CS 240 – Data Structures and Data Management

Module 7: Dictionaries via Hashing

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Based on lecture notes by many previous cs240 instructors

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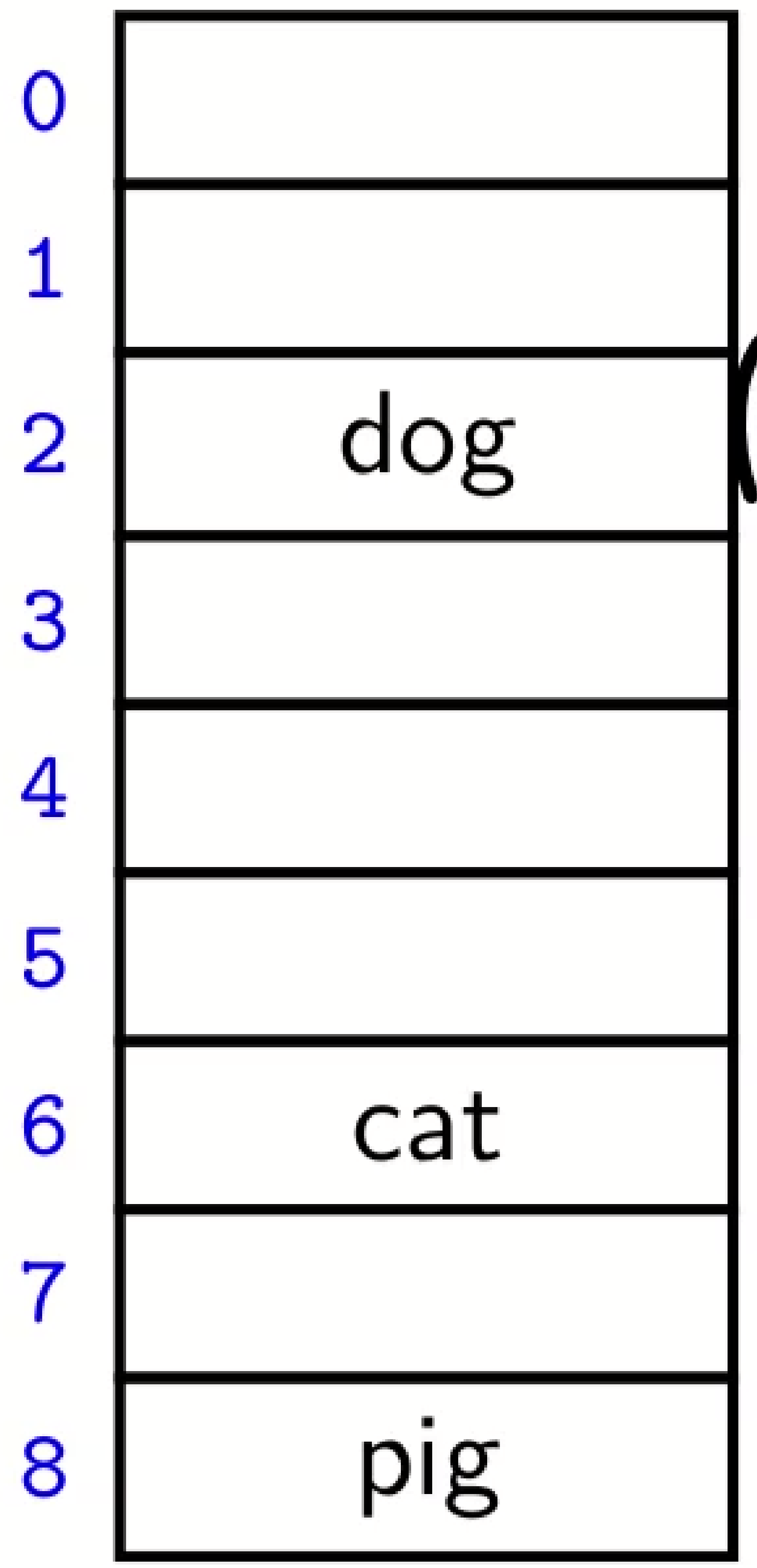
Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.



(2, dog)

(6, cat)

(8, pig)

- *search*(k): Check whether $A[k]$ is `NIL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NIL`

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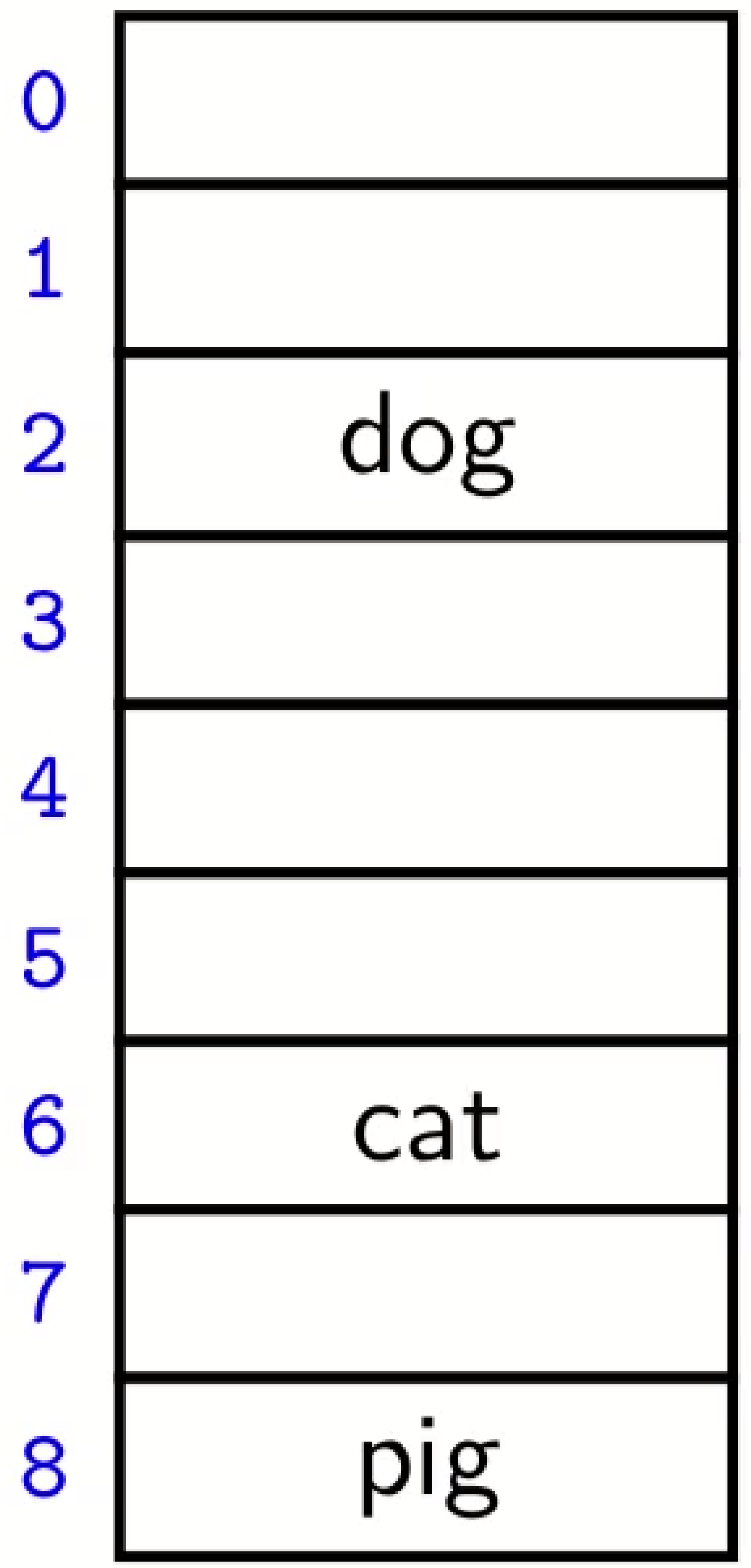


- *search*(k): Check whether $A[k]$ is `NIL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NIL`

Each operation is $\Theta(1)$.
Total space is $\Theta(M)$.

What sorting algorithm does this remind you of?
Bucket Sort

Hashing

Two disadvantages of direct addressing:

- It cannot be used if the keys are not integers.
- It wastes space if M is unknown or $n \ll M$.

Hashing idea: Map (arbitrary) keys to integers in range $\{0, \ldots, M-1\}$ and then use direct addressing.

Details:

- **Assumption:** We know that all keys come from some **universe** U. (Typically $U = \mathbb{N}$.)
- We design a **hash function** $h : U \to \{0, 1, \ldots, M-1\}$. (Commonly used: $h(k) = k \bmod M$. We will see other choices later.)
- Store dictionary in **hash table**, i.e., an array T of size M.
- An item with key k should ideally be stored in **slot** $h(k)$, i.e., at $T[h(k)]$.

Hashing example

$U = \mathbb{N}, M = 11 \qquad h(k) = k \bmod 11.$

The hash table stores keys 7, 13, 43, 45, 49, 92. (Values are not shown).

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

$45 = 44 + 1$

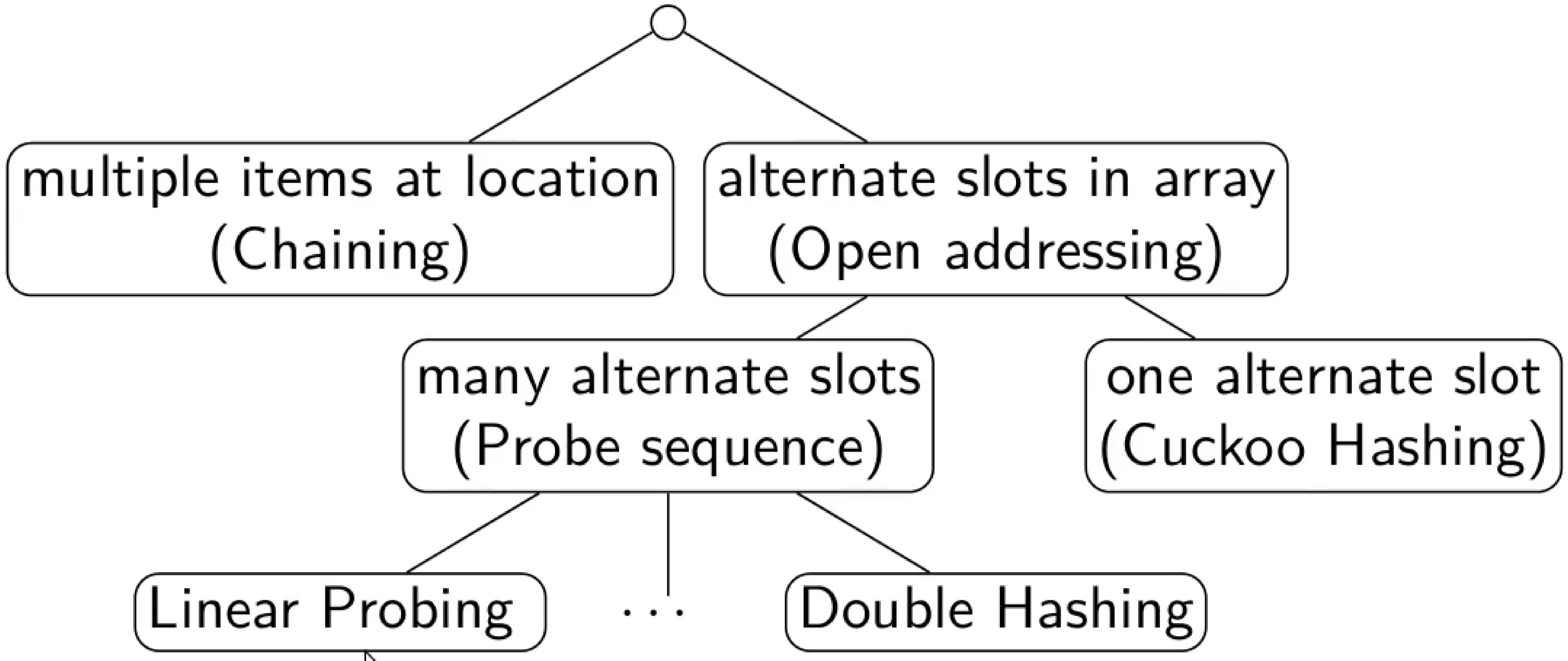
$= 4 \times 11 + 1$

$\rightarrow 45 \bmod 11 = 1$

$\rightarrow h(45) = 1$

Collisions

- Generally hash function h is not injective, so many keys can map to the same integer.
  - For example, $h(46) = 2 = h(13)$ if $h(k) = k \bmod 11$.
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.
- There are many strategies to resolve collisions:



Probability of having a collision?

Suppose we pick n values in $\{0 \ldots M-1\}$, independently, uniform distribution.

$$\text{Prob(no collision)} = \frac{M(M-1)(M-2)\cdots(M-(n-1))}{M^n}$$

$$= 1 \cdot \left(1-\frac{1}{M}\right)\left(1-\frac{2}{M}\right)\cdots\left(1-\frac{n-1}{M}\right)$$

$$\text{Prob(collision)} = 1 - \left(1-\frac{1}{M}\right)\cdots\left(1-\frac{n-1}{M}\right) =: f(n,M)$$

$$f(22,365) < 0.5 \qquad f(23,365) > 0.5$$

Outline

Separate Chaining

Simplest collision-resolution strategy: Each slot stores a **bucket** containing 0 or more KVPs.

- A bucket could be implemented by any dictionary realization (even another hash table!).
- The simplest approach is to use unsorted linked lists for buckets. This is called collision resolution by **separate chaining**.

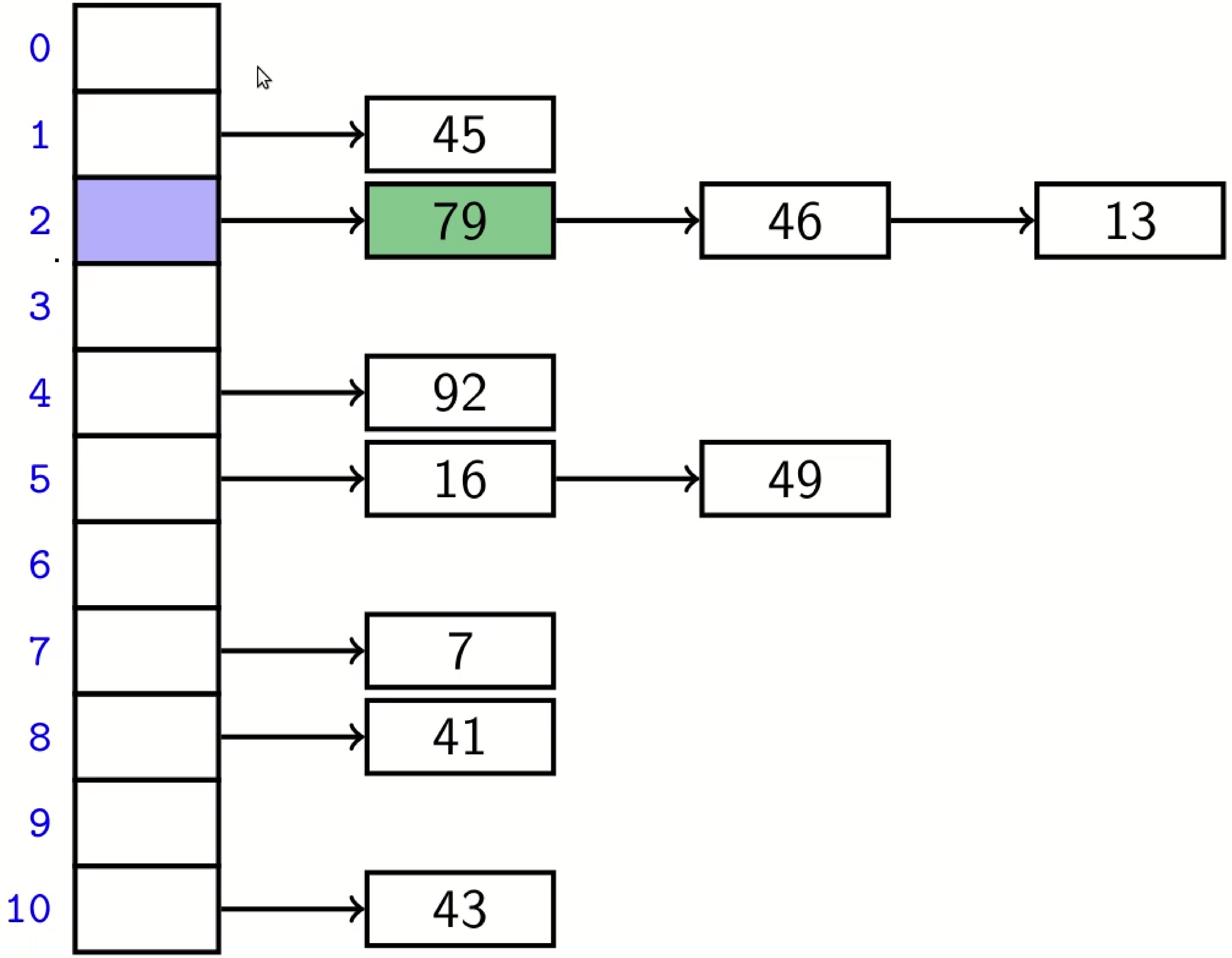
- *search*(k): Look for key k in the list at $T[h(k)]$.
  Apply MTF-heuristic!
- *insert*(k, v): Add (k, v) to the front of the list at $T[h(k)]$.
- *delete*(k): Perform a search, then delete from the linked list.

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(79)

$h(79) = 2$



Complexity of chaining

⚠ We assume that computing $h(k)$ is constant time

Run-times: *insert* takes time $O(1)$.
search and *delete* have run-time $O(1 + \text{size of bucket } T(h(k)))$.

- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)
- However, this does not imply that the *average-case* cost of *search* and *delete* is $O(1 + \alpha)$.
  (If all keys hash to the same slot, then the average bucket-size is still α, but the operations take time $\Theta(n)$ on average.)
- **Uniform Hashing Assumption**: for any key k, and for any $j \in \{0, \ldots, M-1\}$, $h(k) = j$ happens with probability $1/M$, independently of where the other keys hash to.
  (This depends on the input and how we choose the function $\rightsquigarrow$ later.)
- Under this assumption, each key is expected to collide with $\frac{n-1}{M}$ other keys and the average-case cost of *search* and *delete* is hence $O(1 + \alpha)$.

We hash keys $k_1 \dots k_n$. We assume

1) $\forall j$ in $\{0 \dots M-1\}$

$\forall i$ in $\{1 \dots n\}$ $\quad Prob(h(k_i) = j) = \frac{1}{M}$ *

2) $\forall j, j'$ in $\{0 \dots M-1\}$

$\forall i, i'$ in $\{1 \dots n\}$ $i \neq i'$

$Prob(h(k_i) = j$ and $h(k_{i'}) = j')$

$= Prob(h(k_i) = j) \cdot Prob(h(k_{i'}) = j') = \frac{1}{M^2}$.

$\forall i, i'$ in $\{1 \ldots n\}$, $i \neq i'$

$$\mathrm{Prob}(h(k_i) = h(k_{i'})) = \sum_{j=0}^{M-1} \underbrace{\mathrm{Prob}(h(k_i) = h(k_{i'}) = j)}_{1/M^2}$$

$$= 1/M.$$

Let $X_{ii'} := \begin{cases} 0 & \text{if } h(k_i) \neq h(k_{i'}) \\ 1 & \text{if } h(k_i) = h(k_{i'}) \end{cases}$

$\rightarrow$ # collisions involving $k_i = \sum_{i' \neq i} X_{ii'}$

$\rightarrow E(\# \ldots) = \sum_{i \neq i'} E(X_{ii'}) = \sum_{i \neq i'} \frac{1}{M} = \frac{n-1}{M}$

Load factor and re-hashing

- For all collision resolution strategies, the run-time evaluation is done in terms of the *load factor* $\alpha = n/M$.
- We keep the load factor small by **rehashing** when needed:
  - Keep track of n and M throughout operations
  - If α gets too large, create new (twice as big) hash-table, new hash-function(s) and re-insert all items in the new table.
- Rehashing costs $\Theta(\underline{M} + \underline{n})$ but happens rarely enough that we can ignore this term when amortizing over all operations.
- We should also re-hash when α gets too small, so that $M \in \Theta(n)$ throughout, and the space is always $\Theta(n)$.

$\frac{1}{4} \leq \alpha \leq 2 \qquad \frac{1}{4} \leq \frac{n}{M} \leq 2$

Summary: If we maintain $\alpha \in \underline{\Theta(1)}$, then (under the uniform hashing assumption) the average cost for hashing with chaining is $O(1)$ and the space is $\Theta(n)$.

