CS 240 – Data Structures and Data Management

Module 4: Dictionaries

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Based on lecture notes by many previous cs240 instructors

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Outline

1. Dictionaries and Balanced Search Trees
   - ADT Dictionary
   - Review: Binary Search Trees
   - AVL Trees
   - Insertion in AVL Trees
   - Restoring the AVL Property: Rotations

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Dictionary ADT

Dictionary: An ADT consisting of a collection of items, each of which contains

- a *key*
- some *data* (the "value")

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*(k) (also called *findElement*(k))
- *insert*(k, v) (also called *insertItem*(k, v))
- *delete*(k) (also called *removeElement*(k)))
- optional: *closestKeyBefore*, *join*, *isEmpty*, *size*, *etc.*

Examples: symbol table, license plate database

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space (if not, the "value" could be a pointer)
- Keys can be compared in constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$ (except array occasionally needs to resize)

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$ (via binary search)

insert $\Theta(n)$

delete $\Theta(n)$

[5, 1, 10, 2, 7]

[5, 1, -, 2, 7]

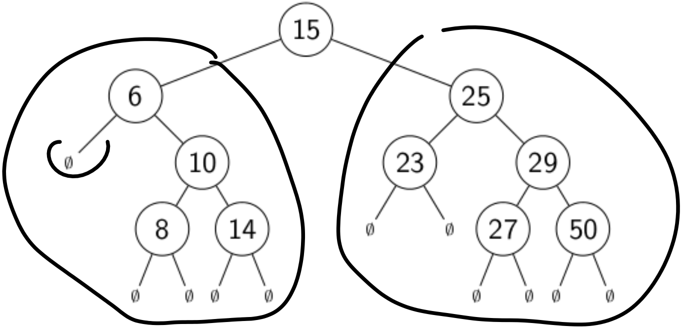
[5, 1, 2, 7]

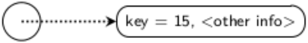
Outline

Binary Search Trees (review)

Structure Binary tree: all nodes have two (possibly empty) subtrees
Every node stores a KVP
Empty subtrees usually not shown

Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.

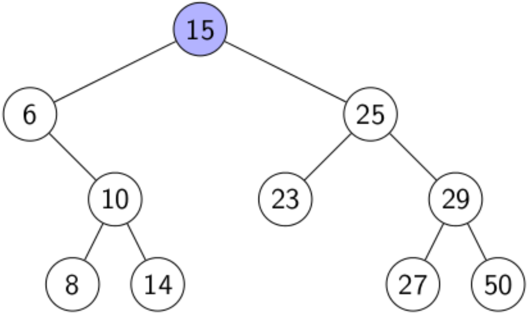


(In our examples we only show the keys, and we show them directly in the node. A more accurate picture would be )

BST as realization of ADT Dictionary

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

Example: *BST::search*(24)



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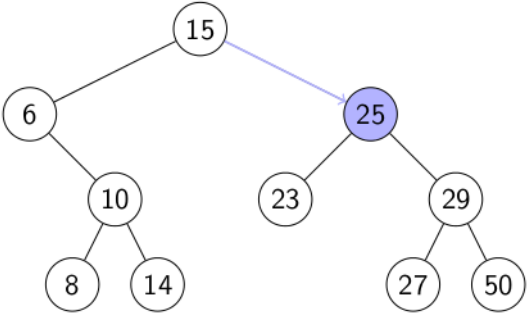

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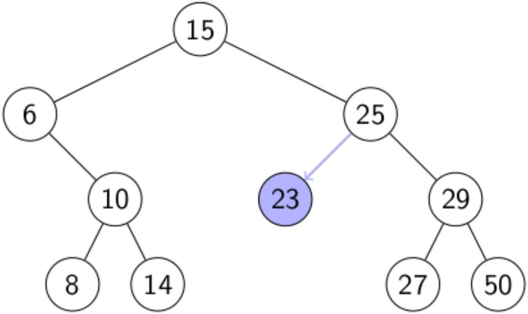

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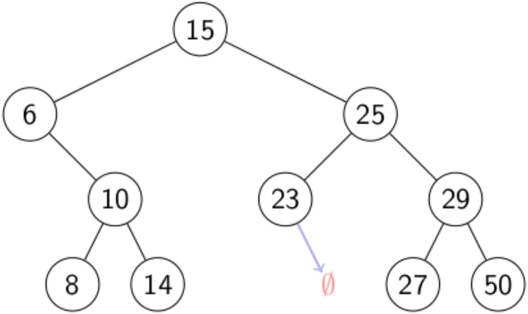
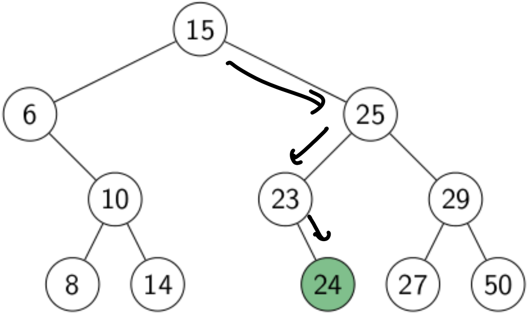


BST as realization of ADT Dictionary

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

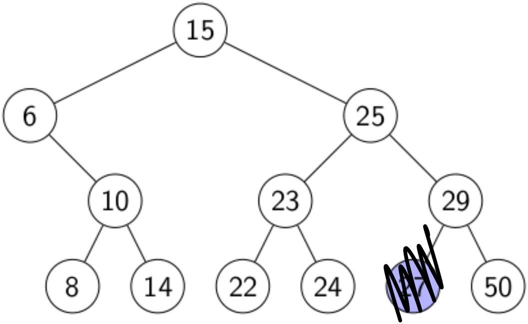
BST::insert(k, v) Search for k, then insert (k, v) as new node

Example: *BST::insert*$(24, v)$



Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.



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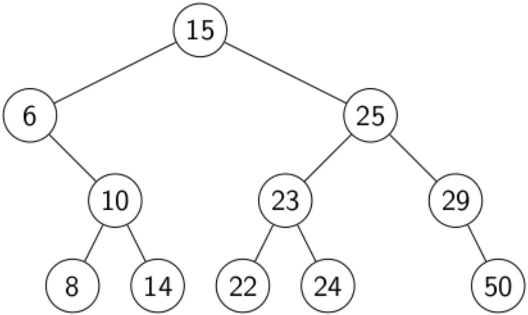

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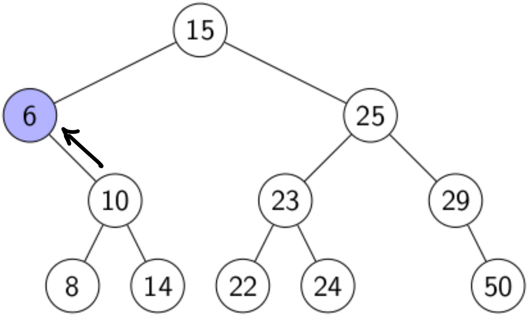

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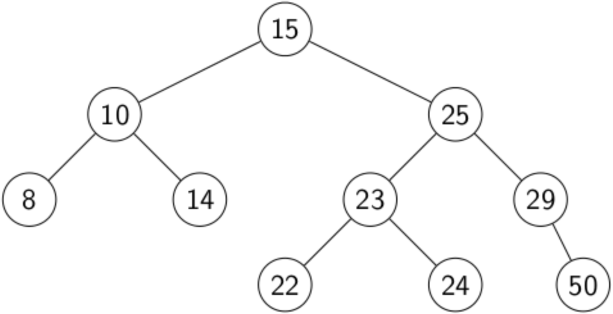

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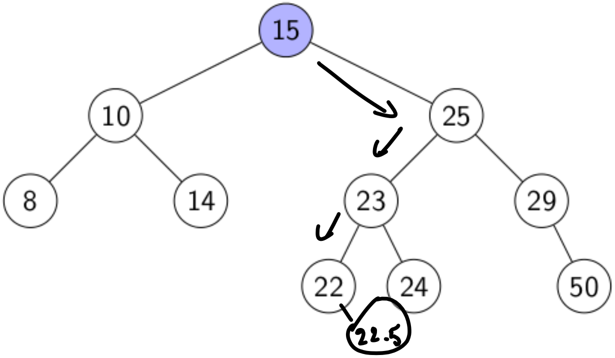

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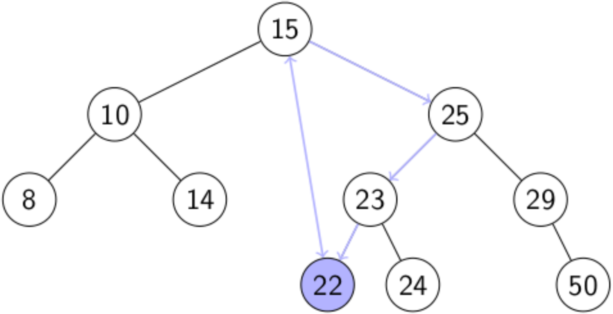

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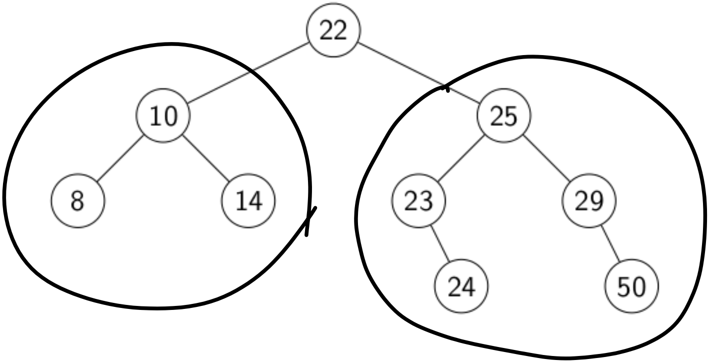
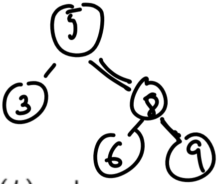


Height of a BST

$h=2$



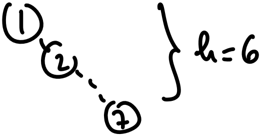
BST::search, *BST::insert*, *BST::delete* all have cost $\Theta(h)$, where
h = height of the tree = max. path length from root to leaf

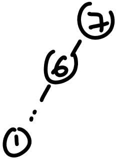
If n items are inserted one-at-a-time, how big is h?

$n = 7$

[1, 2, 3, ... 7]

- Worst-case:





Height of a BST

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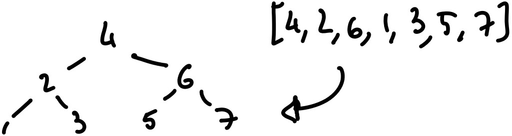
- Worst-case: $n - 1 = \Theta(n)$
- Best-case:



$\Theta(\log(n))$

Height of a BST

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If n items are inserted one-at-a-time, how big is h?

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- Best-case: $\Theta(\log n)$.
  Any binary tree with n nodes has height $\geq \log(n+1) - 1$
- Average-case:

Height of a BST

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- Best-case: $\Theta(\log n)$.
  Any binary tree with n nodes has height $\geq \log(n+1) - 1$
- Average-case: Can show $\Theta(\log n)$

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AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an **AVL Tree** is a BST with an additional **height-balance property** at every node:

> *The heights of the left and right subtree differ by at most 1.*

(The height of an empty tree is defined to be -1.)

Rephrase: If node v has left subtree L and right subtree R, then

$$\textbf{balance}(v) := \textit{height}(R) - \textit{height}(L) \text{ must be in } \{-1, 0, 1\}$$

$balance(v) = -1$ means v is *left-heavy*

$balance(v) = +1$ means v is *right-heavy*

balance (root) = -1 -1 = -2

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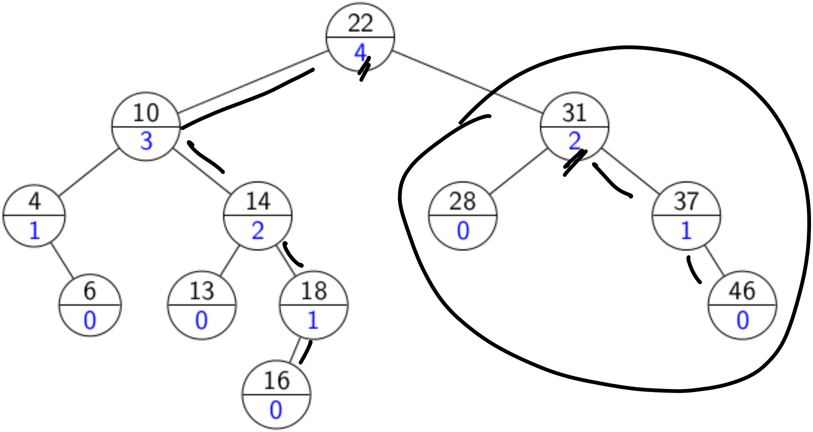
- Need to store at each node v the height of the subtree rooted at it
- Can show: It suffices to store $\mathit{balance}(v)$ instead
  - uses fewer bits, but code gets more complicated

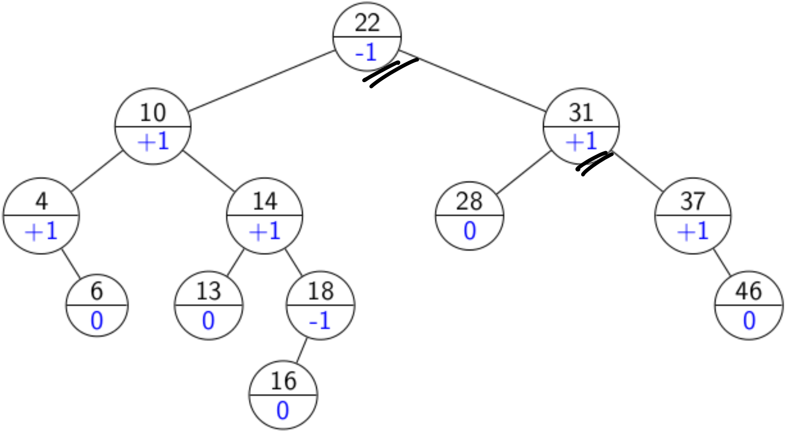
AVL tree example

(The lower numbers indicate the height of the subtree.)



AVL tree example

Alternative: store balance (instead of height) at each node.



Height of an AVL tree

Theorem: An AVL tree on n nodes has $\Theta(\log n)$ height. //
$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$ in the *worst case!*

Proof:

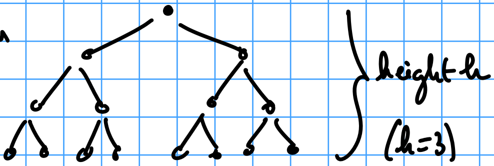
- Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.
- What is a recurrence relation for $N(h)$?
- What does this recurrence relation resolve to?

Claim: the height of <u>any</u> binary search tree with n keys is $\Omega(\log n)$.

Proof: Let h be the height of such a tree.

The number n of keys in the tree is $\leq$

the number of keys in



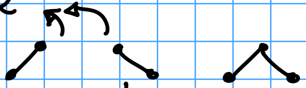
$= 2^{h+1} - 1$ $\quad (h = 3 \Rightarrow 2^{h+1} - 1 = 15)$

$\Rightarrow \quad n \leq 2^{h+1} - 1 \quad \Rightarrow \quad n+1 \leq 2^{h+1}$

$\Rightarrow \quad \log(n+1) - 1 \leq h$

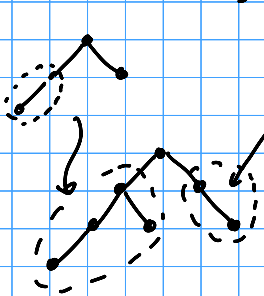
Proof of ① Fix h, let $N(h)$ be the minimum number of nodes in an AVL tree of height h.

$N(-1) = 0$. $N(0) = 1$. $N(1) = 2$

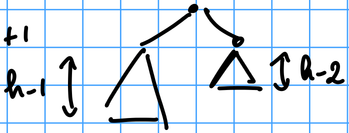


$N(2) = 4$

$N(3) = 4 + 2 + 1 = 7$



$\| \; N(h) = N(h-1) + N(h-2) + 1$



$\Rightarrow N(h) \in \Theta(\varphi^h)$, $\quad \varphi = \frac{1+\sqrt{5}}{2}$

<u>Claim</u> $N(h) \geq \sqrt{2}^{\,h} - 1$, $\quad h \geq -1$.

$h = -1 \qquad N(-1) = 0 \qquad \sqrt{2}^{-1} - 1 < 0 \qquad \checkmark$

$h = 0 \qquad N(0) = 1 \qquad \sqrt{2}^{0} - 1 = 0 \qquad \checkmark$

Assume true for $-1, 0, \ldots, h-2, h-1$. Prove for h.

$N(h) = N(h-1) + N(h-2) + 1 \geq 2N(h-2) + 1 \geq 2\left(\sqrt{2}^{\,h-2} - 1\right) + 1 \qquad 2 = \sqrt{2}^{2}$

$\geq \sqrt{2}^{\,h} \underbrace{-2+1}_{-1}$

Take an AVL tree with n nodes and height h.

$$n \geqslant N(h) \geqslant \sqrt{2}^{h} - 1$$

$$n+1 \geqslant \sqrt{2}^{h}$$

$$\log_{\sqrt{2}}(n+1) \geqslant h \rightarrow \quad h \in O(\log n).$$

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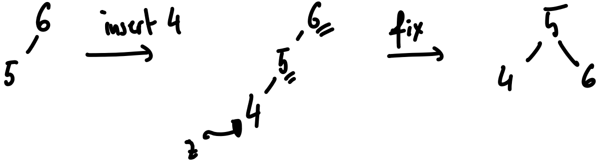
AVL insertion

To perform *AVL::insert*(k, v):

- First, insert (k, v) with the usual BST insertion.
- We assume that this returns the new leaf z where the key was stored.
- Then, move up the tree from z, updating heights.
  - We assume for this that we have parent-links. This can be avoided if *BST::Insert* returns the full path to z.
- If the height difference becomes ± 2 at node z, then z is **unbalanced**. Must re-structure the tree to rebalance.



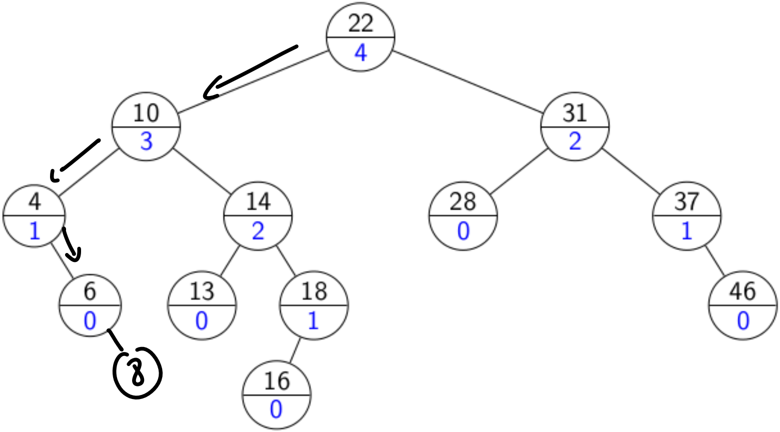
AVL insertion

AVL::insert(k, v)
$z \leftarrow$ *BST::insert*(k, v) // leaf where k is now stored
while (z is not NIL)
  if $(|z.left.height - z.right.height| > 1)$ **then**
    Let y be taller child of z
    Let x be taller child of y
    $z \leftarrow$ *restructure*(x, y, z) // see later
    break // can argue that we are done
  setHeightFromSubtrees(z)
  $z \leftarrow z.parent$

setHeightFromSubtrees(u)
$u.height \leftarrow 1 + \max\{u.left.height, u.right.height\}$

AVL Insertion Example

Example: *AVL::insert*(8)



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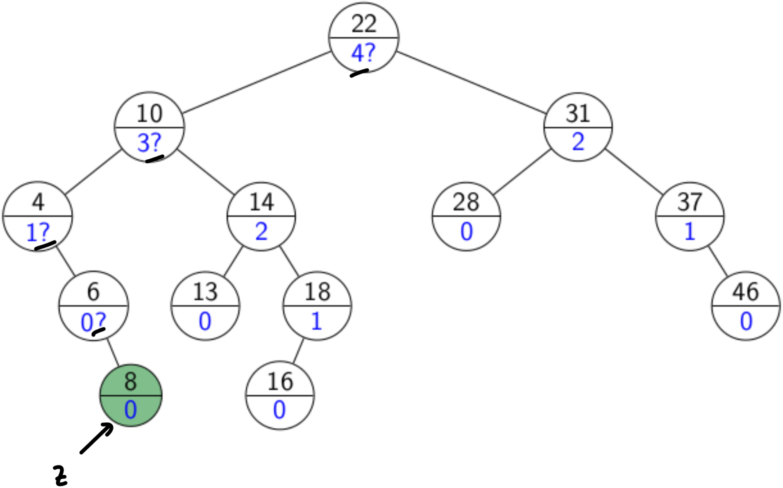

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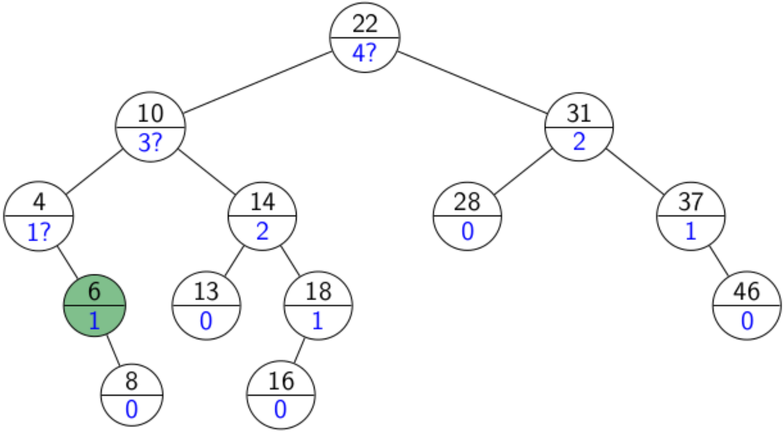

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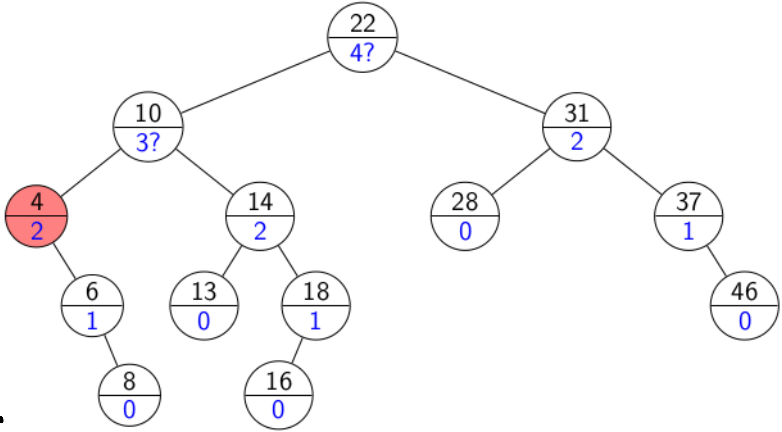

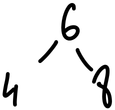

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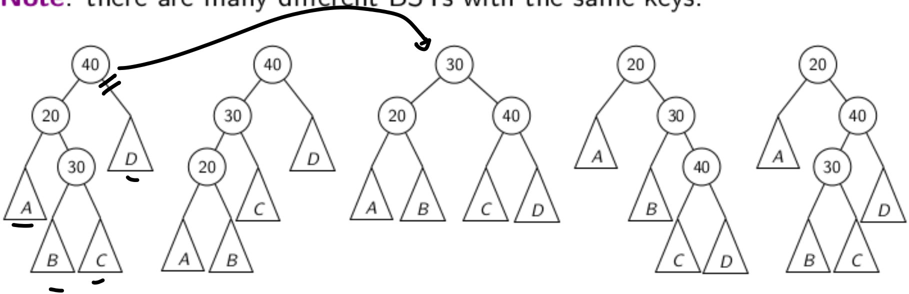
How to "fix" an unbalanced AVL tree

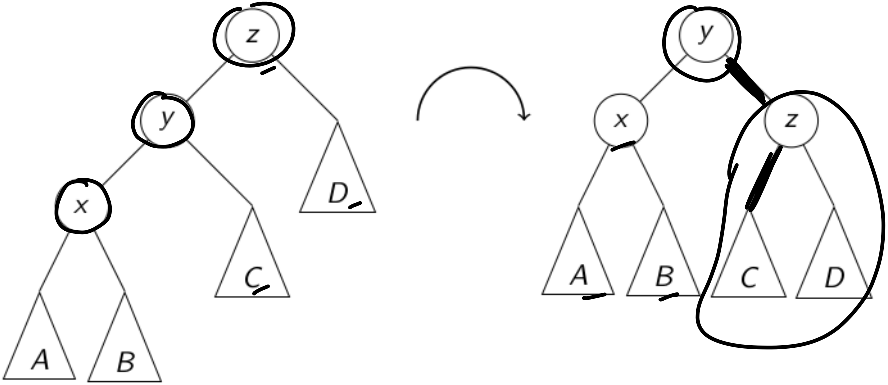
Note: there are many different BSTs with the same keys.



Goal: change the *structure* among three nodes without changing the *order* and such that the subtree becomes balanced.

Right Rotation

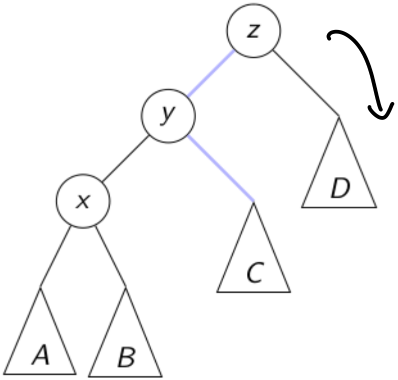
This is a **right rotation** on node z:



rotate-right(z)

1. $y \leftarrow z.left$, $z.left \leftarrow y.right$, $y.right \leftarrow z$
2. *setHeightFromSubtrees*(z), *setHeightFromSubtrees*(y)
3. **return** y // returns new root of subtree

Why do we call this a rotation?



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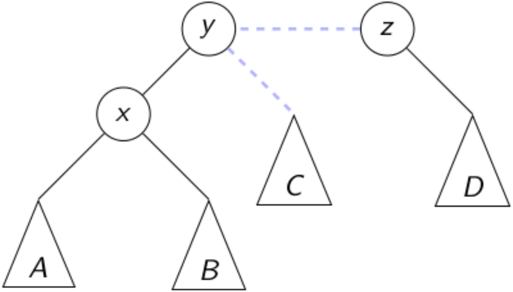

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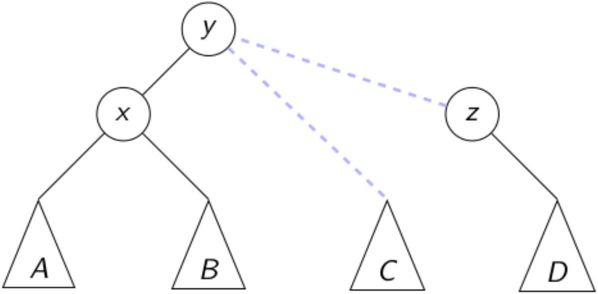

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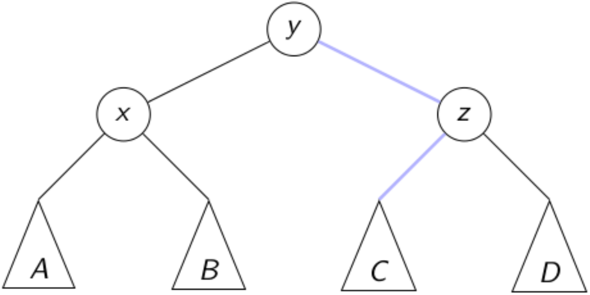
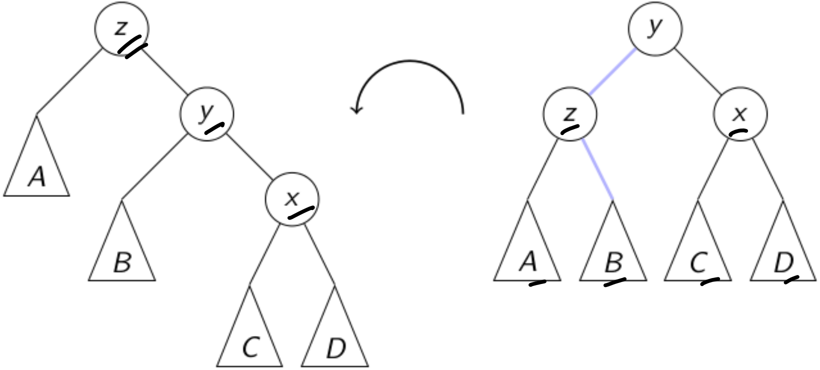


Left Rotation

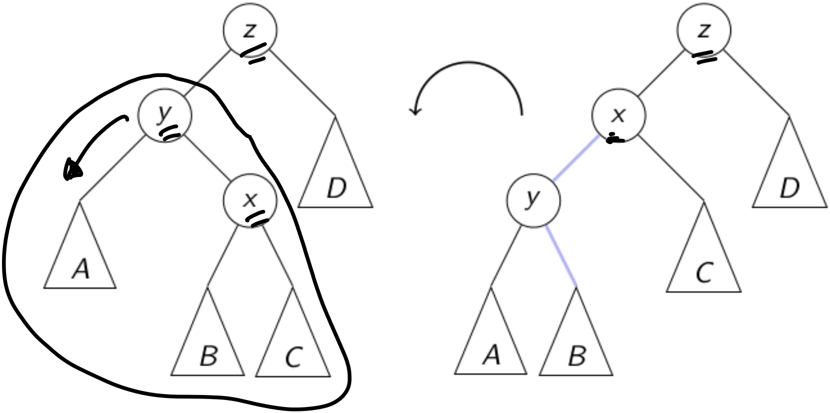
Symmetrically, this is a **left rotation** on node z:



Again, only two links need to be changed and two heights updated.
Useful to fix right-right imbalance.

Double Right Rotation

This is a **double right rotation** on node z:



First, a left rotation at y.

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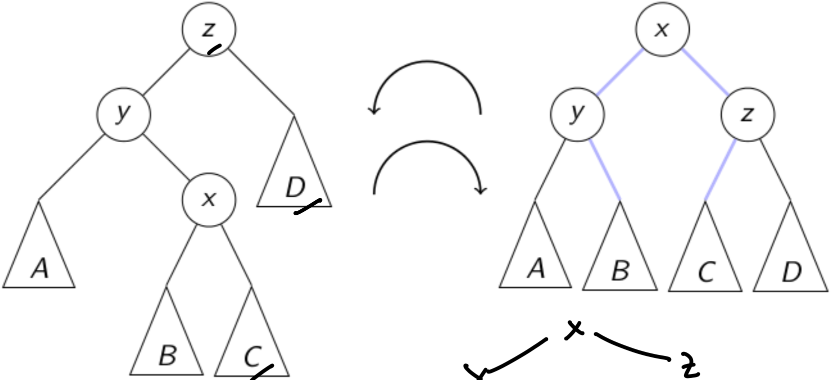
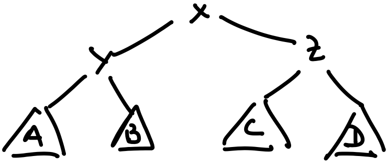
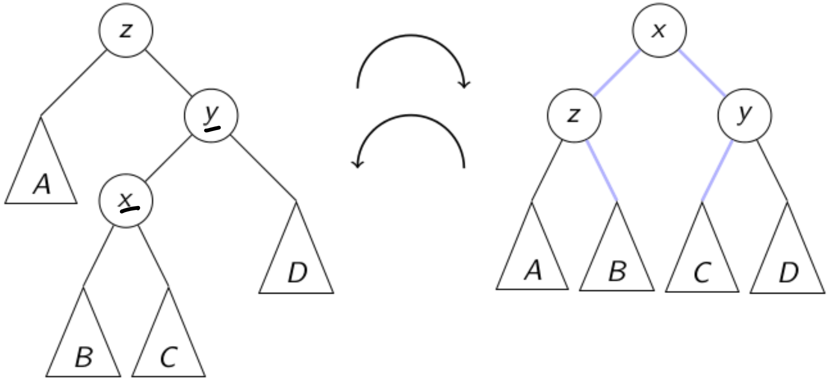




First, a left rotation at y.
Second, a right rotation at z.

Double Left Rotation

Symmetrically, there is a **double left rotation** on node z:

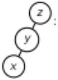


First, a right rotation at y.
Second, a left rotation at z.

Fixing a slightly-unbalanced AVL tree

restructure(x, y, z)
node x has parent y and grandparent z

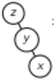
1. **case**



: // Right rotation
  return *rotate-right*(z)

: // Double-right rotation
  z.left ← *rotate-left*(y)
  return *rotate-right*(z)

: // Double-left rotation
  z.right ← *rotate-right*(y)
  return *rotate-left*(z)

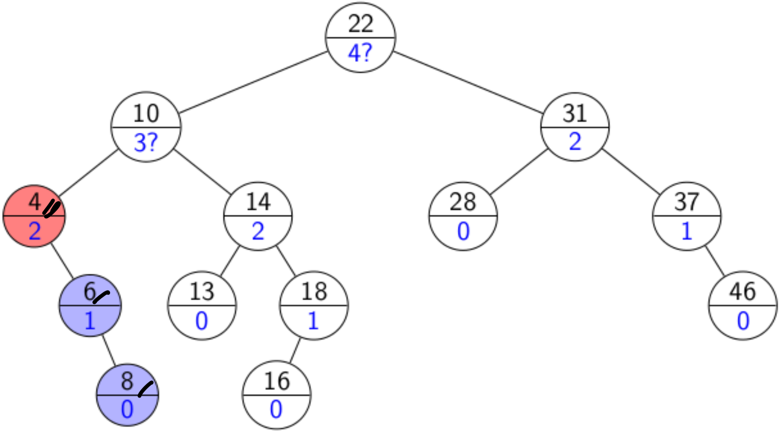


: // Left rotation
  return *rotate-left*(z)

Rule: The middle key of x, y, z becomes the new root.

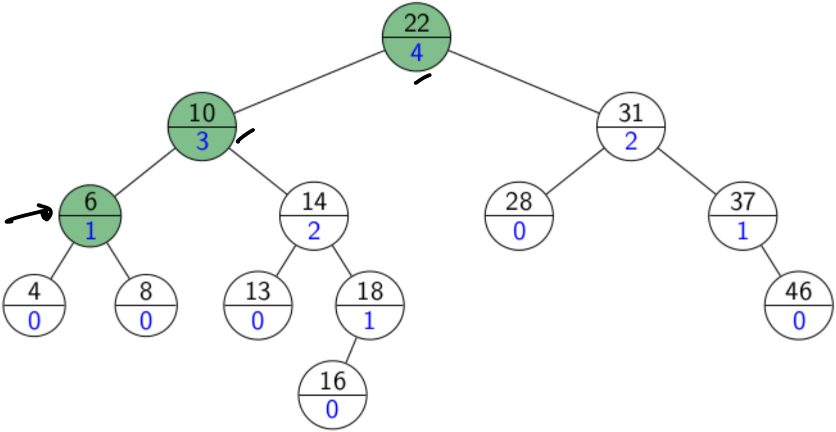
AVL Insertion Example revisited

Example: *AVL::insert*(8)



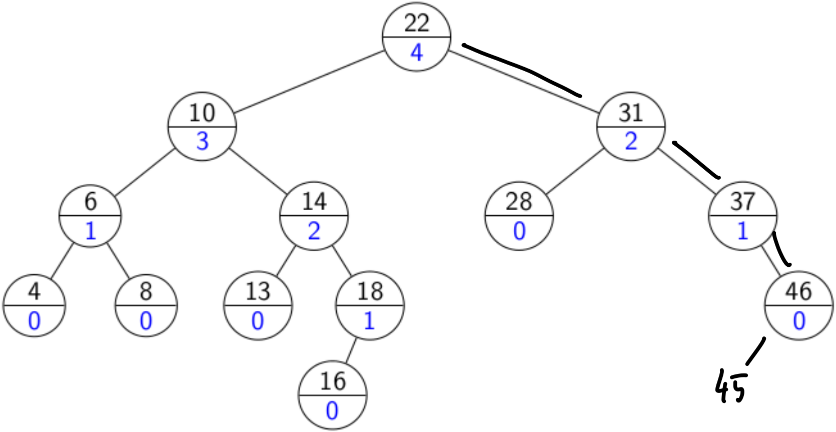
AVL Insertion Example revisited

Example: *AVL::insert*(8)



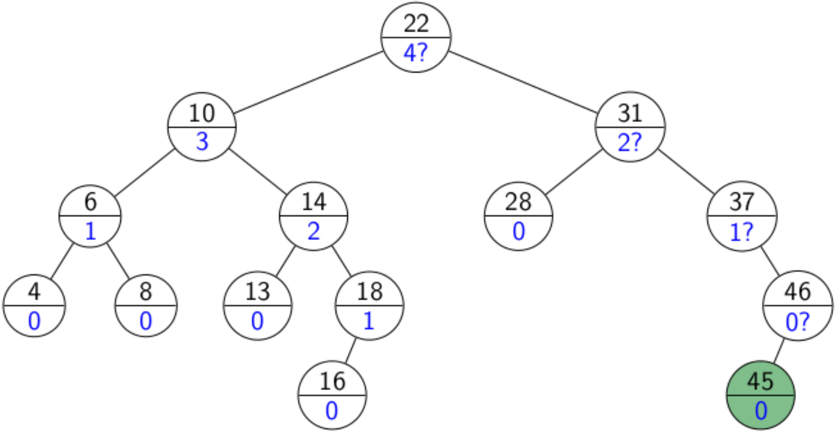
AVL Insertion: Second example

Example: *AVL::insert*(45)



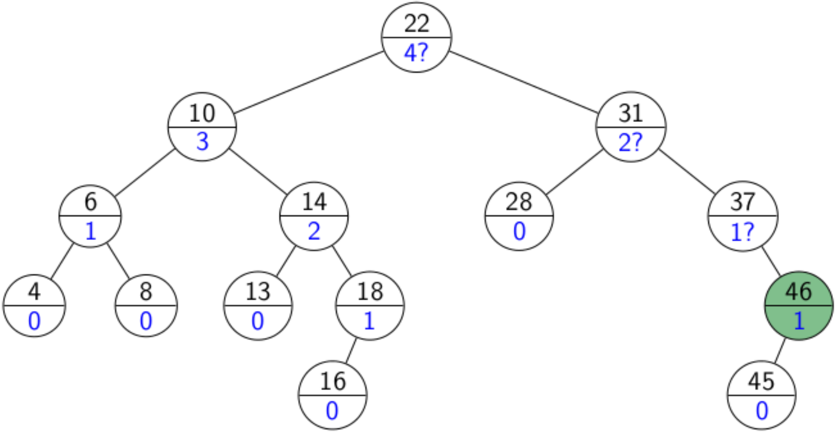
AVL Insertion: Second example

Example: *AVL::insert*(45)



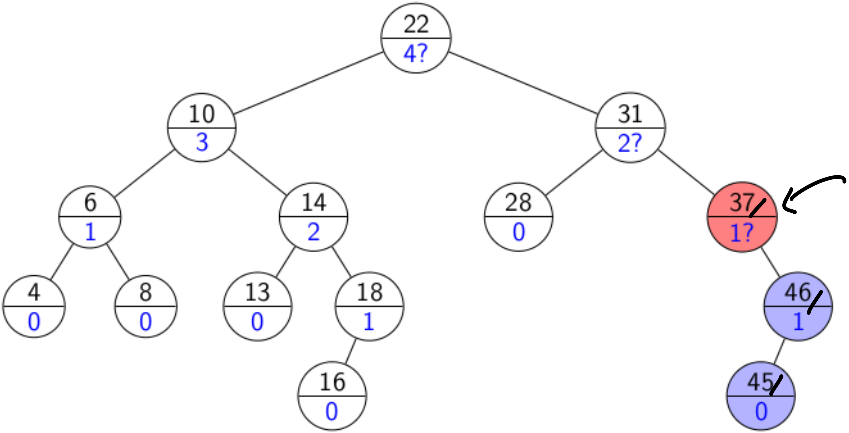
AVL Insertion: Second example

Example: *AVL::insert*(45)



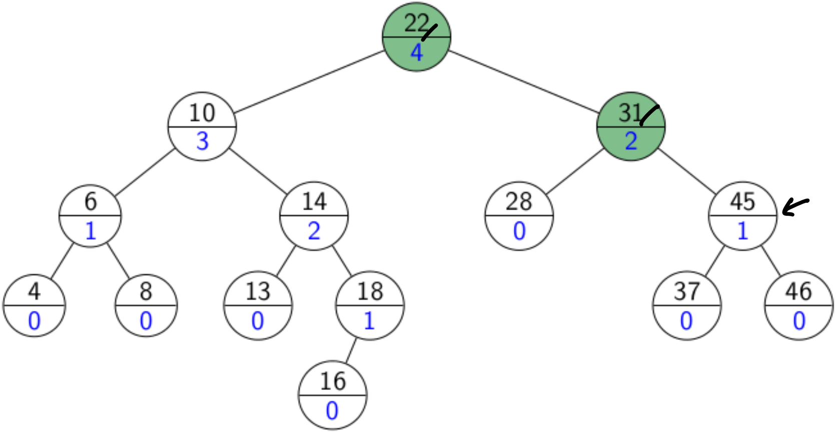
AVL Insertion: Second example

Example: *AVL::insert*(45)



AVL Insertion: Second example

Example: *AVL::insert*(45)



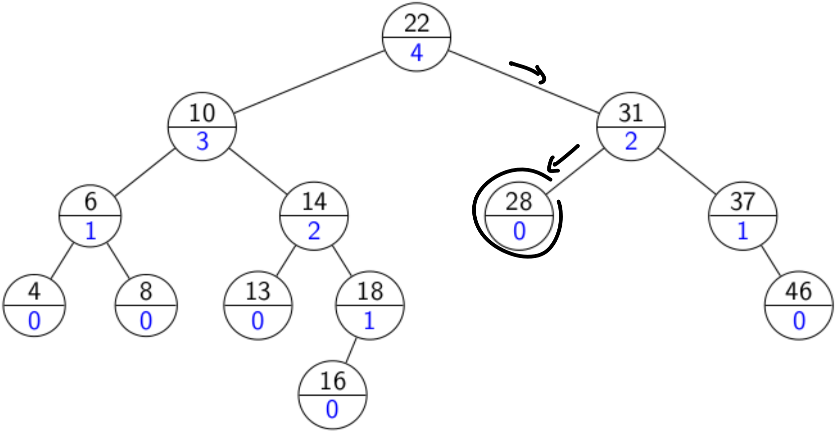
AVL Deletion

Remove the key k with *BST::delete*.
Find node where *structural* change happened.
(This is not necessarily near the node that had k.)
Go back up to root, update heights, and rotate if needed.

```
AVL::delete(k)
    z ← BST::delete(k)
    // Assume z is the parent of the BST node that was removed
    while (z is not NIL)
        if (|z.left.height − z.right.height| > 1) then
            Let y be taller child of z
            Let x be taller child of y (break ties to prefer single rotation)
            z ← restructure(x, y, z)
        // Always continue up the path and fix if needed.
        setHeightFromSubtrees(z)
        z ← z.parent
```

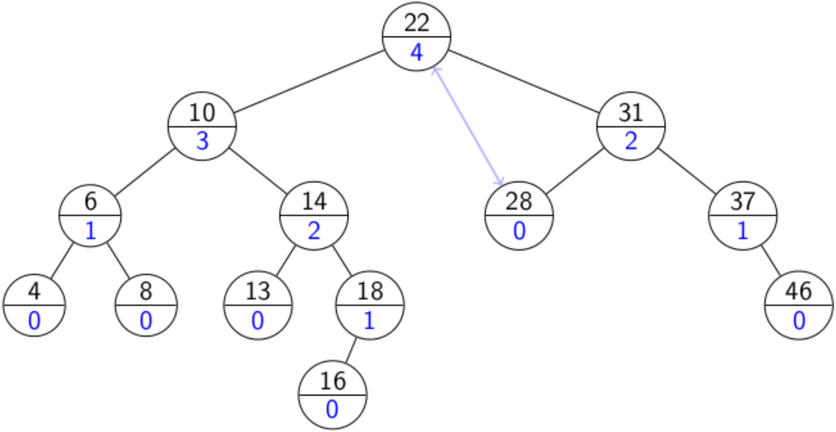
AVL Deletion Example

Example: *AVL::delete*(22)



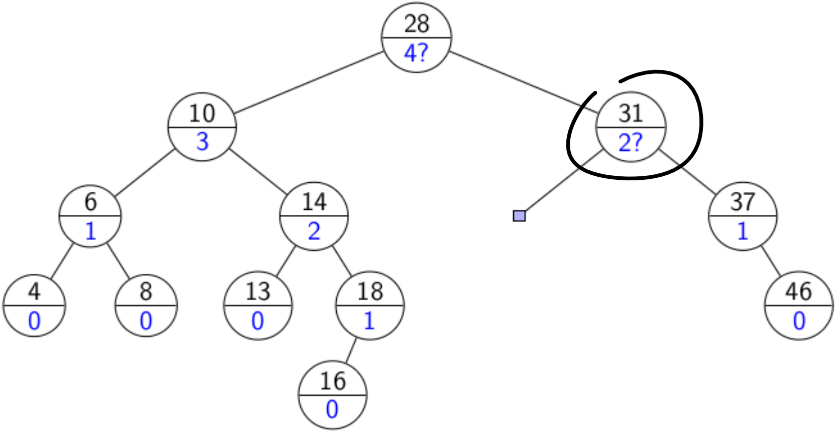
AVL Deletion Example

Example: *AVL::delete*(22)



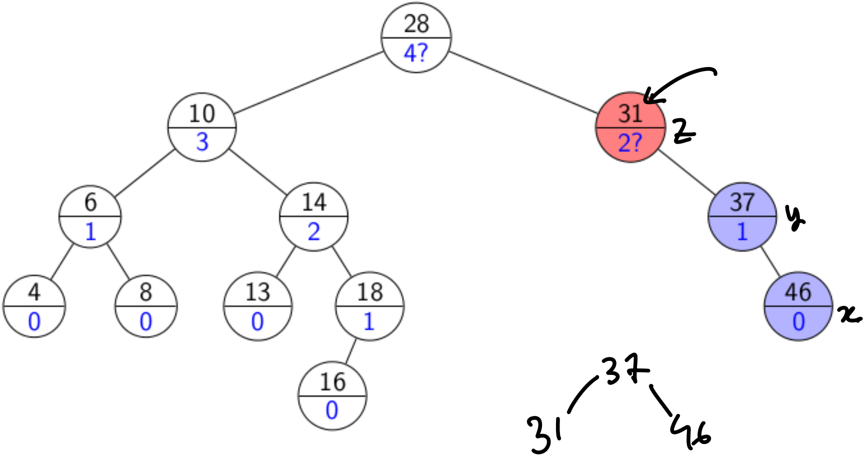
AVL Deletion Example

Example: *AVL::delete*(22)



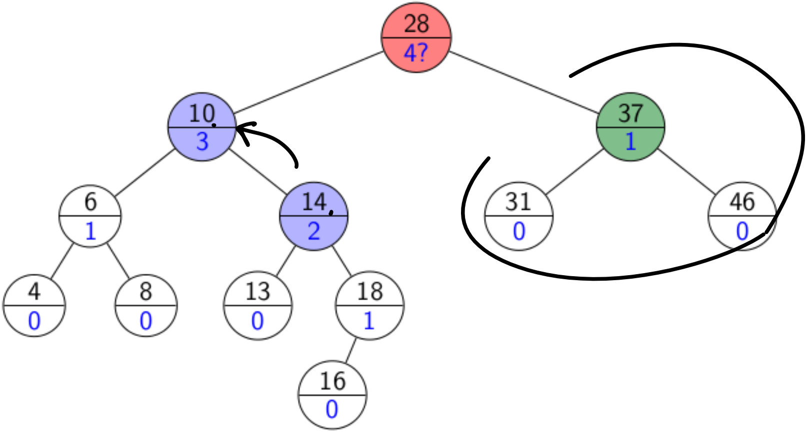
AVL Deletion Example

Example: *AVL::delete*(22)



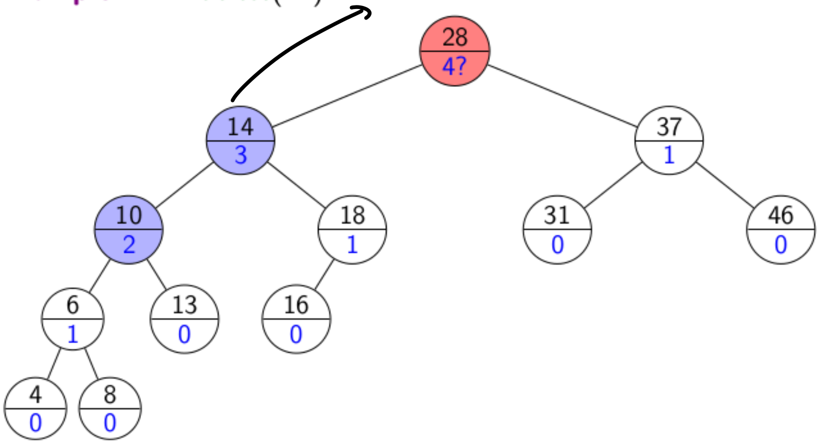
AVL Deletion Example

Example: *AVL::delete*(22)



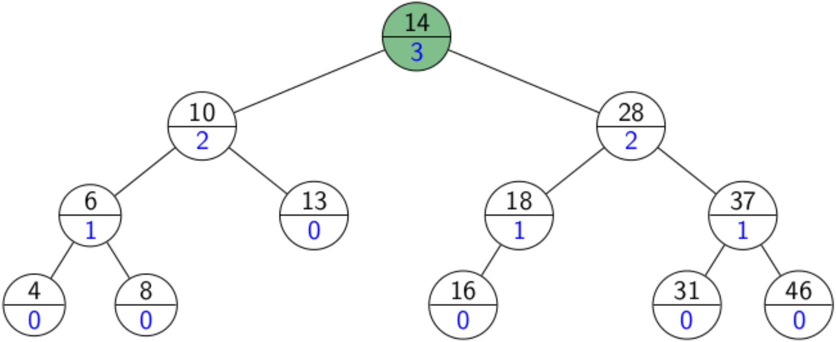
AVL Deletion Example

Example: *AVL::delete*(22)



AVL Deletion Example

Example: *AVL::delete*(22)



AVL Tree Operations Runtime

search: Just like in BSTs, costs $\Theta(height)$

insert: *BST::insert*, then check & update along path to new leaf

- total cost $\Theta(height)$
- *restructure* restores the height of the subtree to what it was,
- so *restructure* will be called *at most once*.

delete: *BST::delete*, then check & update along path to deleted node

- total cost $\Theta(height)$
- *restructure* may be called $\Theta(height)$ times.

Worst-case cost for all operations is $\Theta(height) = \Theta(\log n)$.

But in practice, the constant is quite large.

Claim: Let z be the first non-balanced node we meet after insert

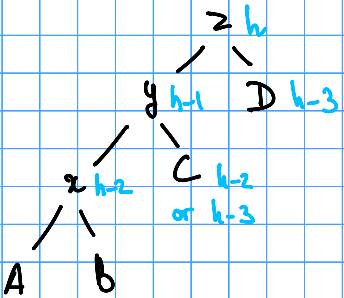
We call T the tree rooted at z

We call T' the tree after restructure.

Then: ① all nodes in T' are balanced

② height (T') = height$(T$ before insert$)$

Proof for right rotation

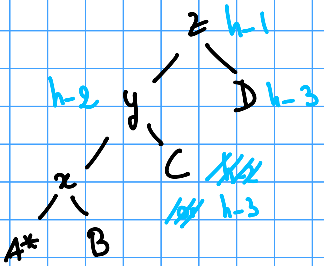


after insert

Let h be the height at z.

(after insert)

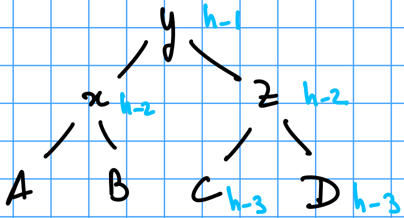
Proof for right rotation



before insert

Let h be the height at z.

(after insert)



after restructure (T')

① all nodes in T' are balanced

② height (T') = h-1 = height (before insert).

FIN.