CS 240 – Data Structures and Data Management

Module 1: Introduction and Asymptotic Analysis

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Based on lecture notes by many previous cs240 instructors

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References: Goodrich & Tamassia 1.1, 1.2, 1.3
Sedgewick 8.2, 8.3

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Outline

1. Introduction and Asymptotic Analysis
   - CS240 Overview
   - Algorithm Design
   - Analysis of Algorithms I
   - Asymptotic Notation
   - Analysis of Algorithms II
   - Example: Analysis of MergeSort
   - Helpful Formulas

Outline

Course Objectives: What is this course about?

- When first learning to program, we emphasize *correctness*: does your program output the expected results?
- Starting with this course, we will also be concerned with *efficiency*: is your program using the computer's resources (typically processor time) efficiently?
- We will study efficient methods of *storing*, *accessing*, and *organizing* large collections of data.
- Typical operations include: *inserting* new data items, *deleting* data items, *searching* for specific data items, *sorting*.
- **Motivating examples:** Digital Music Collection, English Dictionary

Course Objectives: What is this course about?

- We will consider various **abstract data types** (ADTs) and how to implement them efficiently using appropriate **data structures**.
- There is a strong emphasis on mathematical analysis in the course.
- Algorithms are presented using pseudo-code and analyzed using order notation (big-Oh, etc.).

Course Topics

- big-Oh analysis ✓
- priority queues and heaps ✓
- sorting, selection ✓
- binary search trees, AVL trees, B-trees ✓
- skip lists ✓
- hashing ✓
- quadtrees, kd-trees ✓
- range search ✓
- tries ✓
- string matching
- data compression

CS Background

Topics covered in previous courses with relevant sections in [Sedgewick]:

- arrays, linked lists (Sec. 3.2–3.4)
- strings (Sec. 3.6)
- stacks, queues (Sec. 4.2–4.6)
- abstract data types (Sec. 4-intro, 4.1, 4.8–4.9)
- recursive algorithms (5.1)
- binary trees (5.4–5.7)
- sorting (6.1–6.4)
- binary search (12.4)
- binary search trees (12.5)
- probability and expectations (Goodrich & Tamassia, Section 1.3.4)

Outline

Problems (terminology)

First, we must introduce terminology so that we can precisely characterize what we mean by efficiency.

Problem: Given a problem instance, carry out a particular computational task.

Problem Instance: *Input* for the specified problem.

Problem Solution: *Output* (correct answer) for the specified problem instance.

Size of a problem instance: *Size*(I) is a positive integer which is a measure of the size of the instance I.

Example: Sorting problem

input: an array A of integers
output: an array with the same integers in increasing order

size of an input: the length of A

Algorithms and Programs

Algorithm: An algorithm is a *step-by-step process* (e.g., described in pseudo-code) for carrying out a series of computations, given an arbitrary problem instance I.

Solving a problem: An Algorithm A *solves* a problem Π if, for every instance I of Π, A finds (computes) a valid solution for the instance I in finite time.

Program: A program is an *implementation* of an algorithm using a specified computer language.

In this course, our emphasis is on algorithms (as opposed to programs or programming).

Algorithms and Programs

Pseudo-code: a method of communicating an algorithm to another person.

In contrast, a program is a method of communicating an algorithm to a computer.

Pseudo-code

- omits obvious details, e.g. variable declarations,
- has limited if any error detection,
- sometimes uses English descriptions,
- sometimes uses mathematical notation.

Algorithms and Programs

For a problem Π, we can have several algorithms.

For an algorithm $\mathcal{A}$ solving Π, we can have several programs (implementations).

Algorithms in practice: Given a problem Π

1. Design an algorithm $\mathcal{A}$ that solves Π. $\rightarrow$ **Algorithm Design**
2. Assess *correctness* and *efficiency* of $\mathcal{A}$. $\rightarrow$ **Algorithm Analysis**
3. If acceptable (correct and efficient), implement $\mathcal{A}$.

Outline

Efficiency of Algorithms/Programs

- How do we decide which algorithm or program is the most efficient solution to a given problem?
- In this course, we are primarily concerned with the *amount of time* a program takes to run. → **Running Time**
- We also may be interested in the *amount of additional memory* the program requires. → **Auxiliary space**
- The amount of time and/or memory required by a program will depend on *Size(I)*, the size of the given problem instance I.

Running Time of Algorithms/Programs

First option: *experimental studies*

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition.
- Use a method like `clock()` (from `time.h`) to get an accurate measure of the actual running time.
- Plot/compare the results.

Running Time of Algorithms/Programs

Shortcomings of experimental studies

- Implementation may be complicated/costly.
- Timings are affected by many factors: *hardware* (processor, memory), *software environment* (OS, compiler, programming language), and *human factors* (programmer).
- We cannot test all inputs; what are good *sample inputs*?

We want a framework that:

- Does not require implementing the algorithm.
- Is independent of the hardware/software environment.
- Takes into account all input instances.

We need some *simplifications*.

Overview of Algorithm Analysis

We will develop several aspects of algorithm analysis in the next slides.
To overcome dependency on hardware/software:

- Algorithms are presented in structured high-level *pseudo-code* which is language-independent.
- Analysis of algorithms is based on an *idealized computer model*.
- Instead of time, count the number of *primitive operations*.
- The efficiency of an algorithm (with respect to time) is measured in terms of its *growth rate*.

Random Access Machine

Random Access Machine (RAM) model:

- A set of memory cells, each of which stores one item (word) of data. Implicit assumption: memory cells are big enough to hold the items that we store.
- Any *access to a memory location* takes constant time.
- Any *primitive operation* takes constant time. Implicit assumption: primitive operations have fairly similar, though different, running time on different systems
- The *running time* of a program is proportional to the number of memory accesses plus the number of primitive operations.

This is an idealized model, so these assumptions may not be valid for a "real" computer.

Running Time Simplifications

We will simplify our analysis by considering the behaviour of algorithms for large inputs sizes.

- **Example 1:** What is larger, $100n$ or $10n^2$?
- **Example 2:** What is larger, $1000000n + 2000000000000000$ or $0.01n^2$?

- To simplify comparisons, use **order notation**
- Informally: ignore constants and lower order terms

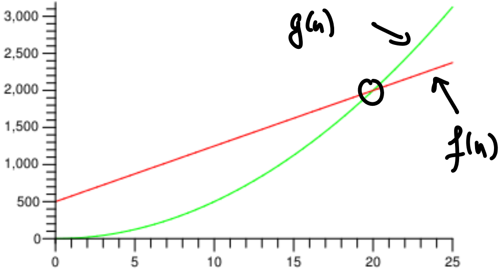
Outline

Order Notation

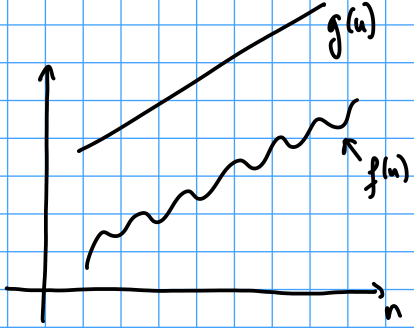
***O*-notation:** $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

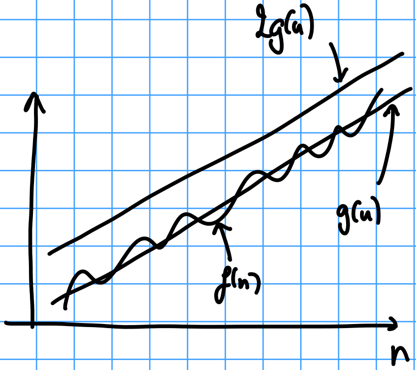
real

Example: $f(n) = 75n + 500$ and $g(n) = 5n^2$ (e.g. $c = 1, n_0 = 20$)



Note: The absolute value signs in the definition are irrelevant for analysis of run-time or space, but are useful in other applications of asymptotic notation.





$\forall n, \quad |f(n)| \leq |g(n)|$

$\rightarrow f(n) \in O(g(n))$

$c = 1, \quad n_0 = 1$

$\forall n, \quad |f(n)| \leq 2|g(n)|$

$\rightarrow f(n) \in O(g(n))$

$c = 2, \quad n_0 = 1$

$f(n) = 75n + 500 \qquad g(n) = 5n^2$

for $n \geq 20$ $\qquad 20 \leq n \xrightarrow{\times 5n}$

$$100n \leq 5n^2 \qquad (*)$$

for $n \geq 20$ $\qquad 20 \leq n \xrightarrow{\times 25}$

$$500 \leq 25n \xrightarrow{+75n}$$

$$75n + 500 \leq 100n \qquad (**)$$

$\Rightarrow$ for $n \geq 20$, $\quad |f(n)| = 75n + 500 \leq 5n^2 = |g(n)|$

Taking $n_0 = 20$ and $c = 1$, this proves that $f(n) \in O(g(n))$

Example of Order Notation

In order to prove that $2n^2 + 3n + 11 \in O(n^2)$ from first principles, we need to find c and n_0 such that the following condition is satisfied:

$$0 \le 2n^2 + 3n + 11 \le c\, n^2 \text{ for all } n \ge n_0.$$

note that not all choices of c and n_0 will work.

$f(n) = 2n^2 + 3n + 11$ $\qquad$ $g(n) = n^2$

$n \geq 1$

$$\begin{array}{rcl} 2n^2 & \leq & 2n^2 \\ 3n & \leq & 3n^2 \\ 11 & \leq & 11n^2 \\ \hline f(n) & & 16n^2 = 16\,g(n) \end{array}$$

+

Taking $n_0 = 1$ and $c = 16$, this proves that $f(n) \in O(g(n))$.

Aymptotic Lower Bound

- We have $2n^2 + 3n + 11 \in O(n^2)$
- But we also have $2n^2 + 3n + 11 \in O(n^{10})$
- We want a *tight* asymptotic bound.

$n^2 \leq n^{10}$

Ω-notation: $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$.

Θ-notation: $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 > 0$ such that $c_1\,|g(n)| \leq |f(n)| \leq c_2\,|g(n)|$ for all $n \geq n_0$.

$$f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))$$

Example of Order Notation

Prove that $f(n) = 2n^2 + 3n + 11 \in \Omega(n^2)$ from first principles.

$\forall n \geq 1 \quad 2n^2 + 3n + 11 \geq n^2$ So taking $n_0 = 1$, $c = 1$

this proves that $f(n) \in \Omega(n^2)$

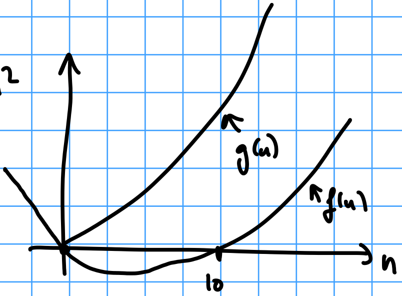
Prove that $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$ from first principles.

Prove that $\log_b(n) \in \Theta(\log n)$ for all $b > 1$ from first principles.

$\log_b(n) = \frac{\log(n)}{\log(b)}$. So taking $n_0 = 1$ and $c_1 = c_2 = \frac{1}{\log(b)}$,

this proves that $\log_b(n) \in \Theta(\log n)$

$f(n) = \frac{1}{2}n^2 - 5n \qquad g(n) = n^2$



For $n \geq 20$, $\quad n^2 \geq 20n$ $\quad \div 4$

$$\frac{1}{4}n^2 \geq 5n$$

$$-5n \geq -\frac{1}{4}n^2 \quad + \frac{1}{2}n^2$$

$$f(n) \geq \frac{1}{2}n^2 = \frac{1}{2}g(n).$$

Taking $n_0 = 20$ and $c = \frac{1}{2}$, this proves that $f(n) \in \Omega(g(n))$.

Strictly smaller/larger asymptotic bounds

- We have $f(n) = 2n^2 + 3n + 11 \in \Theta(n^2)$.
- How to express that $f(n)$ is *asymptotically strictly smaller* than n^3?

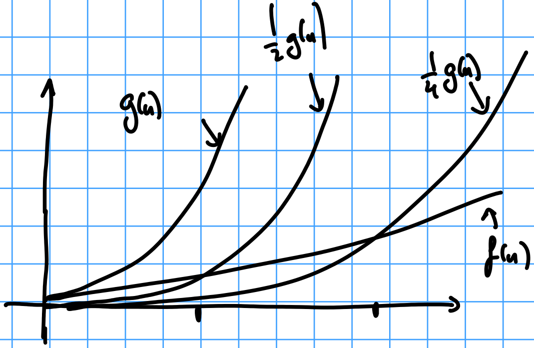
***o*-notation:** $f(n) \in o(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

***ω*-notation:** $f(n) \in \omega(g(n))$ if $g(n) \in o(f(n))$.

- Rarely proved from first principles.

$\iff \forall c > 0 \ \exists n_0 > 0$ such that

$\forall n \geq n_0 \quad |g(n)| \leq c\,|f(n)|$



$g(n)$
$\frac{1}{2}g(n)$
$\frac{1}{4}g(n)$
$f(n)$

$f(n) = 2000\, n^2$ $\qquad$ $g(n) = n^n$

Let $c > 0$. We need to find n_0 such that

$\forall n \geq n_0$ $\qquad 2000\, n^2 \leq c\, n^n \quad (*)$

$(*)$ is equivalent to $\quad 2000 \leq c\, n^{n-2}$

For $n \geq 3$, $\quad n \leq n^{n-2}$

For $\underline{n \geq 3}$ and $\underline{n \geq \frac{2000}{c}}$, $\quad \frac{2000}{c} \leq n$

$\Rightarrow 2000 \leq c\,n \Rightarrow 2000 \leq c\, n^{n-2}$

Taking $n_0 = \max\left(3, \frac{2000}{c}\right)$, we see that $f(n) \in o(g(n))$.

Algebra of Order Notations

Identity rule: $f(n) \in \Theta(f(n))$

Transitivity:

- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$✓
- If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.

Maximum rules: Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Then:

- $O(f(n) + g(n)) = O(\max\{f(n), g(n)\})$ ✓
- $\Omega(f(n) + g(n)) = \Omega(\max\{f(n), g(n)\})$

Suppose that $f(n) \in O(g(n))$ $\quad (*)$
$g(n) \in O(h(n))$ $\quad (**)$

$(*) \Leftrightarrow \exists\, c, n_0$ s.t. $\forall n \geq n_0 \quad |f(n)| \leq c\,|g(n)|$

$(**) \Leftrightarrow \exists\, c', n_0'$ s.t $\forall n \geq n_0' \quad |g(n)| \leq c'\,|h(n)|$

$n_0'' = \max(n_0, n_0')$. Then, for $n \geq n_0''$

$$|f(n)| \leq c\,|g(n)| \leq c c'\,|h(n)|$$

This proves that $f(n) \in O(h(n))$.

$$\overset{(*)}{a(n) \in O(\max(f(n), g(n)))} \iff \overset{(**)}{a(n) \in O(f(n)+g(n))}.$$

① Suppose $(*)$. $\exists\, n_1, c$ such that $\forall n \geq n_1$, $|a(n)| \leq c\,|\max(f(n), g(n))|$.

$\forall n \geq \max(n_0, n_1)$ $\quad |a(n)| \leq c \max(f(n), g(n))$

because $f(n)$ and $g(n) > 0$, $\quad \max(f(n), g(n)) \leq f(n)+g(n)$.

[$\max(\ldots)$ is either $f(n)$ or $g(n)$. Suppose it is $f(n)$.

Then $f(n) \leq f(n)+g(n)$ because $g(n) > 0$.]

$\Rightarrow \quad |a(n)| \leq c\,(f(n)+g(n)) = c\,|f(n)+g(n)|$.

$$\underset{(*)}{a(n) \in O(\max(f(n), g(n)))} \iff \underset{(**)}{a(n) \in O(f(n)+g(n))}.$$

② Suppose $(**)$. $\exists\, n_1, c$ such that for $n \geq n_1$,

$$|a(n)| \leq c\,|f(n)+g(n)|.$$

For $n \geq \max(n_0, n_1)$, $|a(n)| \leq c\,(f(n)+g(n))$

$$f(n)+g(n) \leq \underbrace{\max(f(n),g(n)) + \max(f(n),g(n))}_{2\max(f(n),g(n))}$$

$$= 2\,|\max(f(n), g(n))|.$$

$$|a(n)| \leq 2c\,|\max(f(n), g(n))|.$$

Techniques for Order Notation

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Suppose that

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \text{(in particular, the limit exists).}$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \quad ✓ \\ \Theta(g(n)) & \text{if } 0 < L < \infty \quad ✓ \\ \omega(g(n)) & \text{if } L = \infty. \quad ✓ \end{cases}$$

The required limit can often be computed using *l'Hôpital's rule*. Note that this result gives *sufficient* (but not necessary) conditions for the stated conclusions to hold.

Example 1

Let $f(n)$ be a polynomial of degree $d \geq 0$:

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

for some $c_d > 0$.

Then $f(n) \in \Theta(n^d)$:

1) $f(n) = n^d \underbrace{\left(c_d + \underbrace{\frac{c_{d-1}}{n} + \dots + \frac{c_1}{n^{d-1}} + \frac{c_0}{n^d}}_{\to 0 \text{ when } n \to \infty} \right)}_{\to c_d \text{ when } n \to \infty}$. In particular,

$f(n) > 0$ for n large enough.

Example 1

Let $f(n)$ be a polynomial of degree $d \geq 0$:

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

for some $c_d > 0$.

Then $f(n) \in \Theta(n^d)$:

2) $$\frac{f(n)}{n^d} = c_d + \frac{c_{d-1}}{n} + \cdots + \frac{c_1}{n^{d-1}} + \frac{c_0}{n^d}.$$

The limit of $\frac{f(n)}{n^d}$ when $n \to \infty$ exists and is >0, $<\infty$.

By the limit rule, we get $f(n) \in \Theta(n^d)$.

Example 2

Prove that $n(2 + \sin n\pi/2)$ is $\Theta(n)$. Note that $\lim_{n\to\infty}(2 + \sin n\pi/2)$ does not exist.

$\forall\, n \geq 1$

$$-1 \leq \sin\left(\frac{n\pi}{2}\right) \leq 1 \quad +2$$

$$1 \leq 2 + \sin\left(\frac{n\pi}{2}\right) \leq 3$$

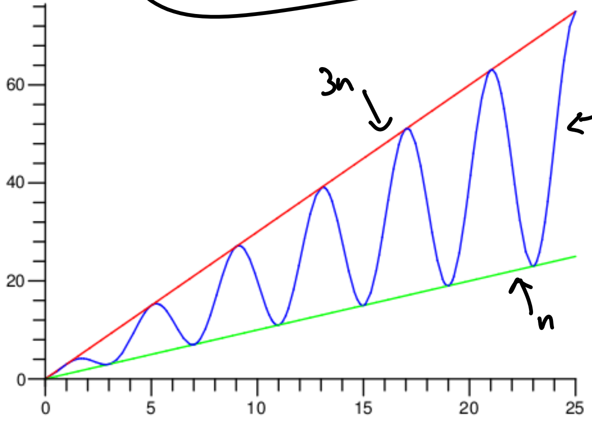
$$n \leq n\left(2 + \sin\left(\frac{n\pi}{2}\right)\right) \leq 3n$$

Taking $n_0 = 1$ and $c_1 = 1$, $c_2 = 3$, this proves $n\left(2+\sin\left(\frac{n\pi}{2}\right)\right) \in \Theta(n)$.

But $\lim\limits_{n\to\infty} \dfrac{n\left(2+\sin\left(\frac{n\pi}{2}\right)\right)}{n} = \lim\limits_{n\to\infty} 2 + \sin\left(\frac{n\pi}{2}\right)$ does not exist.

Example 2

Prove that $n(2+\sin n\pi/2)$ is $\Theta(n)$. Note that $\lim_{n\to\infty}(2+\sin n\pi/2)$ does not exist.



Relationships between Order Notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$ ✓
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$

- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

$f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$

Proof of $\Rightarrow$. By assumption, $\exists\, n_0, c_1, c_2 > 0$ such that

$\forall n \geq n_0,$ $$c_1 |g(n)| \leq |f(n)| \leq c_2 |g(n)|$$

$\div c_2$ $\qquad$ $\div c_1$

$$\frac{1}{c_2} |f(n)| \leq |g(n)| \leq \frac{1}{c_1} |f(n)|.$$

So $g(n) \in \Theta(f(n))$.

Growth Rates

- If $f(n) \in \Theta(g(n))$, then the *growth rates* of $f(n)$ and $g(n)$ are the *same*.
- If $f(n) \in o(g(n))$, then we say that the growth rate of $f(n)$ is *less than* the growth rate of $g(n)$.
- If $f(n) \in \omega(g(n))$, then we say that the growth rate of $f(n)$ is *greater than* the growth rate of $g(n)$.
- Typically, $f(n)$ may be "complicated" and $g(n)$ is chosen to be a very simple function.

Example 3

Compare the growth rates of $\log n$ and n.

Now compare the growth rates of $(\log n)^c$ and n^d (where $c > 0$ and $d > 0$ are arbitrary numbers).

L'hopital: if $\lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = \infty$

and $\lim_{n \to \infty} \frac{f'(n)}{g'(n)} = L$ then $\lim_{n \to \infty} \frac{f(n)}{g(n)} = L$

1) $f(n) = \log(n) = \frac{\ln(n)}{\ln(2)}$ $\qquad g(n) = n$

$f'(n) = \frac{1}{n} \cdot \frac{1}{\ln(2)}$ $\qquad g'(n) = 1$

$\frac{f'(n)}{g'(n)} = \frac{1}{n \ln(2)} \to 0$ when $n \to \infty$

so (l'hopital) $\frac{f(n)}{g(n)} \to 0$ when $n \to \infty$

so (limit rule) $f(n) = o(g(n))$.

2) $f(n) = \log(n)$ and $g(n) = n^a$ $\qquad a > 0$

$$f'(n) = \frac{1}{n\ln(2)} \qquad g'(n) = a\, n^{a-1}$$

so $\dfrac{f'(n)}{g'(n)} = \dfrac{1}{a\, n^a \ln(2)} \to 0$ when $n \to \infty$. $\quad \log(n) \in o(n^a)$.

3) $f(n) = \log(n)^c$ and $g(n) = n^d$ $\qquad c, d > 0$

$\dfrac{f(n)}{g(n)} = \dfrac{\log(n)^c}{n^d} = \left(\dfrac{\log(n)}{n^{d/c}}\right)^c$ so $\lim\limits_{n \to \infty} \dfrac{f(n)}{g(n)} = 0$ so $f(n) \in o(g(n))$.

$\hookrightarrow \lim\limits_{n \to \infty} \dfrac{\log(n)}{n^{d/c}} = 0$

Common Growth Rates

Commonly encountered growth rates in analysis of algorithms include the following (in increasing order of growth rate):

- $\Theta(1)$ (*constant complexity*),
- $\Theta(\log n)$ (*logarithmic complexity*),
- $\Theta(n)$ (*linear complexity*),
- $\Theta(n \log n)$(*linearithmic*),
- $\Theta(n \log^k n)$, for some constant k (*quasi-linear*),
- $\Theta(n^2)$ (*quadratic complexity*),
- $\Theta(n^3)$ (*cubic complexity*),
- $\Theta(2^n)$ (*exponential complexity*).

How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$
- logarithmic complexity: $T(n) = c \log n$
- linear complexity: $T(n) = cn$
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$
- quadratic complexity: $T(n) = cn^2$
- cubic complexity: $T(n) = cn^3$
- exponential complexity: $T(n) = c2^n$

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- constant complexity: $T(n) = c$ $\qquad \rightsquigarrow T(2n) = c$.
- logarithmic complexity: $T(n) = c \log n$
- linear complexity: $T(n) = cn$
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$
- quadratic complexity: $T(n) = cn^2$
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- constant complexity: $T(n) = c$ $\rightsquigarrow T(2n) = c$.
- logarithmic complexity: $T(n) = c \log n$ $\rightsquigarrow T(2n) = T(n) + c$.
- linear complexity: $T(n) = cn$
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$
- quadratic complexity: $T(n) = cn^2$
- cubic complexity: $T(n) = cn^3$
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How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$ $\quad\leadsto T(2n) = c$.
- logarithmic complexity: $T(n) = c \log n$ $\quad\leadsto T(2n) = T(n) + c$.
- linear complexity: $T(n) = cn$ $\quad\leadsto T(2n) = 2T(n)$.
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$
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- exponential complexity: $T(n) = c2^n$ $\rightsquigarrow$ $T(2n) = (T(n))^2/c$.

$\hookrightarrow T(n+1) = c \cdot 2^{n+1} = 2T(n)$

Outline

Techniques for Algorithm Analysis

- Goal: Use asymptotic notation to simplify run-time analysis.
- Running time of an algorithm depends on the *input size* n.

```
Test1(n)
1.    sum ← 0
2.    for i ← 1 to n do
3.        for j ← i to n do
4.            sum ← sum + (i − j)^2
5.    return sum
```

- Identify *primitive operations* that require $\Theta(1)$ time.
- The complexity of a loop is expressed as the *sum* of the complexities of each iteration of the loop.
- Nested loops: start with the innermost loop and proceed outwards. This gives *nested summations*.

Let $T_1(n)$ be the cost of $\text{Test}_1(n)$.

$T_1(n) \in \Theta(S_1(n))$, where $S_1(n)$ is the number of times we go through Step 4

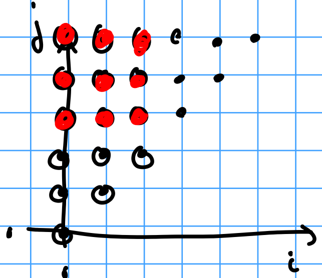
$$S_1(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} 1.$$

① $\sum_{j=i}^{n} 1 = n-i+1 \longrightarrow S_1(n) = \sum_{i=1}^{n} (n-i+1) = n^2 - \frac{n(n+1)}{2} + n$

$$= \frac{n^2}{2} + \frac{n}{2} \in \Theta(n^2).$$

② O and Ω separately

$$S_1(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} 1$$



big O: $S_1(n) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} 1 = n^2 \Rightarrow S_1(n) \in O(n^2).$

big Ω: $S_1(n) \geq \sum_{i=1}^{n/2} \sum_{j=i}^{n} 1 \geq \sum_{i=1}^{n/2} \sum_{j=\frac{n}{2}+1}^{n} 1 = \sum_{i=1}^{n/2} \frac{n}{2} = \frac{n^2}{4}$

$S_1(n) \in \Omega(n^2) \to S_1(n) \in \Theta(n^2)$

Techniques for Algorithm Analysis

Two general strategies are as follows.

Strategy I: Use Θ-bounds *throughout the analysis* and obtain a Θ-bound for the complexity of the algorithm.

Strategy II: Prove a O-bound and a *matching* Ω-bound *separately*.
Use upper bounds (for O-bounds) and lower bounds (for Ω-bound) early and frequently.
This may be easier because upper/lower bounds are easier to sum.

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

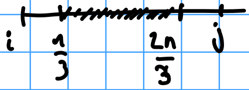
let $T_2(n)$ be the cost of Test2(A, n)

Then $T_2(n) \in \Theta(S_2(n))$, $S_2(n)$ is the number of times we enter Step 6.

① $$S_2(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 = \frac{n(n^2+3n+2)}{6} \in \Theta(n^3).$$

② $$S_2(n) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} 1 = n^3 \quad \text{so} \quad S_2(n) \in O(n^3)$$

$$S_2(n) \geq \sum_{i=1}^{n/3} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 \geq \sum_{i=1}^{n/3} \sum_{j=\frac{2n}{3}}^{n} \sum_{k=i}^{j} 1$$



$$S_2(n) \geq \sum_{i=1}^{n/3} \sum_{j=\frac{2n}{3}}^{n} \frac{n}{3}$$

$$\geq \sum_{i=1}^{n/3} \left(\frac{n}{3}\right)^2$$

$$\geq \left(\frac{n}{3}\right)^3 \quad \rightarrow \quad S_2(n) \in \Omega(n^3)$$

$$\rightarrow \quad S_2(n) \in \Theta(n^3).$$

Complexity of Algorithms

- Algorithm can have different running times on two instances of the same size.

```
Test3(A, n)
A: array of size n
    for i ← 1 to n − 1 do
        j ← i
        while j > 0 and A[j] > A[j − 1] do
            swap A[j] and A[j − 1]
            j ← j − 1
```

A=[3 2 1]
↓
A=[3 2 1]

A=[1 2 3]
i=1
A=[2 1 3]
i=2
A=[2 3 1]
A=[3 2 1]

Let $T_{\mathcal{A}}(I)$ denote the running time of an algorithm $\mathcal{A}$ on instance I.

Worst-case complexity of an algorithm: take the worst I

Average-case complexity of an algorithm: average over I

Complexity of Algorithms

Worst-case complexity of an algorithm: The worst-case running time of an algorithm $\mathcal{A}$ is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ mapping n (the input size) to the *longest* running time for any input instance of size n:

$$T_{\mathcal{A}}(n) = \max\{ T_{\mathcal{A}}(I) : Size(I) = n\}.$$

Average-case complexity of an algorithm: The average-case running time of an algorithm $\mathcal{A}$ is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ mapping n (the input size) to the *average* running time of $\mathcal{A}$ over all instances of size n:

$$T_{\mathcal{A}}^{avg}(n) = \frac{1}{|\{I : Size(I) = n\}|} \sum_{\{I : Size(I)=n\}} T_{\mathcal{A}}(I).$$

O-notation and Complexity of Algorithms

- It is important not to try and make *comparisons* between algorithms using O-notation.

- For example, suppose algorithm $\mathcal{A}_1$ and $\mathcal{A}_2$ both solve the same problem, $\mathcal{A}_1$ has worst-case run-time $O(n^3)$ and $\mathcal{A}_2$ has worst-case run-time $O(n^2)$.

- Observe that we *cannot* conclude that $\mathcal{A}_2$ is more efficient than $\mathcal{A}_1$ for all input!
  1. The worst-case run-time may only be achieved on some instances.
  2. O-notation is an upper bound. $\mathcal{A}_1$ may well have worst-case run-time $O(n)$. If we want to be able to compare algorithms, we should always use Θ-notation.

Outline

Design of MergeSort

if $n=1$, return.

Input: Array A of n integers

- **Step 1:** We split A into two subarrays: A_L consists of the first $\lceil \frac{n}{2} \rceil$ elements in A and A_R consists of the last $\lfloor \frac{n}{2} \rfloor$ elements in A.
- **Step 2:** *Recursively* run *MergeSort* on A_L and A_R.
- **Step 3:** After A_L and A_R have been sorted, use a function *Merge* to merge them into a single sorted array.

MergeSort

MergeSort$(A, \ell \leftarrow 0, r \leftarrow n-1, S \leftarrow \texttt{NIL})$
A: array of size n, $0 \le \ell \le r \le n-1$

    if S is NIL   initialize it as array $S[0..n-1]$
    if $(r \le \ell)$ **then**
        return
    else
        $m = (r + \ell)/2$
        MergeSort(A, ℓ, m, S)
        MergeSort$(A, m+1, r, S)$
        Merge(A, ℓ, m, r, S)

Two tricks to reduce run-time and auxiliary space:

- The recursion uses parameters that indicate the range of the array that needs to be sorted.
- The array used for copying is passed along as parameter.

Merge

Merge(A, ℓ, m, r, S)
$A[0..n-1]$ is an array, $A[\ell..m]$ is sorted, $A[m+1..r]$ is sorted
$S[0..n-1]$ is an array

copy $A[\ell..r]$ into $S[\ell..r]$
int $i_L \leftarrow \ell$; int $i_R \leftarrow m+1$;
for $(k \leftarrow \ell; k \leq r; k{+}{+})$ **do**
  if $(i_L > m)$ $A[k] \leftarrow S[i_R{+}{+}]$
  else if $(i_R > r)$ $A[k] \leftarrow S[i_L{+}{+}]$
  else if $(S[i_L] \leq S[i_R])$ $A[k] \leftarrow S[i_L{+}{+}]$
  else $A[k] \leftarrow S[i_R{+}{+}]$

Merge takes time $\Theta(r - \ell + 1)$, i.e., $\Theta(n)$ time for merging n elements.

$n = 5$
$l = 0$
$r = n-1 = 4$
$m = 2$

$A = [\ 2 \quad 4 \quad 7 \ \vdots \ 5 \quad 6\ ]$

$S = [\ 2 \quad 4 \quad 7 \ \vdots \ 5 \quad 6\ ]$

$\uparrow \quad \uparrow \quad \uparrow \qquad \uparrow \quad \uparrow \quad \uparrow$

$i_L = l = 0 \qquad\qquad i_R = m+1$

$k=0 \qquad A = [\ 2 \ \_ \ \_ \ \_ \ \_\ ] \qquad i_L = 1$

$k=1 \qquad A = [\ 2 \ 4 \ \_ \ \_ \ \_\ ] \qquad i_L = 2$

$k=3 \qquad A = [\ 2 \ 4 \ 5 \ \_ \ \_\ ] \qquad i_R = 4$

$k=4 \qquad A = [\ 2 \ 4 \ 5 \ 6 \ \_\ ] \qquad i_R = 5$

$k=5 \qquad A = [\ 2 \ 4 \ 5 \ 6 \ 7\ ] \qquad i_L = 3$

Analysis of MergeSort

Let $T(n)$ denote the time to run *MergeSort* on an array of length n.

- creating S takes time $\Theta(n)$ ✓
- recursive calls take time $T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$ ✓
- merging takes time $\Theta(n)$ ✓

The **recurrence relation** for $T(n)$ is as follows:

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + \Theta(n) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1. \end{cases}$$

It suffices to consider the following *exact recurrence*, with constant factor c replacing Θ's: (requires proof !)

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

Analysis of MergeSort

- The following is the corresponding **sloppy recurrence** (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- The exact and sloppy recurrences are *identical* when n is a power of 2.
- The recurrence can easily be solved by various methods when $n = 2^j$. The solution has growth rate $T(n) \in \Theta(n \log n)$.
- It is possible to show that $T(n) \in \Theta(n \log n)$ *for all n* by analyzing the exact recurrence.

Let $n = 2^k$ $\qquad T(n) = 2T\left(\frac{n}{2}\right) + cn \qquad T(1) = c$

$$T(2^k) = 2T(2^{k-1}) + c2^k$$
$$= 2\left(2T(2^{k-2}) + c \cdot 2^{k-1}\right) + c2^k$$
$$= 2^2\, T(2^{k-2}) + 2c2^k$$
$$= 2^2\left(2T(2^{k-3}) + c \cdot 2^{k-2}\right) + 2c2^k$$
$$= 2^3\, T(2^{k-3}) + 3c2^k$$
$$= 2^4\, T(2^{k-4}) + 4c2^k$$
$$= \cdots = 2^k\, T(2^{k-k}) + kc2^k$$

$$= cn + \log(n)\, cn.$$

$$\frac{T(n)}{n\log(n)} = \frac{cn\log(n) + cn}{n\log(n)} = c + \frac{c}{\log(n)}$$

limit rule $\Rightarrow T(n) \in \Theta(n \log n)$

Some Recurrence Relations

Recursion	resolves to	example
$T(n) = T(n/2) + \Theta(1)$	$T(n) \in \Theta(\log n)$	Binary search
$T(n) = 2T(n/2) + \Theta(n)$	$T(n) \in \Theta(n \log n)$	Mergesort
$T(n) = 2T(n/2) + \Theta(\log n)$	$T(n) \in \Theta(n)$	Heapify ($\rightarrow$ later)
$T(n) = T(cn) + \Theta(n)$ for some $0 < c < 1$	$T(n) \in \Theta(n)$	Selection ($\rightarrow$ later)
$T(n) = 2T(n/4) + \Theta(1)$	$T(n) \in \Theta(\sqrt{n})$	Range Search ($\rightarrow$ later)
$T(n) = T(\sqrt{n}) + \Theta(1)$	$T(n) \in \Theta(\log \log n)$	Interpolation Search ($\rightarrow$ later)

- Once you know the result, it is (usually) easy to prove by induction.
- Many more recursions, and some methods to find the result, in cs341.

Outline

1. Introduction and Asymptotic Analysis
   - CS240 Overview
   - Algorithm Design
   - Analysis of Algorithms I
   - Asymptotic Notation
   - Analysis of Algorithms II
   - Example: Analysis of MergeSort
   - Helpful Formulas

Order Notation Summary

O-notation: $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $|f(n)| \le c\,|g(n)|$ for all $n \ge n_0$.

Ω-notation: $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $c\,|g(n)| \le |f(n)|$ for all $n \ge n_0$.

Θ-notation: $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 > 0$ such that $c_1\,|g(n)| \le |f(n)| \le c_2\,|g(n)|$ for all $n \ge n_0$.

o-notation: $f(n) \in o(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $|f(n)| \le c\,|g(n)|$ for all $n \ge n_0$.

ω-notation: $f(n) \in \omega(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $c\,|g(n)| \le |f(n)|$ for all $n \ge n_0$.

$|g(n)| \le c\,|f(n)|$ $\quad\quad$ $\hookrightarrow$ $|g(n)| \le \frac{1}{c}\,|f(n)|$

Useful Sums

Arithmetic sequence:

$\sum_{i=0}^{n-1} i = ???$ $\frac{n(n-1)}{2}$ $\qquad \sum_{i=0}^{n-1}(a+di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2)$ if $d \neq 0$.

Geometric sequence:

$\Theta(r^n)$

$$\sum_{i=0}^{n-1} 2^i = ??? \qquad \sum_{i=0}^{n-1} a\,r^i = \begin{cases} a\dfrac{r^n-1}{r-1} & \in \Theta(r^{n-1}) & \text{if } r > 1 \\ na & \in \Theta(n) & \text{if } r = 1 \\ a\dfrac{1-r^n}{1-r} & \in \Theta(1) & \text{if } 0 < r < 1. \end{cases}$$

$= 2^n - 1$

Harmonic sequence:

$\sum_{i=1}^{n} \frac{1}{i} = ???$ $\qquad H_n := \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$

A few more:

$\sum_{i=1}^{n} \frac{1}{i^2} = ???$ $\qquad \sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6} \in \Theta(1)$

$\sum_{i=1}^{n} i^k = ???$ $\qquad \sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$ for $k \geq 0$

Useful Math Facts

Logarithms:

- $c = \log_b(a)$ means $b^c = a$. E.g. $n = 2^{\log n}$.
- $\log(a)$ (in this course) means $\log_2(a)$
- $\log(a \cdot c) = \log(a) + \log(c)$, $\log(a^c) = c \log(a)$
- $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$, $a^{\log_b c} = c^{\log_b a}$
- $\ln(x) =$ natural log $= \log_e(x)$, $\frac{d}{dx} \ln x = \frac{1}{x}$
- concavity: $\alpha \log x + (1-\alpha) \log y \leq \log(\alpha x + (1-\alpha) y)$ for $0 \leq \alpha \leq 1$

Factorial:

- $n! := n(n-1)(n-2) \cdots \cdot 2 \cdot 1 =$ # ways to permute n elements
- $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$

Probability and moments:

- $E[aX] = aE[X]$, $E[X+Y] = E[X] + E[Y]$ (linearity of expectation)