$\frac{n}{2} \leq M \leq 4n \rightarrow M \in \Theta(n)$

Outline

Open addressing

Main idea: Avoid the links needed for chaining by permitting only one item per slot, but allowing a key k to be in multiple slots.

search and *insert* follow a **probe sequence** of possible locations for key k: $\langle h(k,0), h(k,1), h(k,2), \ldots \rangle$ until an empty spot is found.

delete becomes problematic:

- Cannot leave an empty spot behind; the next search might otherwise not go far enough.
- **lazy deletion**: Mark spot as *deleted* (rather than NIL) and continue searching past deleted spots.

Open addressing

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Simplest method for open addressing: *linear probing*
$h(k,i) = (h(k) + i) \bmod M$, for some hash function h.

Linear probing example

$M = 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

$h(k) = k \bmod 11$

$\rightarrow h(k,i) = (k+i) \bmod 11$

Linear probing example

$M = 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

insert(84)

$h(84, 2) = 9$

Linear probing example

$M = 11, \qquad h(k,i) = (h(k) + i) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

delete(43)

$h(43,0) = 10$

Linear probing example

$M = 11, \qquad h(k, i) = (h(k) + i) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

search(20)

$h(20, 2) = 0$

found

Probe sequence operations

probe-sequence::insert$(T, (k, v))$

for $(j = 0; j < M; j{+}{+})$
    if $T[h(k,j)]$ is `NIL` or "deleted"
        $T[h(k,j)] = (k, v)$
        return "success"
return "failure to insert"   // need to re-hash

probe-sequence-search(T, k)

for $(j = 0; j < M; j{+}{+})$
    if $T[h(k,j)]$ is `NIL`
        return "item not found"
    else if $T[h(k,j)]$ has key k
        return $T[h(k,j)]$
    // ignore "deleted" and keep searching
return "item not found"

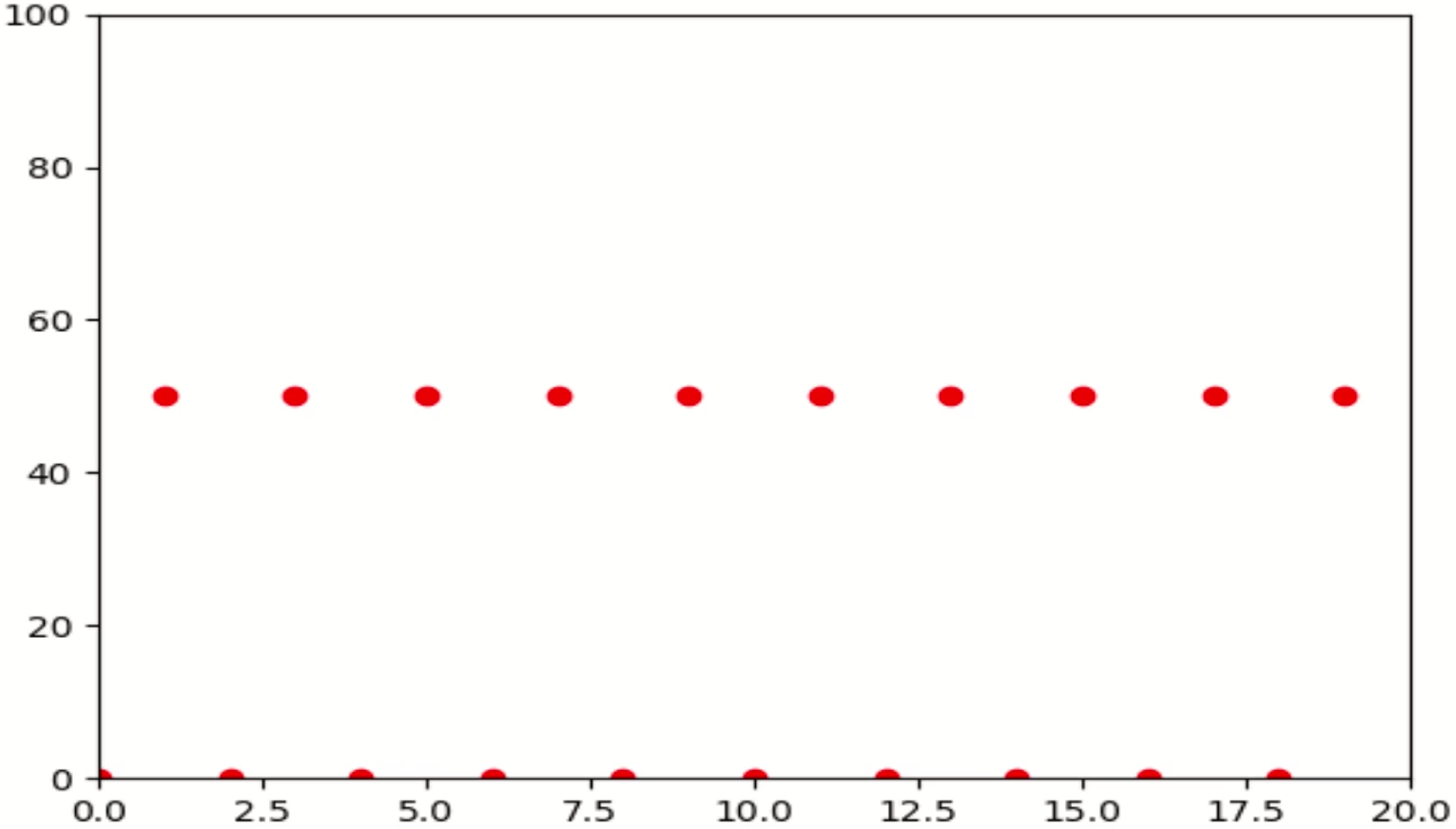
Independent hash functions

$\forall k \ \forall i,j$ in $\{0 \ldots M-1\}$ $\mathrm{Prob}(h_0(k)=i$ and $h_1(k)=j)$

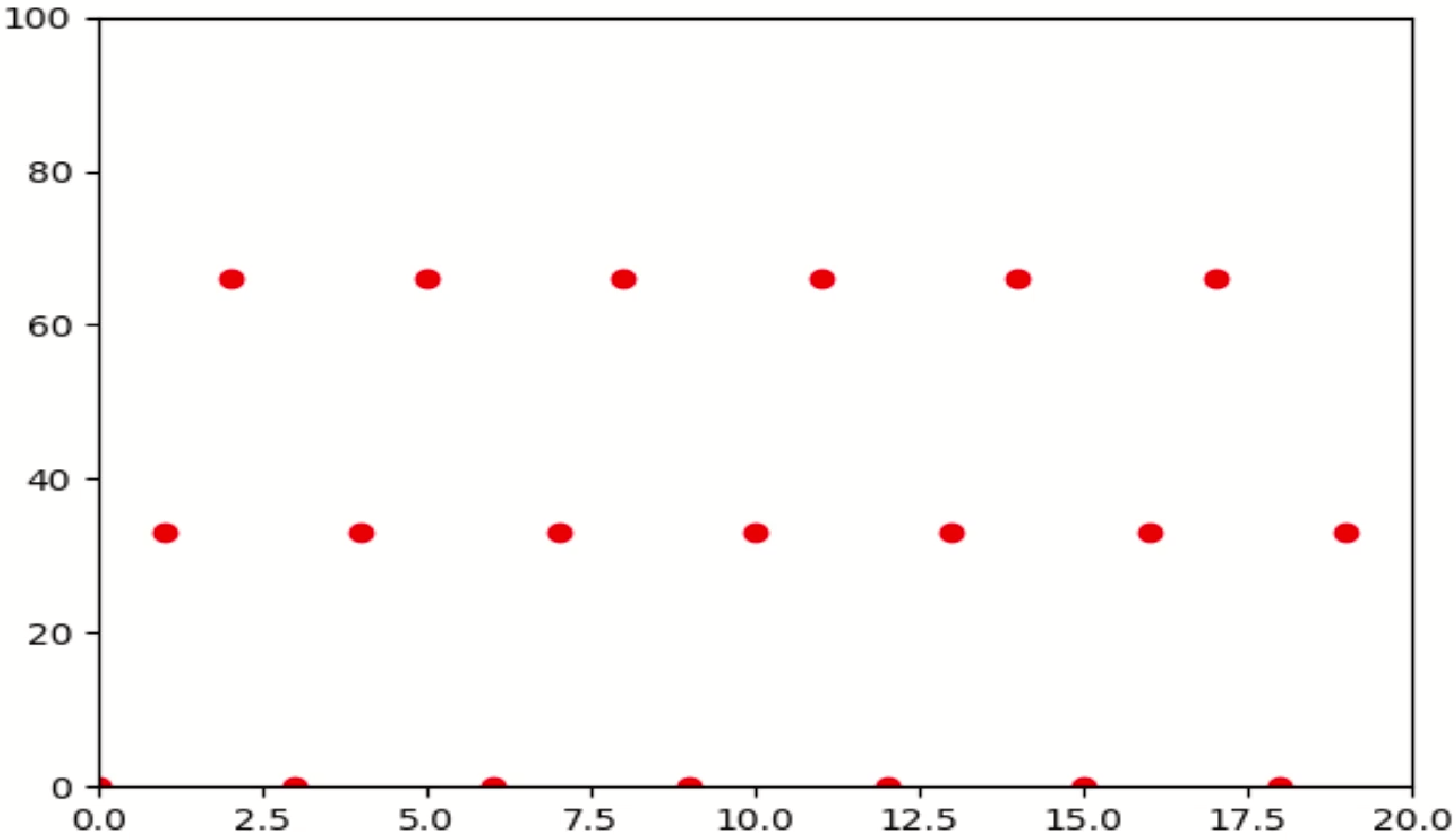
$= \mathrm{Prob}(h_0(k)=i) \cdot \mathrm{Prob}(h_1(k)=j)$

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
- Using two modular hash-functions may often lead to dependencies.
- Better idea: Use *multiplicative method* for second hash function: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
  - A is some floating-point number
  - $kA - \lfloor kA \rfloor$ computes fractional part of kA, which is in $[0, 1)$
  - Multiply with M to get floating-point number in $[0, M)$
  - Round down to get integer in $\{0, \ldots, M-1\}$

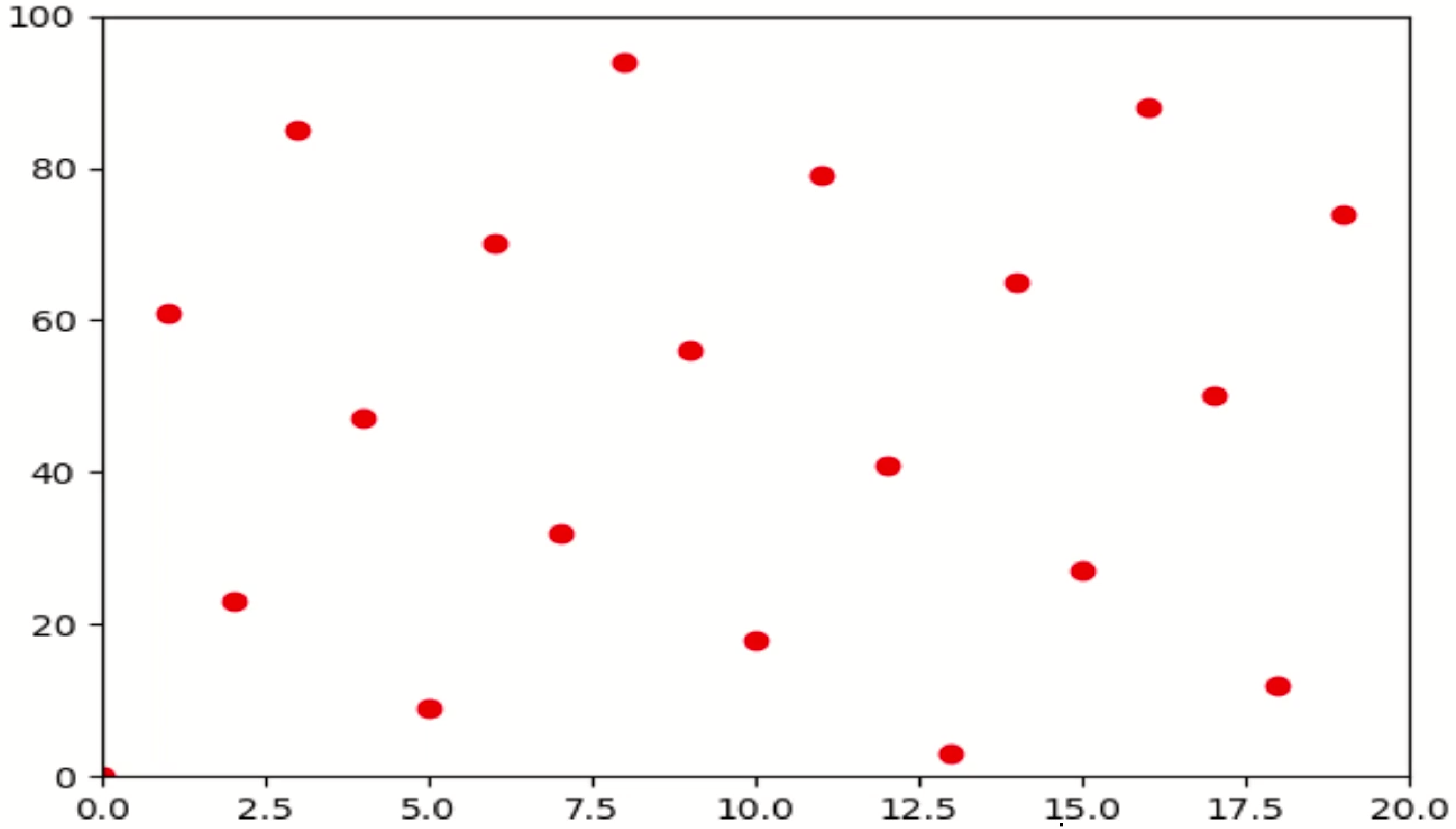
Knuth suggests $A = \varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$.



$A = 0.5 \quad M = 100 \quad h(i),\ i = 0..19$



$A = \frac{1}{3}$ $M = 100$ $h(i)$, $i = 0..19$.



$A = \frac{\sqrt{5}-1}{2}$ $M = 100$ $h(i),\ i = 0..19$

Double Hashing

- Assume we have two hash independent functions h_0, h_1.
- Assume further that $h_1(k) \neq 0$ and that $h_1(k)$ is relative prime with the table-size M for all keys k.
  - Choose M prime.
  - Modify standard hash-functions to ensure $h_1(k) \neq 0$
    E.g. modified multiplication method: $h(k) = 1 + \lfloor (M-1)(kA - \lfloor kA \rfloor) \rfloor$
- **Double hashing**: open addressing with probe sequence

$$h(k, i) = h_0(k) + i \cdot h_1(k) \bmod M$$

- *search*, *insert*, *delete* work just like for linear probing, but with this different probe sequence.

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(41)

$h_0(41) = 8$

$h(41, 0) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$

$h(194, 2) = 3$

0	
1	45
2	13
3	194
4	92
5	49
6	
7	7
8	41
9	
10	43

$M = 16$

k such that $h_0(k) = 3$

$h_1(k) = 4$

$$\begin{tikzcd}
& 3 \arrow[dr] & \\
15 \arrow[ur] & & 7 \arrow[dl] \\
& 11 \arrow[ul] &
\end{tikzcd}$$

Outline

1. Dictionaries via Hashing
   - Hashing Introduction
   - Separate Chaining
   - Probe Sequences
   - Cuckoo hashing
   - Hash Function Strategies

Cuckoo hashing

We use two independent hash functions h_0, h_1 and two tables T_0, T_1.

Main idea: An item with key k can *only* be at $T_0[h_0(k)]$ or $T_1[h_1(k)]$.

- *search* and *delete* then take constant time.
- *insert always* initially puts a new item into $T_0[h_0(k)]$

  If $T_0[h_0(k)]$ is occupied: "kick out" the other item, which we then attempt to re-insert into its alternate position $T_1[h_1(k)]$

  This may lead to a loop of "kicking out". We detect this by aborting after too many attempts.

  In case of failure: rehash with a larger M and new hash functions.

insert may be slow, but is expected to be constant time if the load factor is small enough.

Cuckoo hashing insertion

cuckoo::insert(k, v)

$i \leftarrow 0$
do at most $2n$ times:
  if $T_i[h_i(k)]$ is `NIL`
    $T_i[h_i(k)] \leftarrow (k, v)$
    return "success"
  swap$((k, v), T_i[h_i(k)])$
  $i \leftarrow 1 - i$
return "failure to insert"   // need to re-hash

After $2n$ iterations, there definitely was a loop in the "kicking out" sequence (why?)

In practice, one would stop the iterations much earlier already.

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 0$
$k = 95$

$h_0(k) = 7$
$h_1(k) = 7$

0	44
1	
2	
3	
4	59
5	
6	
7	51
8	
9	92
10	

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

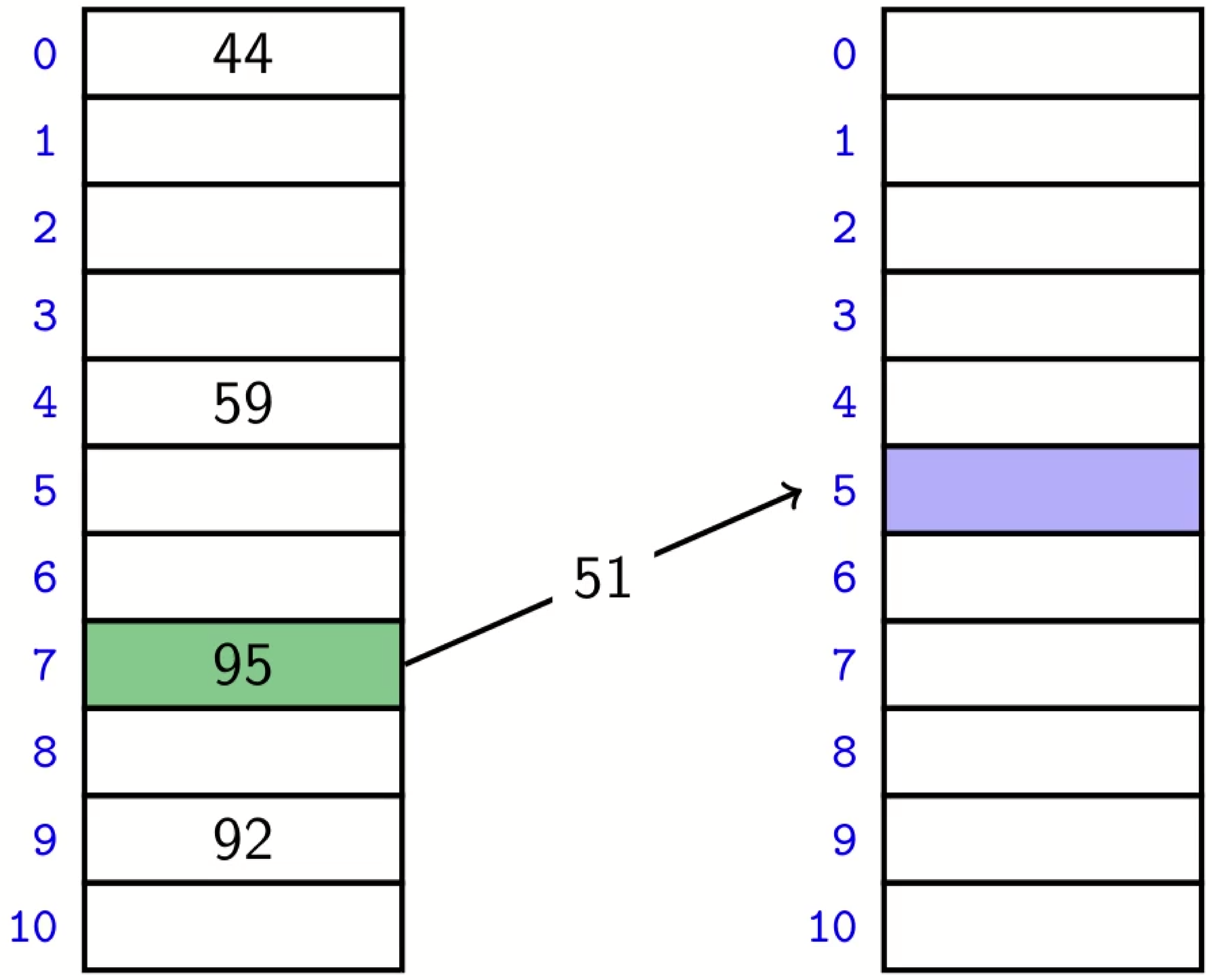
Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 26$

$h_0(k) = 4$
$h_1(k) = 0$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		51
6		
7	95	
8		
9	92	
10		

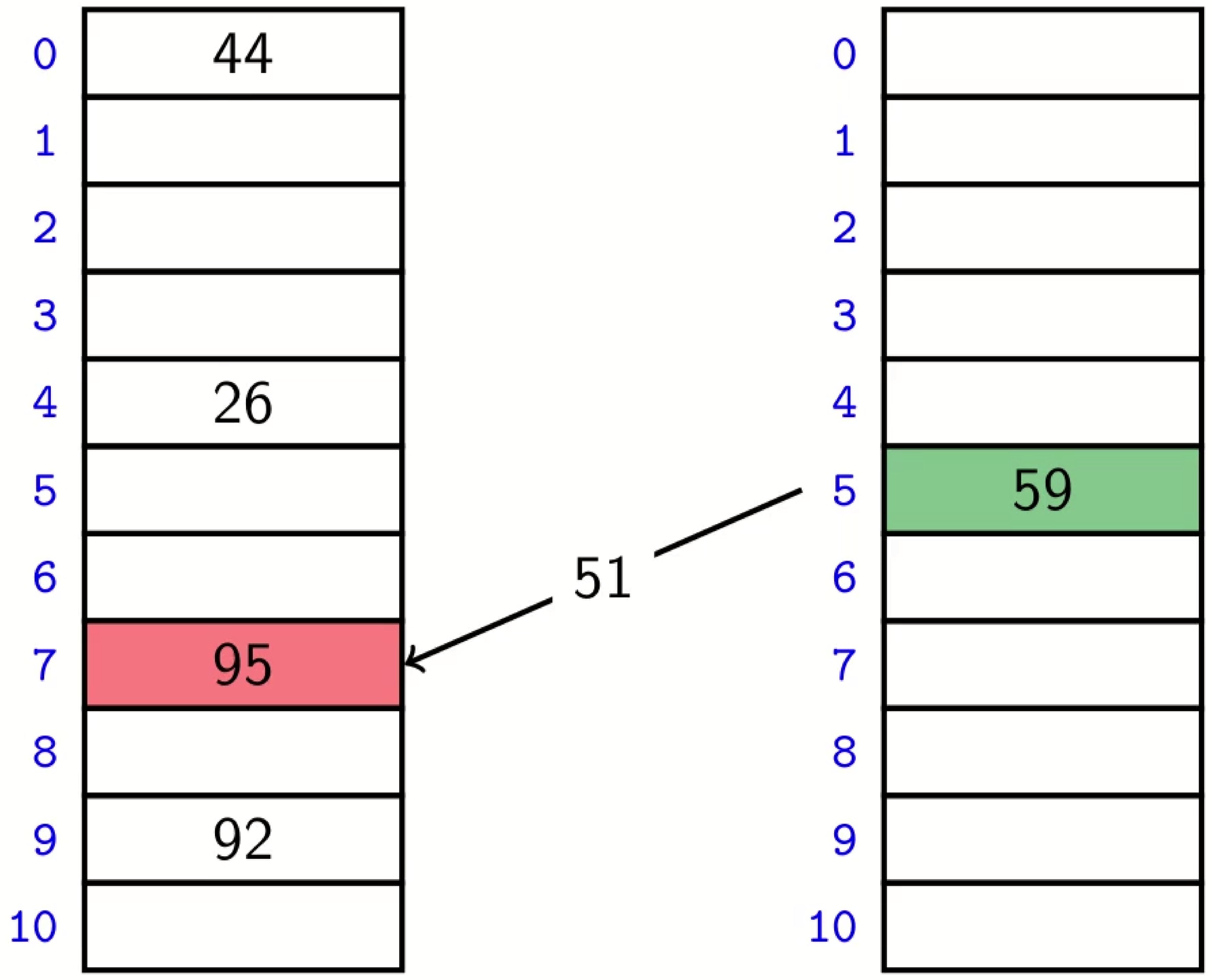
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

search(59)

$h_0(59) = 7$
$h_1(59) = 5$

	T_0
0	44
1	
2	
3	
4	26
5	
6	
7	51
8	
9	92
10	

	T_1
0	
1	
2	
3	
4	
5	59
6	
7	95
8	
9	
10	

Claim: if we enter the insert loop $2n+1$ times,

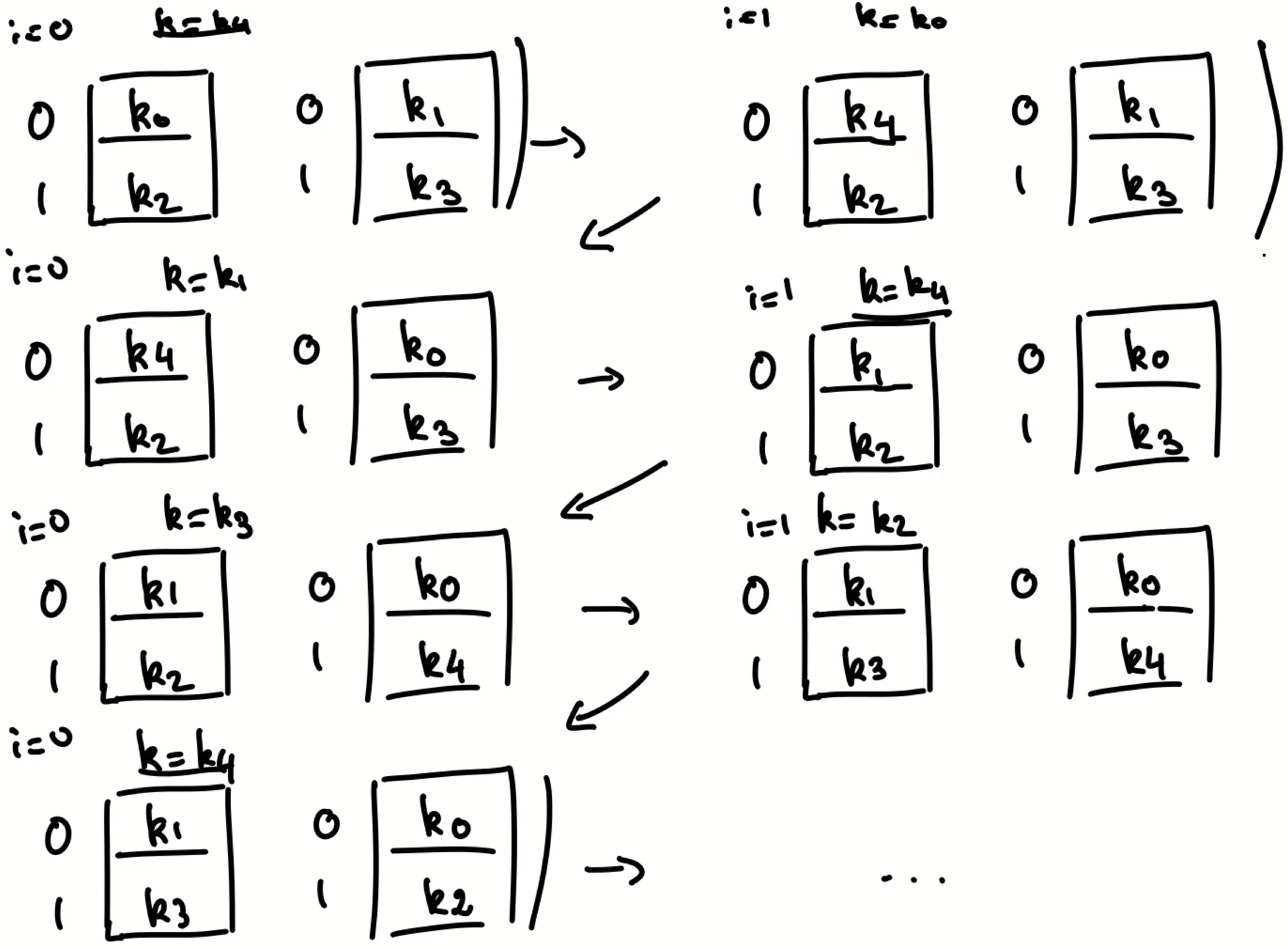
we don't exit the loop.

Proof We have n keys $k_1 \dots k_n$

2 tables

$\rightarrow \exists\, k_i$ such that we insert k_i 3 times.

$\rightarrow$ This will create an infinite loop.


i=0 k=k4
0 k0
1 k2
0 k1
1 k3
i=1 k=k0
0 k4
1 k2
0 k1
1 k3
i=0 k=k1
0 k4
1 k2
0 k0
1 k3
i=1 k=k4
0 k1
1 k2
0 k0
1 k3
i=0 k=k3
0 k1
1 k2
0 k0
1 k4
i=1 k=k2
0 k1
1 k3
0 k0
1 k4
i=0 k=k4
0 k1
1 k3
0 k0
1 k2
. . .

Cuckoo hashing discussions

- The two hash-tables need not be of the same size.
- *Load factor* $\alpha = n/(\text{size of } T_0 + \text{size of } T_1)$ $= n/2M$
- One can argue: If the load factor α is small enough then insertion has $O(1)$ expected run-time.
- This crucially requires $\alpha < \frac{1}{2}$.

There are many possible variations:

- The two hash-tables could be combined into one.
- Be more flexible when inserting: Always consider both possible positions.
- Use $k > 2$ allowed locations (i.e., k hash-functions).

Complexity of open addressing strategies

For any open addressing scheme, we *must* have $\alpha < 1$ (why?).
Cuckoo hashing requires $\alpha < 1/2$.

α is fixed, $\begin{cases} n \to \infty \\ M \to \infty \end{cases}$

Avg.-case costs:	*search (unsuccessful)*	*insert*	*search (successful)*
Linear Probing	$\frac{1}{(1-\alpha)^2}$	$\frac{1}{(1-\alpha)^2}$	$\frac{1}{1-\alpha}$
Double Hashing	$\frac{1}{1-\alpha}$	$\frac{1}{1-\alpha}$	$\frac{1}{\alpha}\log\left(\frac{1}{1-\alpha}\right)$
Cuckoo Hashing	1 (worst-case)	$\frac{\alpha}{(1-2\alpha)^2}$	1 (worst-case)

Summary: All operations have $O(1)$ average-case run-time if the hash-function is uniform and α is kept sufficiently small.
But worst-case run-time is (usually) $\Theta(n)$.

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Choosing a good hash function

- **Goal:** Satisfy uniform hashing assumption (each hash-index is equally likely)
- Proving this is usually impossible, as it requires knowledge of the input distribution and the hash function distribution.
- We can get good performance by choosing a hash-function that is
  - unrelated to any possible patterns in the data, and
  - depends on all parts of the key.
- We saw two basic methods for integer keys:
  - **Modular method**: $h(k) = k \bmod M$.
    We should choose M to be a prime.
  - **Multiplicative method**: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
    for some constant floating-point number A with $0 < A < 1$.

Universal Hashing

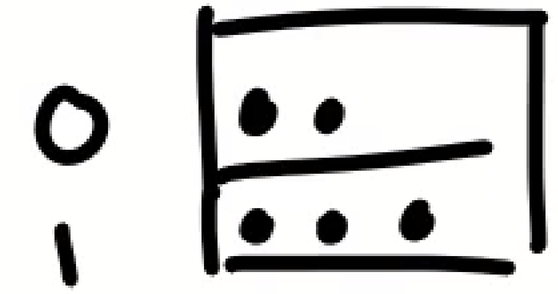
Every hash function *must* do badly for some sequences of inputs:

- If the universe contains at least ~~$M n$~~ keys, then there are n keys that all hash to the same value.

  $M\cdot(n-1)+1$

  $M=2$ $n=3$ $M\cdot(n-1)+1=5$

- For this set of keys, we have the worst case.



Idea: Randomization!

- When initializing or re-hashing, use as hash function

$$h(k) = \underline{((ak+b) \bmod p) \bmod M}$$

  where $p > M$ is a prime number, and a, b are *random* numbers in $\{0, \ldots p-1\}$, $a \neq 0$.

- Can prove: For any (fixed) numbers $x \neq y$, the probability of a collision using this random function h is at most $\frac{1}{M}$.
- Therefore the expected run-time is $O(1)$ if α is kept small enough.

We have again shifted the performance from "bad input" to "bad luck".

Multi-dimensional Data

What if the keys are multi-dimensional, such as strings in Σ^*?

Standard approach is to *flatten* string w to integer $f(w) \in \mathbb{N}$, e.g.

$$
\begin{aligned}
A \cdot P \cdot P \cdot L \cdot E \quad &\rightarrow \quad (65, 80, 80, 76, 69) \quad \text{(ASCII)} \\
&\rightarrow \quad 65R^4 + 80R^3 + 80R^2 + 76R^1 + 68R^0 \\
&\qquad \text{(for some radix } R\text{, e.g. } R = 255\text{)}
\end{aligned}
$$

We combine this with a modular hash function: $h(w) = f(w) \bmod M$

M fixed, R=255

To compute this in $O(|w|)$ time without overflow, use Horner's rule and apply mod early. For exampe, $h(APPLE)$ is

$(((65R+80)R+80)R+76)R+68 \quad \bmod M$

$$
= \left(\left(\left(\left(\left(\left(\left((65R+80) \bmod M \right) R+80 \right) \bmod M \right) R+76 \right) \bmod M \right) R+69 \right) \bmod M
$$

<255

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- Predictable space usage (exactly n nodes)
- Never need to rebuild the entire structure
- Supports ordered dictionary operations (rank, select etc.)

Advantages of Hash Tables

- $O(1)$ operations (if hashes well-spread and load factor small)
- We can choose space-time tradeoff via load factor
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete