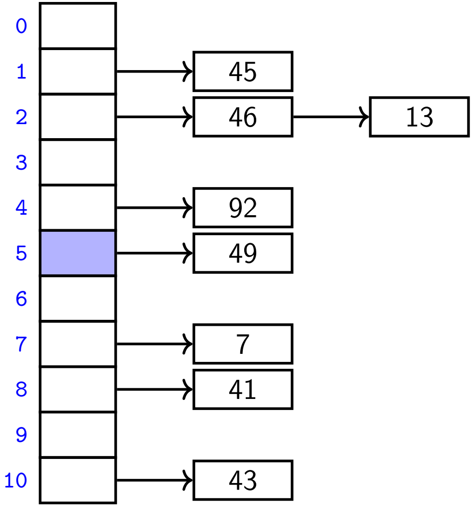


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(16)

$h(16) = 5$



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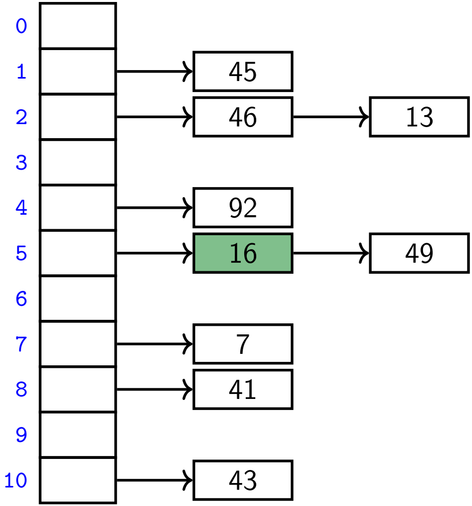
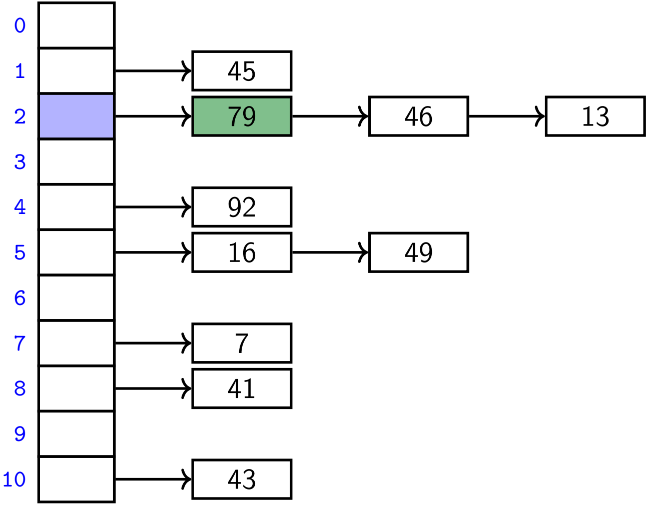


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(79)

$h(79) = 2$



Complexity of chaining

Run-times: *insert* takes time $\Theta(1)$.

search and *delete* have run-time $\Theta\Big(1 + \text{size of bucket } T[h(k)]\Big)$.

- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)
- However, this does not imply that the *average-case* cost of *search* and *delete* is $\Theta(1 + \alpha)$.
  (If all keys hash to the same slot, then the average bucket-size is still α, but the operations take time $\Theta(n)$ on average.)
- We need some assumptions on the hash-functions and the keys!

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- How can we randomize?

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  Assume that the *hash-function* is chosen randomly.
- **Uniform Hashing Assumption**: U is finite and any possible hash-function is equally likely to be chosen as hash-function.

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- Can show:
  - $P\Big(h(k) = i\Big) = \frac{1}{M}$ for any key k and slot i,
  - Hash-values of any two keys are independent of each other.

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- Can show:
  - $P\Big(h(k) = i\Big) = \frac{1}{M}$ for any key k and slot i,
  - Hash-values of any two keys are independent of each other.
- Under this assumption, each key in dictionary is expected to collide with $\frac{n-1}{M}$ other keys and the expected cost of *search* and *delete* is hence $\Theta(1 + \alpha)$.

