Examples from Module 1

Example 1. *Prove that* $2n^2 + 3n + 11 \in O(n^2)$ *from first principles.*

We need to find c and n_0 such that:

$$0 \leq 2n^2 + 3n + 11 \leq cn^2 \quad \text{for all } n \geq n_0$$

The first inequality automatically holds for all $n \geq 0$. For the second, we note that:

$$\begin{aligned} 2n^2 + 3n + 11 &\leq 2n^2 + 3n^2 + 11n^2 \\ &= 16n^2 \end{aligned}$$

So we take $c = 16$ and n_0 to satisfy the inequality.

Example 2. *Prove that* $f(n) = 2n^2 + 3n + 11 \in \Omega(n^2)$ *from first principles.*

We simply take $c = n_0 = 1$, then for all $n \geq n_0$, we have that $cn^2 = n^2 \leq 2n^2 + 2n + 11$.

Example 3. *Prove that* $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$ *from first principles.*

Rearrange by splitting the quadratic term, then factor out n:

$$\begin{aligned} \frac{1}{2}n^2 - 5n &= \frac{1}{4}n^2 + \frac{1}{4}n^2 - 5n \\ &= \frac{1}{4}n^2 + n\left(\frac{1}{4} - 5n\right) \end{aligned}$$

If we have $n \geq 20$, then $\frac{1}{4} - 5n \geq 0$. So taking $c = \frac{1}{4}$ and $n_0 = 20$, we have:

$$\frac{1}{4}n^2 + n\left(\frac{1}{4} - 5n\right) \geq \frac{1}{4}n^2 = cn^2$$

Example 4. *Prove that* $\log_b(n) \in \Theta(\log n)$ *for all* $b > 1$ *from first principles.*

Set $c_1 = c_2 = \frac{1}{\log b}$ and use the change of base formula:

$$\begin{aligned} c_1 \log n &= \frac{\log n}{\log b} \\ &= \log_b n \\ &= \frac{\log n}{\log b} \\ &= c_2 \log n \end{aligned}$$

Example 5. *Let* $f(n)$ *be a polynomial of degree* $d \geq 0$*:*

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

for some $c_d > 0$.
Prove $f(n) \in \Theta(n^d)$.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{n^d} &= \lim_{n\to\infty} \left(\frac{\sum_{i=0}^{d} c_i n^i}{n^d} \right) \\
&= \sum_{i=0}^{d} \lim_{n\to\infty} \frac{c_i n^i}{n^d} \\
&= \lim_{n\to\infty} \frac{c_d n^d}{n^d} + \sum_{i=0}^{d-1} \lim_{n\to\infty} \frac{c_i n^i}{n^d}
\end{aligned}
$$

For $i < d$, $\frac{c_i n^i}{n^d} = \frac{c_i}{n^{d-i}}$ goes to 0 as $n \to \infty$, so:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{c_d n^d}{n^d} + \sum_{i=0}^{d-1} \lim_{n\to\infty} \frac{c_i n^i}{n^d} &= \lim_{n\to\infty} \frac{c_d n^d}{n^d} \\
&= c_d
\end{aligned}
$$

Because the limit is a positive constant, $f(n) \in \Theta(n^d)$.

Example 6. *Prove that $n(2 + \sin n\pi/2)$ is $\Theta(n)$.*

Because $\lim_{n\to\infty}(2 + \sin n\pi/2)$ does not exist, we can't use the limit theorem.
But $|\sin x| \le 1$, so:

$$
\begin{aligned}
1 \le 2 + \sin n\pi/2 &\le 3 \\
n \le n(2 + \sin n\pi/2) &\le 3n
\end{aligned}
$$

Example 7. *Compare the growth rates of $f(n) = \log n$ and $g(n) = n$*

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\log n}{n} &= \lim_{n\to\infty} \frac{\ln n / \ln 2}{n} && \text{Change of base} \\
&= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{\ln n}{n} \\
&= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{1/n}{1} && \text{l'Hôpital's rule} \\
&= 0
\end{aligned}
$$

Thus $\log n \in o(n)$.

Example 8. *Compare the growth rates of $f(n) = (\log n)^c$ and $g(n) = n^d$ for integer $c, d > 0$.*

Lemma.

$$
\lim_{n\to\infty} \frac{(\ln n)^c}{n^d} = 0
$$

Proof. Proof by induction on c, base case is in the previous example.

For $c \geq 2$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\ln n)^c}{n^d} &= \lim_{n\to\infty} \frac{c(\ln n)^{c-1}/n}{dn^{d-1}} && \text{l'Hôpital's rule w/ chain rule} \\
&= \frac{c}{d} \lim_{n\to\infty} \frac{(\ln n)^{c-1}}{n^d} \\
&= \frac{c}{d} 0 && \text{Induction} \\
&= 0
\end{aligned}$$

$\square$

So for the functions in the example:

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\log n)^c}{n^d} &= \frac{1}{(\ln 2)^c} \lim_{n\to\infty} \frac{(\ln n)^c}{n^d} && \text{Change of base} \\
&= 0 && \text{Lemma}
\end{aligned}$$

Thus $(\log n)^c \in o(n^d)$.