Load factor and re-hashing

- For all collision resolution strategies, the run-time evaluation is done in terms of the *load factor* $\alpha = n/M$.
- We keep the load factor small by **rehashing** when needed:
  - Keep track of n and M throughout operations
  - If α gets too large, create new (twice as big) hash-table, new hash-function(s) and re-insert all items in the new table.
- Rehashing costs $\Theta(M + n)$ but happens rarely enough that we can ignore this term when amortizing over all operations.
- We should also re-hash when α gets too small, so that $M \in \Theta(n)$ throughout, and the space is always $\Theta(n)$.

Summary: If we maintain $\alpha \in \Theta(1)$, then (under the uniform hashing assumption) the expected cost for hashing with chaining is $O(1)$ and the space is $\Theta(n)$.

Outline

Open addressing

Main idea: Avoid the links needed for chaining by permitting only one item per slot, but allowing a key k to be in multiple slots.

search and *insert* follow a **probe sequence** of possible locations for key k: $\langle h(k,0), h(k,1), h(k,2), \ldots \rangle$ until an empty spot is found.

delete becomes problematic:

- Cannot leave an empty spot behind; the next search might otherwise not go far enough.
- Idea 1: Move later items in the probe sequence forward.
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Simplest method for open addressing: *linear probing*
$h(k,i) = (h(k) + i) \bmod M$, for some hash function h.

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

insert(41)

$h(41, 0) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Linear probing example

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0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

insert(84)

$h(84, 0) = 7$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

insert(84)

$h(84, 1) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

insert(84)

$h(84, 2) = 9$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

insert(20)

$h(20,0) = 9$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

insert(20)

$h(20,1) = 10$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

insert(20)

$h(20, 2) = 0$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

delete(43)

$h(43,0) = 10$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k, i) = (h(k) + i) \bmod 11.$

search(63)

$h(63, 0) = 8$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

search(63)

$h(63, 1) = 9$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

search(63)

$h(63, 2) = 10$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

search(63)

$h(63,3) = 0$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k)+i) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

search(63)

$h(63, 4) = 1$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

search(63)

$h(63, 5) = 2$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k, i) = (h(k) + i) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

search(63)

$h(63, 6) = 3$

not found

Probe sequence operations

probe-sequence::insert$(T, (k, v))$

for $(j = 0; j < M; j{+}{+})$
    if $T[h(k, j)]$ is `NIL` or "deleted"
        $T[h(k, j)] = (k, v)$
        return "success"
return "failure to insert"   // need to re-hash

probe-sequence-search(T, k)

for $(j = 0; j < M; j{+}{+})$
    if $T[h(k, j)]$ is `NIL`
        return "item not found"
    else if $T[h(k, j)]$ has key k
        return $T[h(k, j)]$
    // ignore "deleted" and keep searching
return "item not found"

Independent hash functions

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
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- Using two modular hash-functions may often lead to dependencies.
- Better idea: Use *multiplication method* for second hash function: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
  - A is some floating-point number with $0 < A < 1$
  - $kA - \lfloor kA \rfloor$ computes fractional part of kA, which is in $[0, 1)$
  - Multiply with M to get floating-point number in $[0, M)$
  - Round down to get integer in $\{0, \ldots, M - 1\}$

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  - Multiply with M to get floating-point number in $[0, M)$
  - Round down to get integer in $\{0, \dots, M - 1\}$
- Some observations/suggestions:
  - Multiplying with A is used to scramble the keys.
  - Good scrambling is achieved with $A = \varphi = \frac{\sqrt{5}-1}{2} \approx 0.618033988749....$

Double Hashing

- Assume we have two hash independent functions h_0, h_1.
- Assume further that $h_1(k) \neq 0$ and that $h_1(k)$ is relative prime with the table-size M for all keys k.
  - Choose M prime.
  - Modify standard hash-functions to ensure $h_1(k) \neq 0$
    E.g. modified multiplication method: $h(k) = 1 + \lfloor (M-1)(kA - \lfloor kA \rfloor) \rfloor$
- **Double hashing**: open addressing with probe sequence

$$h(k,i) = \big(h_0(k) + i \cdot h_1(k)\big) \bmod M$$

- *search*, *insert*, *delete* work just like for linear probing, but with this different probe sequence.

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Double hashing example

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insert(41)

$h_0(41) = 8$

$h(41, 0) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

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insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

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insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$

$h(194, 2) = 3$

0	
1	45
2	13
3	194
4	92
5	49
6	
7	7
8	41
9	
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   - Cuckoo hashing
   - Hash Function Strategies

Cuckoo hashing

We use two independent hash functions h_0, h_1 and two tables T_0, T_1.

Main idea: An item with key k can *only* be at $T_0[h_0(k)]$ or $T_1[h_1(k)]$.

- *search* and *delete* then take constant time.
- *insert always* initially puts a new item into $T_0[h_0(k)]$

  If $T_0[h_0(k)]$ is occupied: "kick out" the other item k', which we then attempt to re-insert into its alternate position $T_1[h_1(k')]$

  This may lead to a loop of "kicking out". We detect this by aborting after too many attempts.

  In case of failure: rehash with a larger M and new hash functions.

insert may be slow, but is expected to be constant time if the load factor is small enough.

Cuckoo hashing insertion

```
cuckoo::insert(k, v)
    i ← 0
    do at most 2n times:
        if T_i[h_i(k)] is NIL
            T_i[h_i(k)] ← (k, v)
            return "success"
        swap((k, v), T_i[h_i(k)])
        i ← 1 − i
    return "failure to insert"      // need to re-hash
```

Can prove: after $2n$ iterations, there was a loop in the "kicking out" sequence

In practice, one would stop the iterations much earlier already.

Cuckoo hashing example

$M = 11$, $\quad h_0(k) = k \bmod 11$, $\quad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		
6		
7		
8		
9		92
10		

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

T_0

0	44
1	
2	
3	
4	59
5	
6	
7	
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

T_0

0	44
1	
2	
3	
4	59
5	
6	
7	51
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor)\rfloor$

insert(95)

$i = 0$
$k = 95$

$h_0(k) = 7$
$h_1(k) = 7$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		
6		
7	51	
8		
9		92
10		

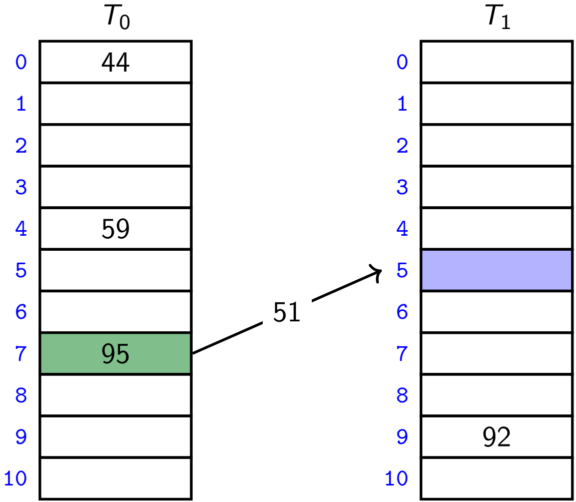
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



Cuckoo hashing example

$M = 11$, $\quad h_0(k) = k \bmod 11$, $\quad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		51
6		
7	95	
8		
9		92
10		

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 26$

$h_0(k) = 4$
$h_1(k) = 0$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		51
6		
7	95	
8		
9		92
10		

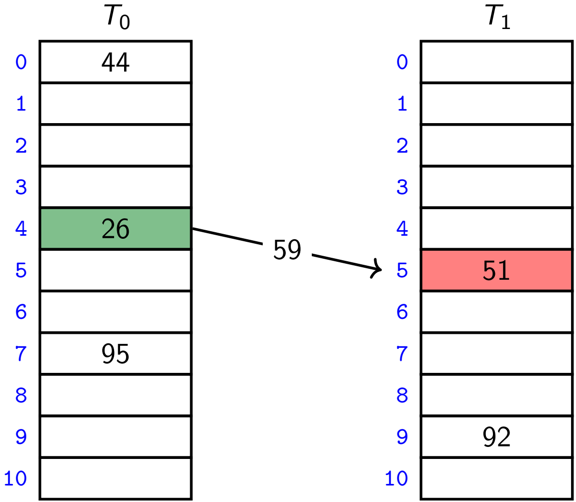
Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 59$

$h_0(k) = 4$
$h_1(k) = 5$



Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
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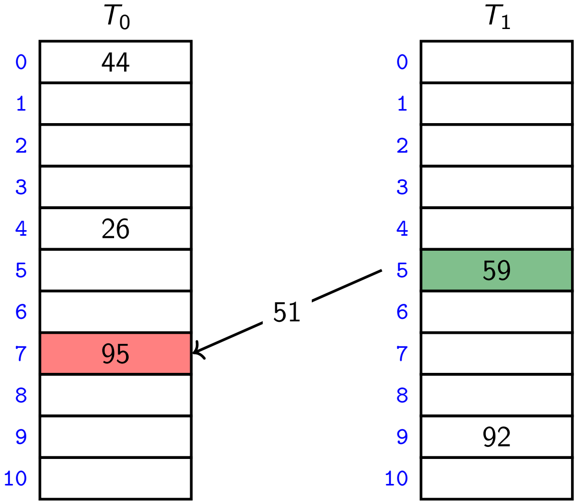
$h_1(k) = 5$



Cuckoo hashing example

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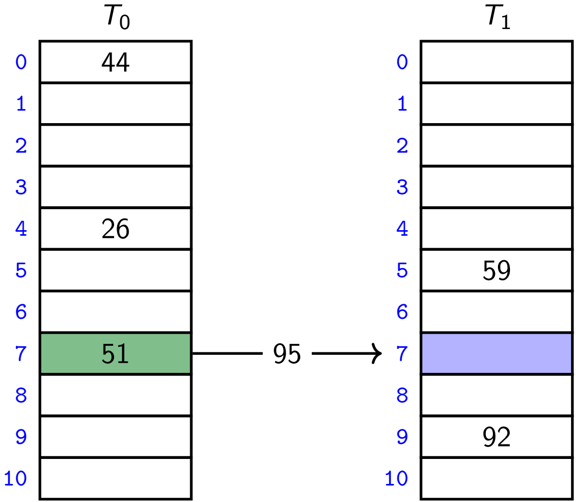

Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 95$

$h_0(k) = 4$
$h_1(k) = 7$

	T_0	T_1
0	44	
1		
2		
3		
4	26	
5		59
6		
7	51	95
8		
9		92
10		

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

search(59)

$h_0(59) = 4$
$h_1(59) = 5$

	T_0		T_1
0	44	0	
1		1	
2		2	
3		3	
7	26	4	
5		5	59
6		6	
7	51	7	95
8		8	
9		9	92
10		10	

Cuckoo hashing example

$M = 11$, $\qquad h_0(k) = k \bmod 11$, $\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

delete(59)

$h_0(59) = 4$
$h_1(59) = 5$

T_0

0	44
1	
2	
3	
7	26
5	
6	
7	51
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	
6	
7	95
8	
9	92
10	

Cuckoo hashing discussions

- The two hash-tables need not be of the same size.
- *Load factor* $\alpha = n/(\text{size of } T_0 + \text{size of } T_1)$
- One can argue: If the load factor α is small enough then insertion has $O(1)$ expected run-time.
- This crucially requires $\alpha < \frac{1}{2}$.

Cuckoo hashing discussions

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- This crucially requires $\alpha < \frac{1}{2}$.

There are many possible variations:

- The two hash-tables could be combined into one.
- Be more flexible when inserting: Always consider both possible positions.
- Use $k > 2$ allowed locations (i.e., k hash-functions).

Complexity of open addressing strategies

For any open addressing scheme, we *must* have $\alpha \leq 1$ (why?).
For the analysis, we require $\alpha < 1$ and Cuckoo hashing requires $\alpha < 1/2$.

Expected # probes $\leq$	*search (unsuccessful)*	*insert*	*search (successful)*
Linear Probing	$\frac{1}{(1-\alpha)^2}$	$\frac{1}{(1-\alpha)^2}$	$\frac{1}{1-\alpha}$ (on avg. over keys)
Double Hashing	$\frac{1}{1-\alpha} + o(1)$	$\frac{1}{1-\alpha} + o(1)$	$\frac{1}{1-\alpha} + o(1)$
Cuckoo Hashing	1 (worst-case)	$\frac{\alpha}{(1-2\alpha)^2}$	1 (worst-case)

Summary: All operations have $O(1)$ expected run-time if hash-function chosen uniformly and α is kept sufficiently small.
But for fixed hash-function the worst-case run-time is (usually) $\Theta(n)$.

Outline

Choosing a good hash function

- Satisfying the uniform hashing assumption is impossible: There are too many hash functions; we would not know how to look up $h(k)$.
- We need to compromise:
  - Choose a hash-function that is easy to compute.
  - But aim for $P(h(k) = i) = \frac{1}{M}$ w.r.t. key-distribution.

Choosing a good hash function

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- We need to compromise:
  - Choose a hash-function that is easy to compute.
  - But aim for $P(h(k) = i) = \frac{1}{M}$ w.r.t. key-distribution.
- If all keys are used equally often, then this can be doable. But in practice keys are not used equally often.
- We can get good performance by choosing a hash-function that is
  - unrelated to any possible patterns in the data, and
  - depends on all parts of the key.
- We saw two basic methods for integer keys:
  - **Modular method**: $h(k) = k \bmod M$.
    We should choose M to be a prime.
  - **Multiplicative method**: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
    for some constant floating-point number A with $0 < A < 1$.

Carter-Wegman's universal hashing

Even better: Randomization that uses easy-to-compute hash-functions.

- Requires: all keys are in $\{0, \ldots, p-1\}$ for some (big) prime p.
- Choose $M < p$ arbitrarily, power of 2 is ok.
- Choose two *random* numbers a, b in $\{0, \ldots p-1\}$, $a \neq 0$
- Use as hash-function

$$h(k) = \big((ak + b) \bmod p\big) \bmod M$$

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$$h(k) = ((ak + b) \bmod p) \bmod M$$

- Clearly $h(k)$ can be computed in $O(1)$ time.
- Choosing h in this way does not satisfy uniform hashing assumption, but we can prove that two keys collide with probability at most $\frac{1}{M}$.
- This is enough to prove the expected run-time bounds for chaining.

Multi-dimensional Data

What if the keys are multi-dimensional, such as strings in Σ^*?

Standard approach is to *flatten* string w to integer $f(w) \in \mathbb{N}$, e.g.

$$\begin{aligned} A \cdot P \cdot P \cdot L \cdot E &\rightarrow (65, 80, 80, 76, 69) \quad \text{(ASCII)} \\ &\rightarrow 65R^4 + 80R^3 + 80R^2 + 76R^1 + 69R^0 \\ &\quad \text{(for some radix } R\text{, e.g. } R = 255) \end{aligned}$$

We combine this with a modular hash function: $h(w) = f(w) \bmod M$

To compute this in $O(|w|)$ time without overflow, use Horner's rule and apply mod early. For exampe, $h(APPLE)$ is

$$\left(\left(\left(\left(\left(\left(65R+80\right) \bmod M\right)R+80\right) \bmod M\right)R+76\right) \bmod M\right)R+69\right) \bmod M$$

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- Predictable space usage (exactly n nodes)
- Never need to rebuild the entire structure
- Supports ordered dictionary operations (rank, select etc.)

Advantages of Hash Tables

- $O(1)$ operation cost (if hash-function random and load factor small)
- We can choose space-time tradeoff via load factor
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete