Module 1: Introduction and Asymptotic Analysis

CS 240 – Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Outline

- CS240 overview
  - course objectives
  - course topics
- Introduction and Asymptotic Analysis
  - algorithm design
  - pseudocode
  - measuring efficiency
  - asymptotic analysis
  - analysis of algorithms
  - analysis of recursive algorithms
  - helpful formulas

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Course Objectives

- When first learn to program, emphasize *correctness*
  - does program output the expected results?
- This course is also concerned with *efficiency*
  - does program use computer resources efficiently?
    - processor time, memory space
- Strong emphasis on mathematical analysis of efficiency
- Will study efficient methods of *storing*, *accessing*, and performing *operations* on large collections of data

Course Objectives

- New abstract data types (ADTs)
  - how to implement ADT efficiently using appropriate data structures
    - typical operations in data structures
      - *inserting* new data items
      - *deleting* data items
      - *searching* for specific data items
- Algorithms
  - presented in pseudocode
  - analyzed using order notation (big-Oh, etc.)

Course Topics

▪ asymptotic (big-Oh) analysis	mathematical tool for efficiency
▪ priority queues and heaps ▪ sorting, selection ▪ binary search trees, AVL trees, B-trees	twists on data structures and algorithms you already know
▪ skip lists ▪ hashing	makes efficient dictionaries in practice
▪ quadtrees, kd-trees ▪ range search	searching data in multiple dimensions
▪ tries	special dictionary for strings
▪ string matching ▪ data compression	useful for unstructured data

CS Background

- Topics covered in previous courses with relevant sections [Sedgewick]
  - arrays, linked lists (Sec. 3.2–3.4)
  - strings (Sec. 3.6)
  - stacks, queues (Sec. 4.2–4.6)
  - abstract data types (Sec. 4-intro, 4.1, 4.8–4.9)
  - recursive algorithms (5.1)
  - binary trees (5.4–5.7)
  - sorting (6.1–6.4)
  - binary search (12.4)
  - binary search trees (12.5)
  - probability and expectation (Goodrich & Tamassia, Section 1.3.4)

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Algorithm Design Terminology

- **Problem:** given a problem instance, carry out a particular computational task
    - sort an input array A
- **Problem Instance:** *input* for the specified problem
    - A = [5, 2, 1, 8, 2]
- **Problem Solution:** *output* (correct answer) for the specified problem instance
    - A = [1, 2, 2, 5, 8]
- **Size of a Problem Instance** size(*I*)
    - a positive integer measuring size of instance *I*
    - size(A) = 5
    - often use n to denote instance size
        - often input is array, and instance size is array size

Algorithm Design Terminology

- **Algorithm:** *step-by-step process* (usually described in pseudocode) for carrying out a series of computations, given an arbitrary problem instance *I*
- **Algorithm solving a problem:** algorithm ***A*** *solves* problem ***Π*** if for every instance *I* of ***Π***, ***A*** computes a valid solution in finite time
- **Program:** *implementation* of an algorithm using a specified computer language
- In this course, the emphasis is on algorithms
  - as opposed to programs or programming

Algorithms in Practice

- For a problem ***Π***, can have many algorithms
- Given a problem ***Π***
  1. **Algorithm Design:** design algorithm **A** that solves ***Π***
  2. **Algorithm Analysis:** assess *correctness* and *efficiency* of ***A***
  3. **Implementation**: if acceptable (correct and efficient), implement ***A***
     - many possible programs implementing ***A***

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Pseudocode

- Pseudocode is a method of communicating algorithm to a human
  - whereas program (implementation) is a method of communicating algorithm to a computer

```
Test3(A, n)
A: array of size n
for i ← 1 to n − 1 do
    j ← i
    while j > 0 and A[j] > A[j − 1] do
        swap A[j] and A[j − 1]
        j ← j − 1
```

- Pseudocode
  - preferred language for describing algorithms
  - omits obvious details, e.g. variable declarations
  - sometimes uses English descriptions
  - has limited if any error detection
  - sometimes uses mathematical notation

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - indentation replaces braces
- Expressions
  - $\leftarrow$ assignment
  - == equality testing
  - $\boldsymbol{n}^2$ superscripts and other mathematical formatting allowed
- Method declaration
  - **Algorithm *method* (*arg, arg*…)**
  - Input …
  - Output …

Algorithm *arrayMax*(*A*, *n*)
  Input: array *A* of *n* integers
  Output: maximum element of *A*
  $currentMax \leftarrow \boldsymbol{A}[0]$
  for $\boldsymbol{i} \leftarrow 1$ **to** $\boldsymbol{n} - 1$ **do**
    if $A[i] > currentMax$ **then**
      $currentMax \leftarrow A[i]$
  return *currentMax*

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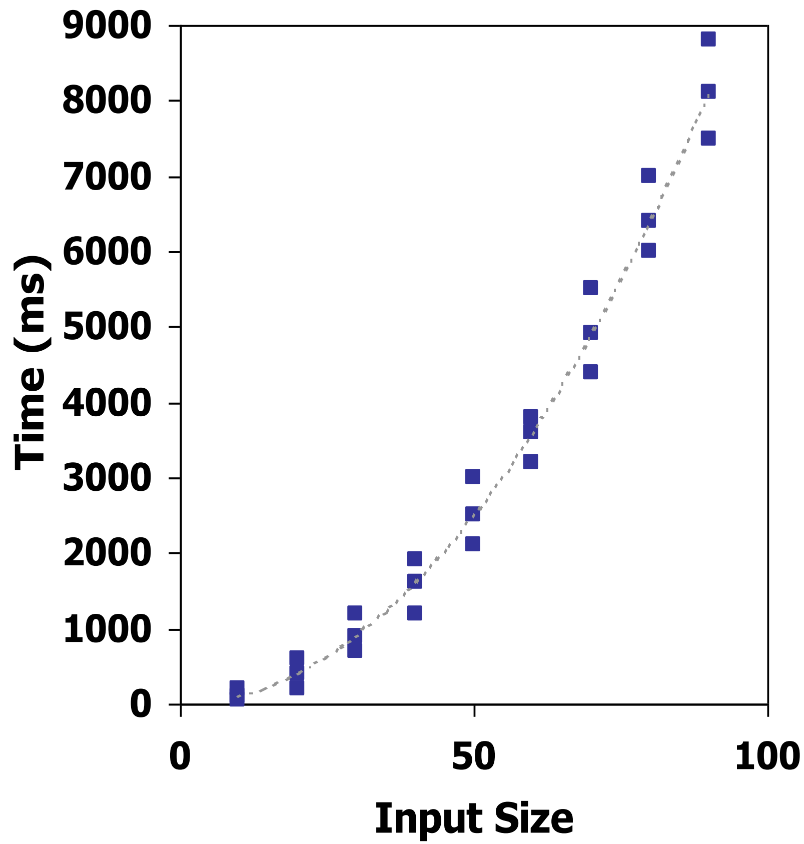
Efficiency of Algorithms/Programs

- How decide which algorithm or program is the most efficient for a given problem?
- Efficiency
  - **time:** *amount of time* program takes to run
    - also called time complexity
  - **space:** *amount of memory* program requires
    - also called space complexity
- Efficiency depends on *size(I)*, size of a given problem instance *I*
  - efficiency is a function of input size
- Primarily concerned with time efficiency in this course

Running Time of Algorithms/Programs

- One option: *experimental studies*
  - write program implementing the algorithm
  - run program with inputs of *varying size* and *composition*
  - can use clock() from time.h, to measure running time
  - plot/compare results



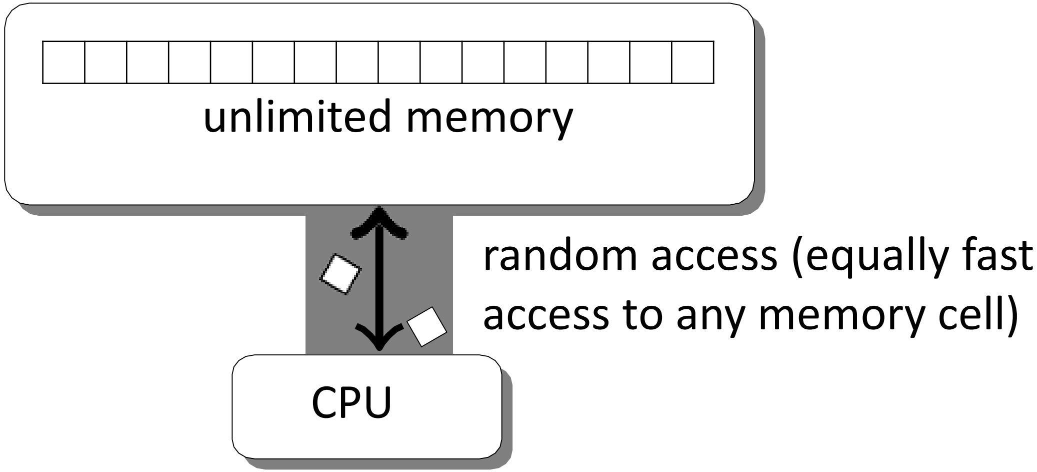
Running Time of Algorithms/Programs

- Shortcomings of experimental studies
  - implementation may be complicated/costly
  - timings are affected by many factors
    - *hardware* (processor, memory)
    - *software environment* (OS, compiler, programming language)
    - *human factors* (programmer)
  - cannot test all inputs, hard to select good *sample inputs*
  - thus cannot easily compare two algorithms/programs
- Want framework that
  - does not require implementing the algorithm
  - independent of hardware/software environment
  - takes into account all possible input instances

Theoretical Framework For Algorithm Analysis

- To overcome dependency on hardware/software
  - write algorithms in pseudo-code
    - language independent
  - “run” algorithms on idealized computer model
    - allows to reason about efficiency

Idealized Computer Model



- **Random Access Machine (RAM)** Model
  - has a set of memory cells, each of which stores one data item
    - memory cells are big enough to hold stored items
  - any *access to a memory location* takes constant time
  - run *primitive operations* on this machine
    - primitive operation takes constant time
- Simplified model
  - most of these assumptions are not valid for a real computer

Theoretical Framework For Algorithm Analysis

- To overcome dependency on hardware/software
  - write algorithms in pseudo-code
    - language independent
  - “run” algorithms on idealized computer model
    - allows to reason about efficiency
  - instead of time, count number of *primitive operations*
    - assume all primitive operations take the same time
  - measure time efficiency of an algorithm in terms of growth rate
    - avoids complicated functions and isolates the factor that effects the efficiency the most for large inputs
- This framework makes many simplifying assumptions
  - makes analysis of algorithms easier

Theoretical Analysis of Running time

- Pseudocode is a sequence of *primitive operations*
- A primitive operation is
  - independent of input size
- Examples of Primitive Operations
  - addition, subtraction, etc.
    - $x \cdot n$ is a primitive operation
    - x^n is not a primitive operation, runtime depends on input size n
  - assigning a value to a variable
  - indexing into an array
  - returning from a method
  - exact definition not important
    - will see why later

Algorithm ***arrayMax***(***A***, ***n***)
  Input: array *A* of *n* integers
  Output: maximum element of *A*
  currentMax $\leftarrow$ ***A***[0]
  for ***i*** $\leftarrow$ 1 **to** ***n*** $-$ 1 **do**
    if *A*[*i*] $>$ *currentMax* **then**
      currentMax $\leftarrow$ *A*[*i*]
  return *currentMax*

- To find running time, count the number of primitive operations
  - as a function of input size $\boldsymbol{n}$

Theoretical Analysis of Running time

- To find running time, count the number of primitive operations $T(\boldsymbol{n})$
  - function of input size $\boldsymbol{n}$

Algorithm ***arrayMax***(***A***, ***n***)	# operations
$currentMax \leftarrow A[0]$	2
for $i \leftarrow 1$ **to** $n - 1$ **do**	
if $A[i] > currentMax$ **then**	
$currentMax \leftarrow A[i]$	
{ increment counter ***i*** }	
return $currentMax$	

Theoretical Analysis of Running time

- To find running time, count the number of primitive operations $T(\boldsymbol{n})$
  - function of input size $\boldsymbol{n}$

Algorithm ***arrayMax***(***A***, ***n***) # operations

currentMax ← *A*[0] 2
for *i* ← 1 **to** *n* – 1 **do**
if *A*[*i*] > *curr*
curr
{ increment coun
return *currentMa*

i ← 1
$n - 1$
$i = 1$, check $i < n - 1$ (enter inside loop)
$i = 2$, check $i < n - 1$ (enter inside loop)
…
$i = n - 1$, check $i < n - 1$(enter inside loop)
$i = n$, check $i < n - 1$(do not enter inside loop)

Total: 2+n

Theoretical Analysis of Running time

- To find running time, count the number of primitive operations $T(\boldsymbol{n})$
  - function of input size $\boldsymbol{n}$

Algorithm ***arrayMax***(***A***, ***n***) # operations

$currentMax \leftarrow A[0]$	2
for $i \leftarrow 1$ **to** $n - 1$ **do**	$2 + \boldsymbol{n}$
if $A[i] > currentMax$ **then**	$2(\boldsymbol{n} - 1)$
$currentMax \leftarrow A[i]$	$2(\boldsymbol{n} - 1)$
{ increment counter ***i*** }	$2(\boldsymbol{n} - 1)$
return $currentMax$	1
	Total: $7\boldsymbol{n} - 1$

Theoretical Analysis of Running time: Multiplicative factors

- Algorithm ***arrayMax*** executes $\boldsymbol{T}(\boldsymbol{n}) = 7\boldsymbol{n} - 1$ primitive operations
- Let a = time taken by fastest primitive operation

  b = time taken by slowest primitive operation
- $\boldsymbol{T}(\boldsymbol{n})$ is bounded by two linear functions

$$a\,(7\boldsymbol{n} - 1) \leq \boldsymbol{T}(\boldsymbol{n}) \leq b(7\boldsymbol{n} - 1)$$

- Changing hardware/software environment affects $\boldsymbol{T}(\boldsymbol{n})$ by a multiplicative constant factor
- $\boldsymbol{T}(\boldsymbol{n}) = const \cdot n$ [ignoring the subtracted constant]
  - $const$ will change depending on software/hardware environment
- Want to say $\boldsymbol{T}(\boldsymbol{n}) = 7\boldsymbol{n} - 1$ is essentially $\boldsymbol{n}$
- **Want to ignore constant multiplicative factors**

Theoretical Analysis of Running time: Lower Order Terms

- Running time on small inputs hardly ever matters
  - consider behaviour of algorithms for large input sizes
  - further simplifies running time analysis
- Consider $\boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{n}^2 + \boldsymbol{n}$
- For large $\boldsymbol{n}$, only the fastest growing factor is important

$$\boldsymbol{T}(100{,}000) = 10{,}000{,}000{,}000 + 100{,}000$$

- **Want to ignore slower growing terms**

Theoretical Analysis of Running time

- Thus we want to ignore
  - multiplicative constant factors
  - lower-order (slower growing) terms
- This means focusing on the *growth rate* of the function
  - $10\boldsymbol{n}^2 + 100\boldsymbol{n}$ has growth rate of $\boldsymbol{n^2}$
  - $10\boldsymbol{n} + 10$ has growth rate of $\boldsymbol{n}$
- Asymptotic analysis (i.e. order notation) gives tools to formally focus on the growth rate

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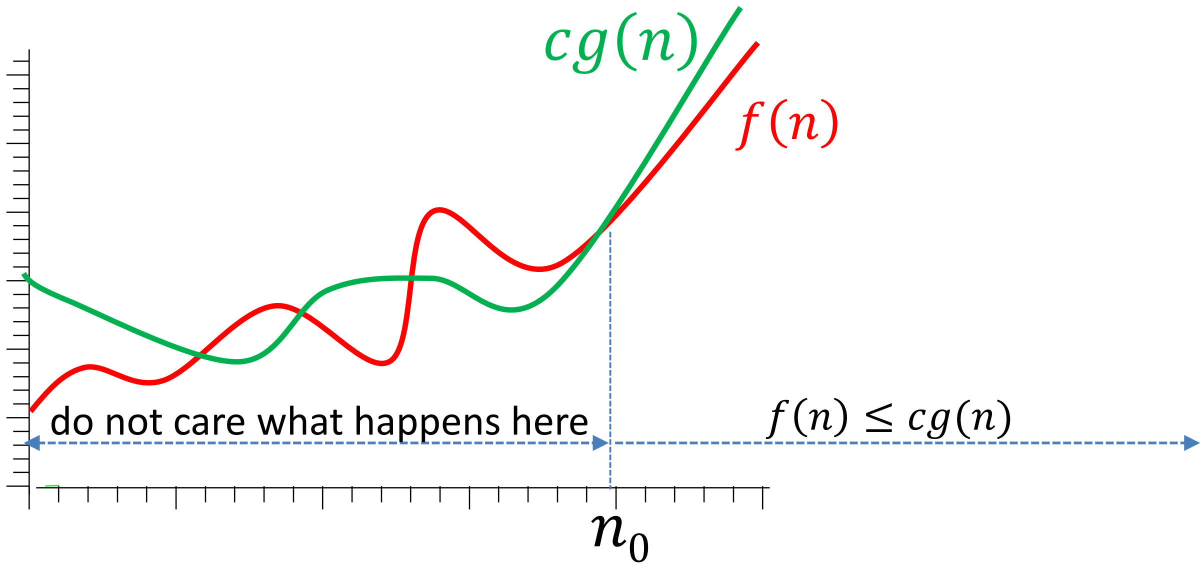
Order Notation: big-Oh

- Bound from above by function expressing "growth rate"

$f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ s.t.

$$|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0$$

a set of functions



- Need c to "get rid" of multiplicative constant in the growth rate
  - cannot say $5n^2 \leq n^2$, but can say $5n^2 \leq cn^2$ for some constant c
- Absolute value signs are not relevant for analysis of run-time or space, but useful in other applications of asymptotic notation

big-Oh Example

O-notation

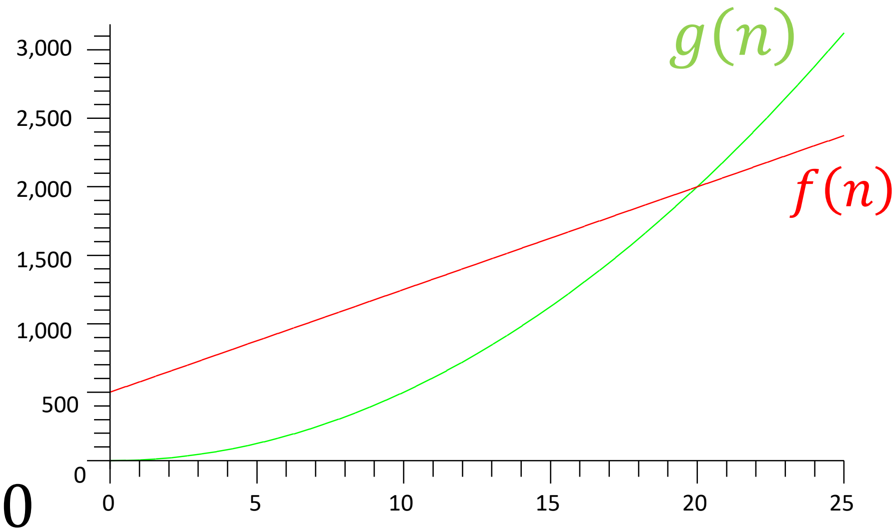
$f(n) \in O\big(g(n)\big)$ if there exist constants $c > 0$ and $n_0 \geq 0$ s.t.

$|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

- Example:

  $f(n) = 75n + 500$

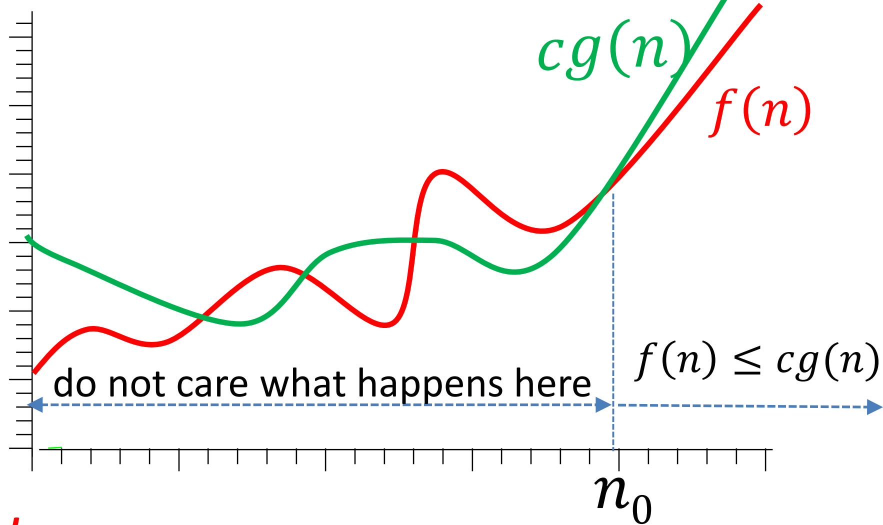
  $g(n) = 5n^2$



- Take $c = 1, n_0 = 20$
- Can also take $c = 10, n_0 = 30$

Order Notation: big-Oh

$f(n) \in O(g(n))$
if there exist constants $c > 0$ and $n_0 \geq 0$
s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$



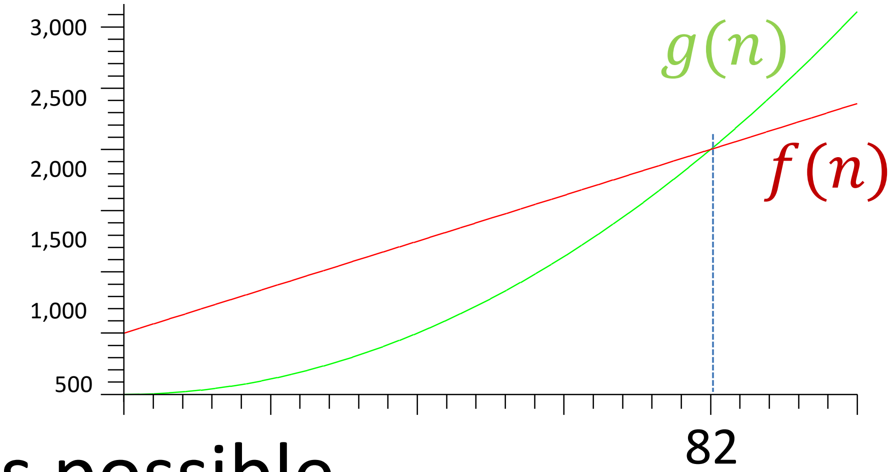
- Big-O gives asymptotic *upper bound*
  - $f(n) \in O(g(n))$ means function $f(n)$ is "bounded" above by function $g(n)$
    1. eventually, for large enough n
    2. ignoring multiplicative constant
  - Growth rate of $f(n)$ is slower or the same as growth rate of $g(n)$
- Use big-O to bound the growth rate of algorithm
  - $f(n)$ for running time
  - $g(n)$ for growth rate
    - should choose $g(n)$ as simple as possible
- Saying $f(n)$ is $O(g(n))$ is equivalent to saying $f(n) \in O(g(n))$

Order Notation: big-Oh

$f(n) \in O(g(n))$

if there exist constants $c > 0$ and $n_0 \geq 0$

s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$



- Choose $g(n)$ as simple as possible
- Previous example: $f(n) = 75n + 500$, $g(n) = 5n^2$
- Simpler function for growth rate: $g(n) = n^2$
- Can show $f(n) \in O(g(n))$ as follows
  - set $f(n) = g(n)$ and solve quadratic equation
  - intersection point is $n = 82$
  - take $c = 1, n_0 = 82$

Order Notation: big-Oh

$f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ s. t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

- Do not have to solve quadratic equation
- $f(n) = 75n + 500,\ g(n) = n^2$
- Show $f(n) \in O(g(n))$
- For all $n \geq 1$

$$75n \leq 75n \cdot n = 75n^2$$

  - Side note: for $0 < n < 1$

$$75n > 75n \cdot n = 75n^2$$

Order Notation: big-Oh

$f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

- Do not have to solve quadratic equation
- $f(n) = 75n + 500,\ g(n) = n^2$
- Show $f(n) \in O(g(n))$

$$75n + 500 \leq 75n^2 + 500n^2 = 575n^2$$

$$\text{for all } n \geq 1$$

- So take $c = 575, n_0 = 1$

Order Notation: big-Oh

$f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$
s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

- Better (i.e. "tighter") bound on growth
  - can bound $f(n) = 75n + 500$ by slower growth than n^2
- $f(n) = 75n + 500,\ g(n) = n$
- Show $f(n) \in O(g(n))$

$$75n + 500 \leq 75n + 500n = 575n$$

for all $n \geq 1$

- So take $c = 575, n_0 = 1$

More big-O Examples

- Prove that

$$2n^2 + 3n + 11 \in O(n^2)$$

- Need to find $c > 0$ and $n_0 \geq 0$ s.t.

$$2n^2 + 3n + 11 \leq cn^2 \text{ for all } n \geq n_0$$

$$2n^2 + 3n + 11 \leq 2n^2 + 3n^2 + 11n^2 = 16n^2$$

$$\text{for all } n \geq 1$$

- So take $c = 16, n_0 = 1$

More big-O Examples

- Prove that

$$2n^2 - 3n + 11 \in O(n^2)$$

- Need to find $c > 0$ and $n_0 \geq 0$ s.t.

$$2n^2 - 3n + 11 \leq cn^2 \text{ for all } n \geq n_0$$

$$2n^2 - 3n + 11 \leq 2n^2 + 0 + 11n^2 = 13n^2$$

$$\text{for all } n \geq 1$$

- Take $c = 13, n_0 = 1$

More big-O Examples

- Have to be careful with logs
- Prove that

$$2n^2 \log n + 3n \in O(n^2 \log n)$$

- Need to find $c > 0$ and $n_0 \geq 0$ s.t.
  $2n^2 \log n + 3n \leq cn^2 \log n$ for all $n \geq n_0$

$$2n^2 \log n + 3n \leq 2n^2 \log n + 3n^2 \log n \leq 5n^2 \log n$$

~~for all $n \geq 1$~~

for all $n \geq 2$

- Take $c = 5, n_0 = 2$

Theoretical Analysis of Running time

- To find running time, count the number of primitive operations $T(\boldsymbol{n})$
  - function of input size $\boldsymbol{n}$
- Last step: express the running time using asymptotic notation

Algorithm *arrayMax*(*A*, *n*) # operations

currentMax ← *A*[0] c_1

for *i* ← 1 **to** *n* − 1 **do**

if *A*[*i*] > *currentMax* **then** $c_2 n$

currentMax ← *A*[*i*]

{ increment counter ***i*** }

return *currentMax* c_3

Total: $c_1 + c_3 + c_2 n$ which is $O(n)$

Theoretical Analysis of Running time

- To find running time, count the number of primitive operations $T(\boldsymbol{n})$
  - function of input size $\boldsymbol{n}$
- Last step: express the running time using asymptotic notation

Algorithm ***arrayMax*(*A*, *n*)** # operations

$currentMax \leftarrow A[0]$

for $i \leftarrow 1$ **to** $n - 1$ **do**

  if $A[i] > currentMax$ **then**

    $currentMax \leftarrow A[i]$

{ increment counter ***i*** }

return *currentMax*

cn

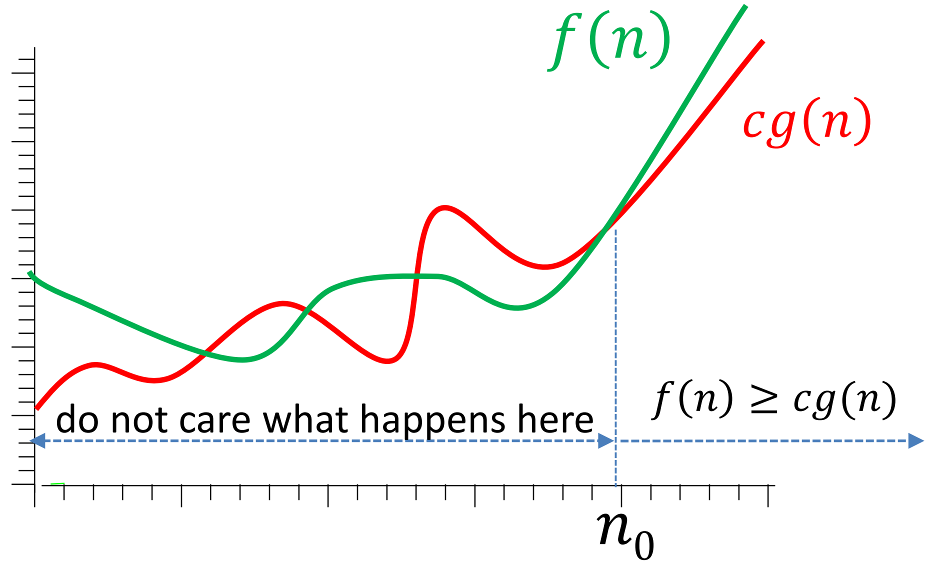
c

Total: $c + cn$ which is $O(n)$

Need for Asymptotic Tight bound

- $2n^2 + 3n + 11 \in O(n^2)$
- But also $2n^2 + 3n + 11 \in O(n^{10})$
    - this is a true but hardly a useful statement
    - if I say I have less than a million $ in my pocket, it is a true, but useless statement
    - i.e. this statement does not give a tight upper bound
    - a bound is tight if it uses the slowest growing function possible
- Want an asymptotic notation that guarantees a tight bound
- For tight bound, also need asymptotic *lower bound*

Aymptotic Lower Bound



- Ω-notation (asymptotic lower bound)

$$f(n) \in \Omega(g(n)) \text{ if there exist constants } c > 0 \text{ and } n_0 \geq 0$$
$$\text{s.t. } |f(n)| \geq c|g(n)| \text{ for all } n \geq n_0$$

- $f(n) \in \Omega\big(g(n)\big)$ means function $f(n)$ is asymptotically bounded below by function $g(n)$
  1. eventually, for large enough n
  2. ignoring multiplicative constant
- Growth rate of $f(n)$ is larger or the same as growth rate of $g(n)$

Asymptotic Lower Bound

$f(n) \in \Omega(g(n))$ if $\exists$ constants $c > 0,\ n_0 \geq 0$ s.t. $|f(n)| \geq c|g(n)|$ for $n \geq n_0$

- Prove that $2n^2 + 3n + 11 \in \Omega(n^2)$
- Find $c > 0$ and $n_0 \geq 0$ s.t.

$$2n^2 + 3n + 11 \geq cn^2 \quad \text{for all } n \geq n_0$$

$$2n^2 + 3n + 11 \geq 2n^2 \quad \text{for all } n \geq 0$$

- Take $c = 2, n_0 = 0$

Asymptotic Lower Bound

$f(n) \in \Omega(g(n))$ if $\exists$ constants $c > 0,\ n_0 \geq 0$ s.t. $|f(n)| \geq c|g(n)|$ for $n \geq n_0$

- Prove that $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$
  - $\frac{1}{2}n^2 - 5n < 0$ for $0 < n < 10$
  - since we ignore absolute value in the derivation, we need to ensure $f(n)$ is actually positive
  - for positivity of $f(n)$, make sure to take $n_0 \geq 10$
- Need to find c and n_0 s.t. $\frac{1}{2}n^2 - 5n \geq cn^2$ for all $n \geq n_0$
- Unlike before, cannot 'drop' lower growing term, as $\frac{1}{2}n^2 - 5n \leq \frac{1}{2}n^2$
- Need $\frac{1}{2}n^2 - 5n \geq cn^2$

an^2 bn^2 positive for large enough n

for large enough n

$$\frac{1}{2}n^2 - 5n \geq an^2 + (bn^2 - 5n) \geq an^2$$

Asymptotic Lower Bound

$f(n) \in \Omega(g(n))$ if $\exists$ constants $c > 0,\ n_0 \geq 0$ s.t. $|f(n)| \geq c|g(n)|$ for $n \geq n_0$

- For positivity of $f(n)$, make sure to take $n_0 \geq 10$
- Need to find c and n_0 s.t. $\frac{1}{2}n^2 - 5n \geq cn^2$ for all $n \geq n_0$
- Rewrite

$$\frac{1}{2}n^2 - 5n = \frac{1}{4}n^2 + \frac{1}{4}n^2 - 5n = \frac{1}{4}n^2 + \underbrace{\left(\frac{1}{4}n^2 - 5n\right)}_{\geq 0,\ \text{if } n \geq 20} \geq \frac{1}{4}n^2$$

so take $n_0 \geq 20$

- Take $c = \frac{1}{4}, n_0 = 20$
  - n_0 happened to be bigger than 10, as needed, automatically

Tight Asymptotic Bound

- Θ-notation

  $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0, n_0 \geq 0$ s.t.
  $c_1|g(n)| \leq |f(n)| \leq c_2|g(n)|$ for all $n \geq n_0$

- Easy to prove that

  $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

- Therefore, to show that $f(n) \in \Theta(g(n))$, it is enough to show
  1. $f(n) \in O(g(n))$
  2. $f(n) \in \Omega(g(n))$
     - that's why we said that for tight bound, we also need lower bound
- $f(n) \in \Theta\big(g(n)\big)$ means $f(n)$, $g(n)$ have equal growth rates

Tight Asymptotic Bound

- Proved previously that
  - $2n^2 + 3n + 11 \in O(n^2)$
  - $2n^2 + 3n + 11 \in \Omega(n^2)$
- Thus $2n^2+3n + 11 \in \Theta(n^2)$
- Ideally, should use Θ to determine growth rate of algorithm
  - $f(n)$ for running time
  - $g(n)$ for growth rate
- Sometimes determining tight bound is hard, so big-O is used

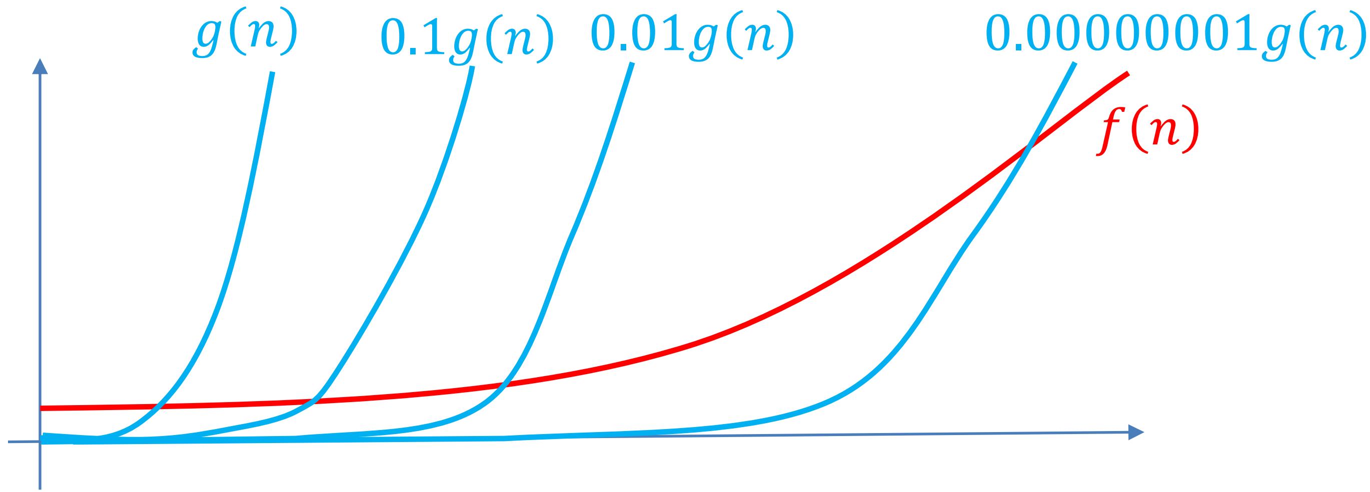
Tight Asymptotic Bound

Prove that $\log_b n \in \Theta(\log n)$ for $b > 1$

- Find $c_1, c_2 > 0, n_0 \geq 0$ s.t. $c_1 \log n \leq \log_b n \leq c_2 \log n$ for all $n \geq n_0$
- $\log_b n = \frac{1}{\log b} \log n$
- $\frac{1}{\log b} \log n \leq \log_b n \leq \frac{1}{\log b} \log n$
- Since $b > 1, \log b > 0$
- Take $c_1 = c_2 = \frac{1}{\log b}$ and $n_0 = 1$

Strictly Smaller Asymptotic Bound

- $f(n) = 2n^2 + 3n + 11 \in \Theta(n^2)$
- How to say $f(n)$ is asymptotically strictly smaller than $g(n) = n^3$?



- *o*-notation

  $f(n) \in o(g(n))$ if **for any constant** $c > 0$, there exists a constant $n_0 \geq 0$ s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

- Meaning: f grows much slower than g

Strictly Larger Asymptotic Bound

- ω-notation

  $f(n) \in \omega(g(n))$ if **for any constant** $c > 0$, there exists a constant $n_0 \geq 0$ s.t. $|f(n)| \geq c|g(n)|$ for all $n \geq n_0$

- Meaning: f grows much faster than g

Strictly Smaller Proof Example

$f(n) \in o(g(n))$ if **for any** $c > 0$, there exists $n_0 \geq 0$ s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

Prove that $5n \in o(n^2)$

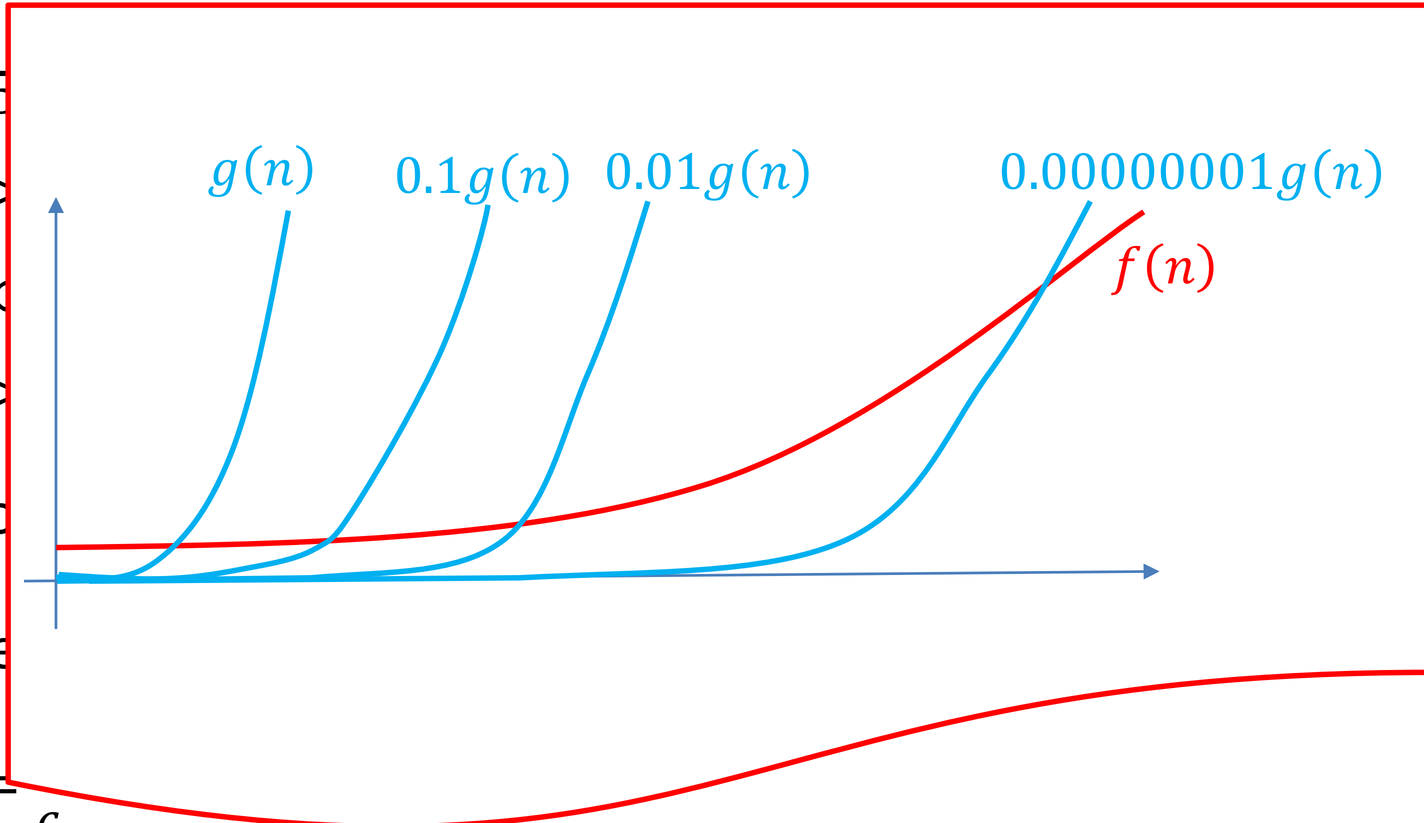
- Given $c > 0$ need to find n_0 s.t. $5n \leq cn^2$ for all $n \geq n_0$
- Dividing both sides by n , this is equivalent to the statement below
- Given $c > 0$ need to find n_0 s.t. $5 \leq cn$ for all $n \geq n_0$
    - holds for $n \geq \frac{5}{c}$
- Therefore, $5n \leq cn^2$ for $n \geq \frac{5}{c}$
- Take $n_0 = \frac{5}{c}$
- Note that for o-proofs, n_0 will usually depend on c

Strictly Smaller Proof Example

$f(n) \in o(g(n))$ if **for any** $c > 0$, there exists $n_0 \geq 0$ s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$

Prove that 5

- Given $c >$
- Dividing b ... ow
- Given $c >$
  - ho
- Therefore
- Take $n_0 =$ c
- Note that for o-proofs, n_0 will usually depend on c



Limit Theorem for Order Notation

- So far had proofs for order notation from the *first principles*
    - i.e. from the definition
- There is a useful limit theorem for order notation
- Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$
- Suppose that $\mathrm{L} = \lim_{n\to\infty} \frac{f(n)}{g(n)}$
- Then $f(n) \in \begin{cases} o\big(g(n)\big) & if\ L = 0 \\ \Theta\big(g(n)\big) & if\ 0 < L < \infty \\ \omega\big(g(n)\big) & if\ L = \infty \end{cases}$
- The required limit can often be computed using l'Hopital's rule
- Theorem gives sufficient but not necessary conditions

Example 1

Let $f(n)$ be a polynomial of degree $d \geq 0$ with $c_d > 0$

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

Then $f(n) \in \Theta(n^d)$

Proof:

$$\lim_{n\to\infty} \frac{f(n)}{n^d} = \lim_{n\to\infty} \left(\frac{c_d n^d}{n^d} + \frac{c_{d-1} n^{d-1}}{n^d} + \cdots + \frac{c_0}{n^d} \right)$$

$$= \underbrace{\lim_{n\to\infty} \left(\frac{c_d n^d}{n^d} \right)}_{= c_d} + \underbrace{\lim_{n\to\infty} \left(\frac{c_{d-1} n^{d-1}}{n^d} \right)}_{= 0} + \cdots + \underbrace{\lim_{n\to\infty} \left(\frac{c_0}{n^d} \right)}_{= 0}$$

$$= c_d \quad > 0$$

Example 2

- Compare growth rates of $\log n$ and n

$$\lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\frac{\ln n}{\ln 2}}{n} = \lim_{n\to\infty} \frac{\frac{1}{\ln 2 \cdot \mathrm{n}}}{1} = \lim_{n\to\infty} \frac{1}{\mathrm{n} \cdot \ln 2} = 0$$

L'Hopital rule

- $\log n \in o(n)$

Example 3

- Prove $(\log n)^a \in o(n^d)$, for any (big) $a > 0$, (small) $d > 0$

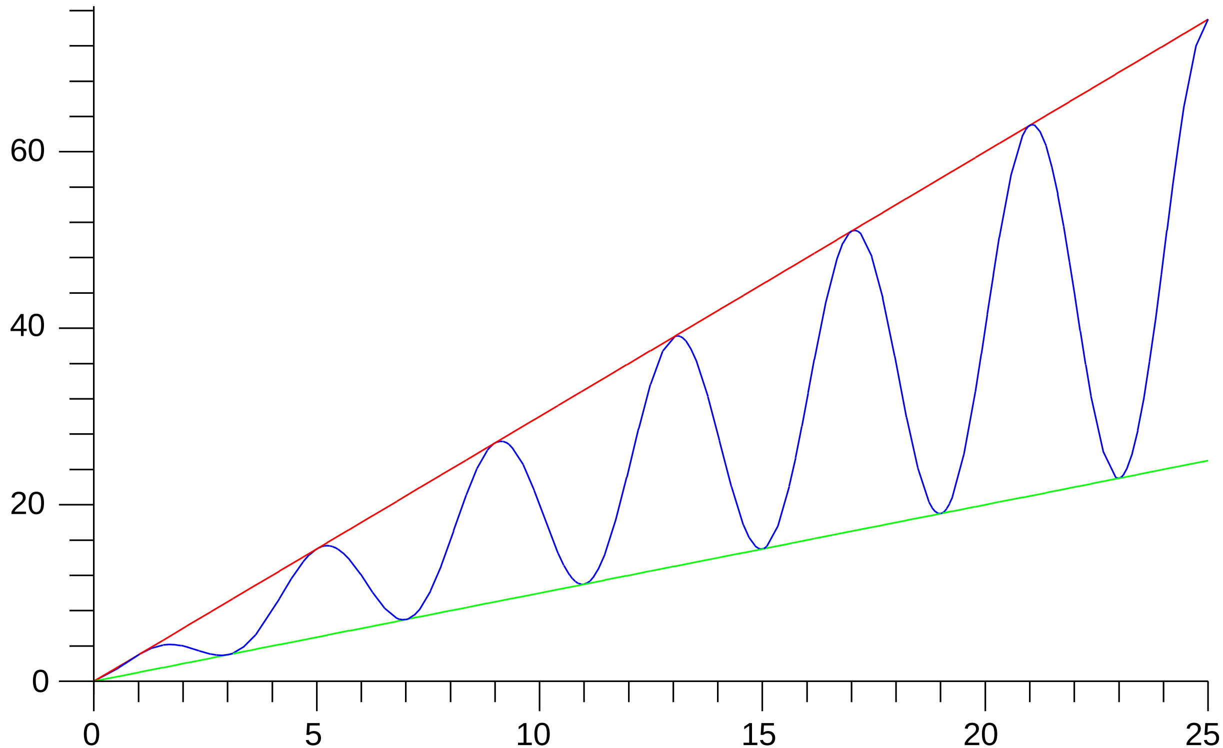
1) Prove (by induction):
$$\lim_{n\to\infty} \frac{\ln^{k} n}{n} = 0 \text{ for any integer } k$$
   - Base case $k = 1$ is proven on previous slide
   - Inductive step: suppose true for $k - 1$
   - $$\lim_{n\to\infty} \frac{\ln^{k} n}{n} \underset{\text{L'Hopital rule}}{=} \lim_{n\to\infty} \frac{\frac{1}{n} k\, ln^{k-1} n}{1} = k \lim_{n\to\infty} \frac{ln^{k-1} n}{n} = 0$$

2) Prove $\lim_{n\to\infty} \frac{\ln^{a} n}{n^d} = 0$
   - $$\lim_{n\to\infty} \frac{\ln^{a} n}{n^d} = \left(\lim_{n\to\infty} \frac{\ln^{a/d} n}{n}\right)^d \leq \left(\lim_{n\to\infty} \frac{\ln^{\lceil a/d \rceil} n}{n}\right)^d = 0$$

3) Finally $\lim_{n\to\infty} \frac{(\log n)^a}{n^d} = \lim_{n\to\infty} \frac{\left(\frac{\ln n}{ln2}\right)^a}{n^d} = \left(\frac{1}{ln2}\right)^a \lim_{n\to\infty} \frac{(\ln n)^a}{n^d} = 0$

Example 4

- Sometimes limit does not exist, but can prove from first principles
- Let $f(n) = n(2 + \sin n\pi/2)$
- Prove that $f(n)$ is $\Theta(n)$



Example 4

- Let $f(n) = n(2 + \sin n\pi/2)$, prove that $f(n)$ is $\Theta(n)$
- Proof:

$$-1 \le sin(\text{any number}) \le 1$$

$$f(n) \le n(2 + 1) = 3n \quad \text{for all } n \ge 1$$

$$n = n(2-1) \le f(n) \quad \text{for all } n \ge 1$$

$$n \le f(n) \le 3n \quad \text{for all } n \ge 1$$

Use $c_1 = 1, c_2 = 3, n_0 = 1$

Order notation Summary

- $f(n) \in \Theta(g(n))$: growth rates of f and g are the same
- $f(n) \in o(g(n))$: growth rate of f is less than growth rate of g
- $f(n) \in \omega(g(n))$: growth rate of f is greater than growth rate of g
- $f(n) \in O(g(n))$: growth rate of f is the same or less than growth rate of g
- $f(n) \in \Omega(g(n))$: growth rate of f is the same or greater than growth rate of g

Relationship between Order Notations

One can prove the following relationships

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

Algebra of Order Notations

- The following rules are easy to prove

1. **Identity rule:** $f(n) \in \Theta\big(f(n)\big)$
2. **Transitivity**
   - if $f(n) \in O\big(g(n)\big)$ and $g(n) \in O\big(h(n)\big)$ then $f(n) \in O\big(h(n)\big)$
   - if $f(n) \in \Omega\big(g(n)\big)$ and $g(n) \in \Omega\big(h(n)\big)$ then $f(n) \in \Omega\big(h(n)\big)$
3. **Maximum rules**

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$, then

a) $f(n) + g(n) \in \Omega(max\{f(n), g(n)\})$

b) $f(n) + g(n) \in O(max\{f(n), g(n)\})$

Proof:

a) $max\{f(n), g(n)\} =$ either $f(n)$ or $g(n) \leq f(n) + g(n)$

b) $$\begin{aligned} f(n) + g(n) &= max\{f(n), g(n)\} + min\{f(n), g(n)\} \\ &\leq max\{f(n), g(n)\} + max\{f(n), g(n)\} \\ &= 2max\{f(n), g(n)\} \end{aligned}$$

Abuse of Notation

- Normally, say $f(n) \in \Theta(g(n))$ because $\Theta(g(n))$ is a set
- Sometimes convenient to abuse of notation, i.e.
  - $f(n) = n^2 + \Theta(n)$
    - $f(n)$ is a quadratic function plus a linear term
  - $f(n) = n^2 + O(n)$
    - $f(n)$ is a quadratic function plus a term that grows slower or at the same rate as a linear function
  - $f(n) = n^2 + O(1)$
    - $f(n)$ is a quadratic function plus a constant
  - $f(n) = n^2 + o(1)$
    - $f(n)$ is a quadratic function plus a term that goes to 0
    - example: $f(n) = n^2 + 1/n$

Common Growth Rates

- Commonly encountered growth rates in increasing order of growth
  - $\Theta(1)$ *constant complexity*
  - $\Theta(\log n)$ *logarithmic complexity*
  - $\Theta(n)$ *linear complexity*
  - $\Theta(n\log n)$ *linearithmic*
  - $\Theta(n\log^k n)$ *quasi-linear* (k is constant, i.e. independent of the problem size)
  - $\Theta(n^2)$ *quadratic complexity*
  - $\Theta(n^3)$ *cubic complexity*
  - $\Theta(2^n)$ *exponential complexity*

How Growth Rates Affect Running Time

- How running time affected when problem size doubles ($n \rightarrow 2n$)
  - constant complexity: $T(n) = c$ $\qquad T(2n) = c$
  - logarithmic complexity: $T(n) = c\log n$ $\qquad T(2n) = T(n) + c$
  - linear complexity: $T(n) = cn$ $\qquad T(2n) = 2T(n)$
  - linearithmic: $T(n) = cn\log n$ $\qquad T(2n) = 2T(n) + 2cn$
  - quadratic complexity: $T(n) = cn^2$ $\qquad T(2n) = 4T(n)$
  - cubic complexity: $T(n) = cn^3$ $\qquad T(2n) = 8T(n)$
  - exponential complexity: $T(n) = c2^n$ $\qquad T(2n) = \frac{1}{c}T^2(n)$

Comparison of Growth Rates

n	log(n)	n	nlog(n)	n^2	n^3	2^n
8	3	8	24	64	512	256
16	4	16	64	256	4096	65536
32	5	32	160	1024	32768	4.3×10^9
64	6	64	384	4096	262144	1.8×10^{19}
128	7	128	896	16384	2097152	3.4×10^{38}
256	8	256	2048	65536	16777218	1.2×10^{77}

Outline

- CS240 overview
    - Course objectives
    - Course topics
- Introduction and Asymptotic Analysis
    - algorithm design
    - pseudocode
    - measuring efficiency
    - analysis of algorithms
    - analysis of recursive algorithms
    - helpful formulas

Techniques for Algorithm Analysis

- Goal: Use asymptotic notation to simplify run-time analysis
- Running time of an algorithm depends on the *input size* n

```
Test1(n)
    sum ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← sum + (i − j)^2
    return sum
```

- Identify *primitive operations* that require $\Theta(1)$ (i.e. constant) time
- Loop complexity expressed as *sum* of complexities of each iteration
- Nested loops: start with the innermost loop and proceed outwards
- This gives *nested summations*

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    return sum
```

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Techniques for Algorithm Analysis

- Goal: Use asymptotic notation to simplify run-time analysis
- Running time of an algorithm depends on the *input size* n

Test1(n)

$sum \leftarrow 0$
for $i \leftarrow 1$ **to** n **do**
  for $j \leftarrow i$ **to** n **do**
    $sum \leftarrow sum + (i - j)^2$
return sum

$$\sum_{j=i}^{n} c$$

- Identify *primitive operations* that require constant, i.e. $\Theta(1)$ time
- Loop complexity expressed as *sum* of complexities of each iteration
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```

$$\sum_{i=1}^{n} \sum_{j=i}^{n} c$$

- Identify *primitive operations* that require $\Theta(1)$ time
- Loop complexity expressed as *sum* of complexities of each iteration
- Nested loops: start with the innermost loop and proceed outwards
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- Goal: Use asymptotic notation to simplify run-time analysis
- Running time of an algorithm depends on the *input size* n

Test1(n)

```
sum ← 0
for i ← 1 to n do
    for j ← i to n do
        sum ← sum + (i − j)^2
return sum
```

$$\sum_{i=1}^{n}\sum_{j=i}^{n} c \; + c$$

- Identify *primitive operations* that require $\Theta(1)$ time
- Loop complexity expressed as *sum* of complexities of each iteration
- Nested loops: start with the innermost loop and proceed outwards
- This gives *nested summations*

Techniques for Algorithm Analysis

```
Test1(n)
    sum ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← sum + (i − j)^2
    return sum
```

- Derived complexity as $$c + \sum_{i=1}^{n} \sum_{j=i}^{n} c$$
- Some textbooks will write this as $$c_1 + \sum_{i=1}^{n} \sum_{j=i}^{n} c_2$$
- Or as $$1 + \sum_{i=1}^{n} \sum_{j=i}^{n} 1$$
- Now need to work out the sum

Sums: Review

summand

$$\sum_{j=1}^{n} 1 = \quad 1 \quad +1 \quad +1 \quad \ldots \quad +1 \quad = n$$

$j = 1$ $\quad$ $j = 2$ $\quad$ $j = 3$ $\quad$ ... $\quad$ $j = n$

index of summation

$$\sum_{j=i}^{n} 1 = \quad 1 \quad +1 \quad \ldots \quad +1 \quad = n - i + 1$$

$j = i$ $\quad$ $j = i + 1$ $\quad$ ... $\quad$ $j = n$

$k = i - i + 1 = 1$ $\quad$ $k = i + 1 - i + 1 = 2$ $\quad$ $k = n - i + 1 = n - i + 1$

Sums: Review

$$\sum_{j=i}^{n} (n - e^x) = n - e^x \quad + n - e^x \quad \ldots \quad + n - e^x = (n - i + 1)(n - e^x)$$

$$j = i \qquad j = i + 1 \qquad \ldots \qquad j = n$$

Sums: Review

$$S = \sum_{i=1}^{n} i = \quad 1 \qquad + 2 \qquad + 3 \qquad \dots \qquad + n$$

$$i = 1 \qquad i = 2 \qquad i = 3 \qquad \dots \qquad i = n$$

$$\begin{array}{rcccccc}
 & & n+1 & n+1 & n+1 & & n+1 \\
 & S = & 1 & +2 & +3 & \cdots & +n \\
+ & S = & n & +(n-1) & +(n-2) & \cdots & +1 \\
\hline
\end{array}$$

$$2S = (n+1)n$$

$$S = \sum_{i=1}^{n} i = \frac{1}{2}(n+1)n$$

Sums: Review

$$S = \sum_{i=a}^{b} i = \quad a \qquad + (a+1) \qquad \cdots \qquad + b$$

$$i = a \qquad i = a+1 \qquad \cdots \qquad i = b$$

$$a+b \qquad a+b \qquad\qquad a+b$$

$$\begin{array}{rllll} & S = & a & + (a+1) & \cdots & + b \\ + & S = & b & + (b-1) & \cdots & + a \\ \hline \end{array}$$

$$2S = (a+b)(b-a+1)$$

$$S = \sum_{i=a}^{b} i = \frac{1}{2}(a+b)(b-a+1)$$

Techniques for Algorithm Analysis

```
Test1(n)
    sum ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← sum + (i − j)^2
    return sum
```

$$c+\sum_{i=1}^{n}\sum_{j=i}^{n}c \quad = c+\sum_{i=1}^{n}c(n-i+1)$$

$$= c \;+c\sum_{i=1}^{n}n \;-c\sum_{i=1}^{n}i \;+c\sum_{i=1}^{n}1$$

$$= c \;+cn^2-c\frac{(n+1)n}{2}+cn = c\frac{n^2}{2}+c\frac{n}{2}+c$$

- Complexity of algorithm Test1 is $\Theta(n^2)$

Techniques for Algorithm Analysis

- Two general strategies
  1. Use Θ-bounds *throughout the analysis* and obtain Θ-bound for the complexity of the algorithm
  2. Prove a *O*-bound and a *matching* Ω-bound *separately*
     - use upper bounds (for *O*-bounds) and lower bounds (for Ω-bound) early and frequently
     - easier because upper/lower bounds are easier to sum

Techniques for Algorithm Analysis

- First strategy

```
Test2(A, n)
1.    max ← 0
2.    for i ← 1 to n do
3.        for j ← i to n do
4.            sum ← 0
5.            for k ← i to j do
6.                sum ← A[k]
7.    return max
```

c

Techniques for Algorithm Analysis

- First strategy

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

$$\sum_{k=i}^{j} c$$

Techniques for Algorithm Analysis

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

- First strategy

$$c + \sum_{k=i}^{j} c$$

Techniques for Algorithm Analysis

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

- First strategy

$$\sum_{j=i}^{n} \left(c + \sum_{k=i}^{j} c\right)$$

- Will write instead

$$\sum_{j=i}^{n} \sum_{k=i}^{j} c$$

- This omits lower order term that does not effect Θ-bound

Techniques for Algorithm Analysis

- First strategy

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} c$$

Techniques for Algorithm Analysis

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

- First strategy

$$c + \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} c$$

- Will write instead

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} c$$

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Techniques for Algorithm Analysis

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Test2(A, n)
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    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

- First strategy

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} c =$$

$$c\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 = c\sum_{i=1}^{n} \overbrace{\sum_{j=i}^{n} (j-i+1)}^{1+2+\cdots+(n-i+1)}$$

$$= c\sum_{i=1}^{n} \frac{(n-i+1)(n-i+2)}{2} = \frac{c}{2}\sum_{i=1}^{n} \left(n^2 - (2n+3)i + i^2 + 3n + 2\right)$$

$$= \frac{c}{2}\left(n^3 - (2n+3)\frac{(n+1)n}{2} + \frac{(2n+1)(n+1)n}{6} + 3n^2 + 2n\right)$$

- Test2 is $\Theta(n^3)$

Techniques for Algorithm Analysis

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

- Second strategy, part 1: upper bound
- Add more iterations to make the number of summands in each sum larger and summand independent of summation index

$$c\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j}1 \le c\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}1 = c\sum_{i=1}^{n}\sum_{j=1}^{n}n$$

$$= c\sum_{i=1}^{n}n^2$$

$$= cn^3$$

- Test2 is $O(n^3)$
- Essence of upper bound: made the number of summands in each sum equal to n

Techniques for Algorithm Analysis

- Second strategy, part 2: lower bound

$$c\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j}1 \geq ?$$

- Cannot make number of summands in each sum equal to n
- To get cubic bound, it is sufficient to make the number of summands equal to a *fraction* of n

$$\sum_{i=1}^{n/5}\sum_{j=n/4}^{2n/4}\sum_{k=1}^{n/2}1 = \sum_{i=1}^{n/5}\sum_{j=n/4}^{2n/4}\frac{n}{2} = \frac{n}{2}\sum_{i=1}^{n/5}\sum_{j=n/4}^{2n/4}1$$

$$= \frac{n}{2}\sum_{i=1}^{n/5}\frac{n}{4} = \frac{n}{3}\cdot\frac{n}{4}\sum_{i=1}^{n/5}1 = \frac{n}{3}\cdot\frac{n}{4}\cdot\frac{n}{5} = \frac{n^3}{60}$$

Techniques for Algorithm Analysis

- To decrease number of iterations, increase the lower or increase the upper range bounds, or both

$$\sum_{k=10}^{100} 1 \geq \sum_{k=\mathbf{20}}^{\mathbf{80}} 1$$

$$\sum_{k=i}^{j} 1 \geq \sum_{k=\boldsymbol{i}+\mathbf{1}}^{\boldsymbol{j}-\mathbf{1}} 1$$

Techniques for Algorithm Analysis

- Let $0 < a < 1$

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 \geq \sum_{i=1}^{an} \sum_{j=i}^{n} \sum_{k=i}^{j} 1$$

$$\geq \sum_{i=1}^{an} \sum_{j=an}^{n} \sum_{k=i}^{j} 1$$

$$\geq \sum_{i=1}^{an} \sum_{j=an}^{n} \sum_{k=an}^{an} 1 \in \Theta(n^2)$$

- Not enough iterations for the innermost loop!
- Solution: j should start at a larger value than an

Techniques for Algorithm Analysis

- Let $0 < a, b < 1$, and $a + b = c < 1$

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 \geq \sum_{i=1}^{an} \sum_{j=i}^{n} \sum_{k=i}^{j} 1$$

$$\geq \sum_{i=1}^{an} \sum_{j=an+bn}^{n} \sum_{k=i}^{j} 1$$

$$\geq \sum_{i=1}^{an} \sum_{j=an+bn}^{n} \sum_{k=an}^{an+bn} 1$$

- Plug in $a = 1/3, b = 1/3$ (but any $0 < a, b < 1$ with $a + b = c < 1$ works)

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 \geq \sum_{i=1}^{\boldsymbol{n/3}} \sum_{j=\boldsymbol{2n/3}}^{n} \sum_{k=\boldsymbol{n/3}}^{\boldsymbol{2n/3}} 1 = \sum_{i=1}^{\boldsymbol{n/3}} \sum_{j=\boldsymbol{2n/3}}^{n} \frac{n}{3} = \frac{n^3}{27}$$

- Test2 is $\Omega(n^3)$
- Combined with upper bound, Test2 is $\Theta(n^3)$

Worst Case Time Complexity

- Can have different running times on two instances of equal size

```
Test3(A, n)
A: array of size n
    for i ← 1 to n − 1 do
        j ← i
        while j > 0 and A[j] > A[j − 1] do
            swap A[j] and A[j − 1]
            j ← j − 1
```

- Let $T_A(I)$ be running time of an algorithm *A* on instance *I*
- **Worst-case complexity of an algorithm**: take the worst *I*
- Formal definition: the worst-case running time of algorithm *A* is a function $f : Z^+ \to R$ mapping n (the input size) to the *longest* running time for any input instance of size n

$$T_A(n) = max\{T_A(I) : Size(I) = n\}$$

Worst Case Time Complexity

- Can have different running times on two instances of equal size

```
Test3(A, n)
A: array of size n
      for i ← 1 to n − 1 do
            j ← i
            while j > 0 and A[j] > A[j − 1] do
                  swap A[j] and A[j − 1]
                  j ← j − 1
```

$$\sum_{i=1}^{n-1} \sum_{j=1}^{i} c = \sum_{i=0}^{n-1} ci$$

$$= c(n-1)n/2$$

- **Worst-case complexity of an algorithm**: take worst instance *I*
- $T_{worst}(n) = c(n-1)n/2$
    - this is primitive operation count as a function of input size n
    - after primitive operation count, apply asymptotic analysis
        - $\Theta(n^2)$ or $O(n^2)$ or $\Omega(n^2)$ are all valid statements about the worst case running time

Best Case Time Complexity

```
Test3(A, n)
A: array of size n
    for i ← 1 to n − 1 do
        j ← i
        while j > 0 and A[j] > A[j − 1] do
            swap A[j] and A[j − 1]
            j ← j − 1
```

$$\sum_{i=1}^{n-1} c = c(n-1)$$

- **Best-case complexity of an algorithm**: take the best instance *I*
- Formal definition: the best-case running time of an algorithm *A* is a function $f : Z^+ \rightarrow R$ mapping n (the input size) to the *smallest* running time for any input instance of size n

$$T_A(n) = min\{T_A(I): Size(I) = n\}$$

- $T_{best}(n) = c(n-1)$
    - this is primitive operation count as a function of input size n
    - after primitive operation count, apply asymptotic analysis
        - $\Theta(n)$ or $O(n)$ or $\Omega(n)$ are all valid about best case running time

Best Case Time Complexity

- Note that best-case complexity is a function of input size n
- Have to think of the best instance of size n
  - for Test3, best instance is sorted (decreasing) array A of size n
  - best instance is not an array of size 1

```
Test3(A, n)
A: array of size n
1.    for i ← 1 to n − 1 do
2.        j ← i
3.        while j > 0 and A[j] > A[j − 1] do
4.            swap A[j] and A[j − 1]
5.            j ← j − 1
```

- For ***hasNegative***, best instance is array A of size n where $A[0] < 0$
- Best-case complexity is $\Theta(1)$

```
Algorithm hasNegative(A, n)
  Input: array A of n integers
  found ← false
  i ← 0
  while i < n − 1 and found == false
    if A[i] < 0 then
      found ← true
    i ← i + 1
  return found
```

Average Case Time Complexity

Average-case complexity of an algorithm: The average-case running time of an algorithm *A* is function $f : \mathrm{Z}^+ \rightarrow \mathrm{R}$ mapping n (input size) to the *average* running time of *A* over all instances of size n

$$T_A^{avg}(n) = \frac{1}{|\{I: Size(I) = n\}|} \sum_{I:Size(I)=n} T_A(I)$$

Average vs. Worst vs. Best Case Time Complexity

- Sometimes, best, worst, average time complexities are the same
- If there is a difference, then best time complexity could be overly optimistic, worst time complexity could be overly pessimistic, and average time complexity is most useful
- However, average case time complexity is usually hard to compute
- Therefore, most often, use worst time complexity
    - worst time complexity is useful as it gives bound on the maximum amount of time one will have to wait for the algorithm to complete
    - default in this course
        - unless stated otherwise, whenever we mention time complexity, assume we mean worst case time complexity
- Suppose A has worst and best case complexities $\Theta(n^2)$ and $\Theta(n)$
    - can say complexity of A is $O(n^2)$, implying that A takes at most $O(n^2)$ time, but can have better time, depending on input

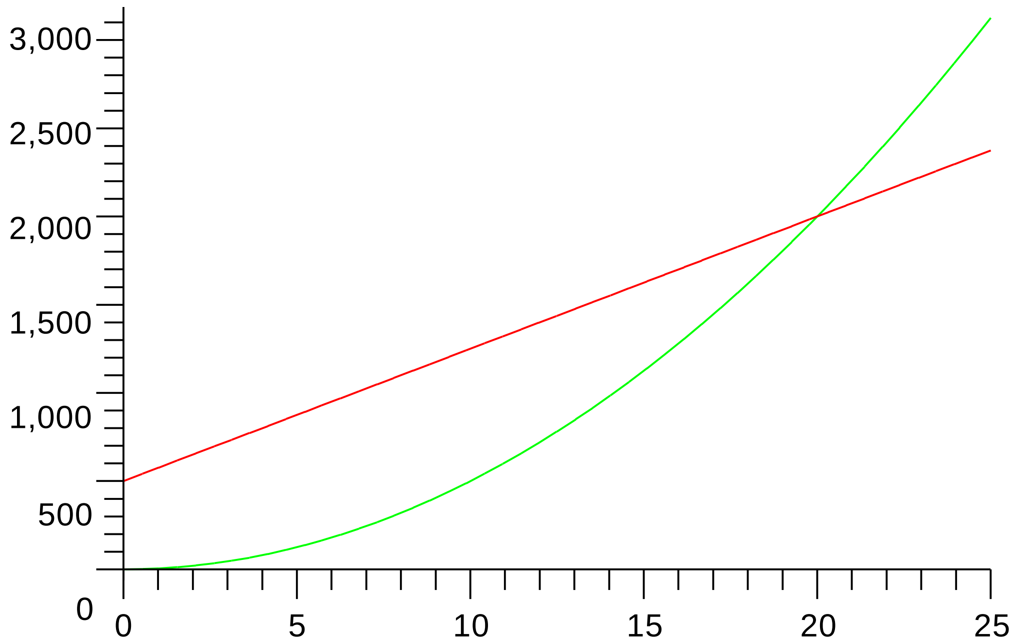
O-notation and Running Time of Algorithms

- It is important not to try make *comparisons* between algorithms using O-notation
- Suppose algorithm ***A*** and ***B*** both solve the same problem
  - ***A*** has worst-case runtime $O(n^3)$
  - ***B*** has worst-case runtime $O(n^2)$
- Cannot conclude that ***B*** is more efficient that ***A*** for all inputs
  1. the worst case runtime may only be achieved on some instances
  2. more importantly, O-notation is only an upper bound, ***A*** could have worst case runtime $O(n)$
- To compare algorithms, should use Θ notation

Running Time: Theory and Practice, Multiplicative Constants

- Algorithm ***A*** has runtime $T(n) = 10000n^2$
- Algorithm ***B*** has runtime $T(n) = 10n^2$
- Theoretical efficiency of ***A*** and ***B*** is the same, $\Theta(n^2)$
- In practice, algorithm ***B*** will run faster (for most implementations)
    - multiplicative constants matter in practice, given two algorithms with the same growth rate
    - but we will not talk about this issue more in this course

Running Time: Theory and Practice, Small Inputs



- Algorithm ***A*** running time $T(n) = 75n + 500$
- Algorithm ***B*** running time $T(n) = 5n^2$
- Then ***B*** is faster for $n \leq 20$
  - will use this fact when talking about practical implementation of recursive sorting algorithms

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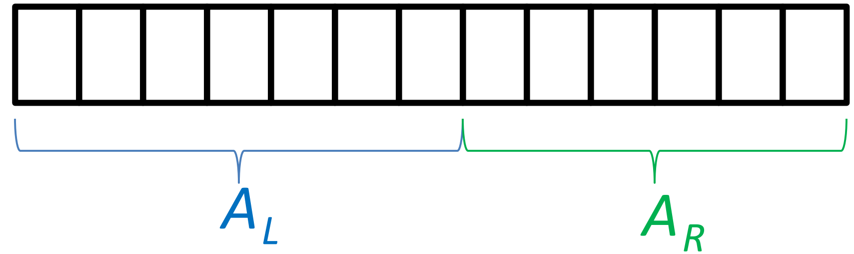
  - analysis of recursive algorithms
  - helpful formulas

Design of MergeSort



Input: Array A of n integers

Step 1: split A into two subarrays

- A_L consists of the first $\left\lceil \frac{n}{2} \right\rceil$ elements
- A_R consists of the last $\left\lfloor \frac{n}{2} \right\rfloor$ elements

Step 2: *Recursively* run *MergeSort* on A_L and A_R

Step 3: Use function *Merge* to merge now sorted A_L and A_R into a single sorted array

MergeSort

MergeSort$(A, \ell \leftarrow 0, r \leftarrow n-1, S \leftarrow NIL)$

A: array of size n, $0 \leq \ell \leq r \leq n-1$

    if S is NIL   initialize it as array $S[0..n-1]$

    if $(r \leq \ell)$ **then**

        return

    else

        $m = (r + \ell)/2$

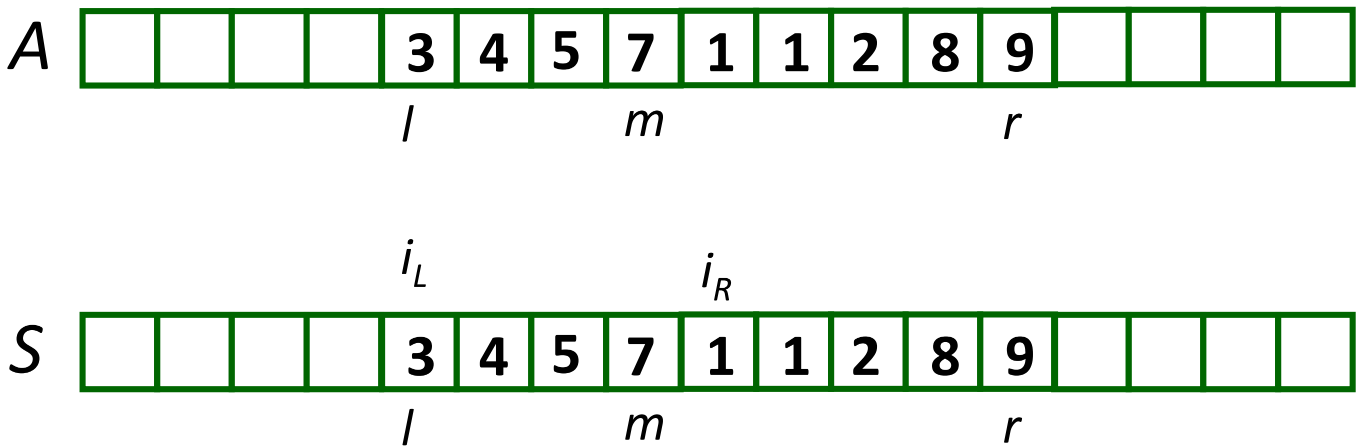
        MergeSort(A, ℓ, m, S)

        MergeSort$(A, m+1, r, S)$

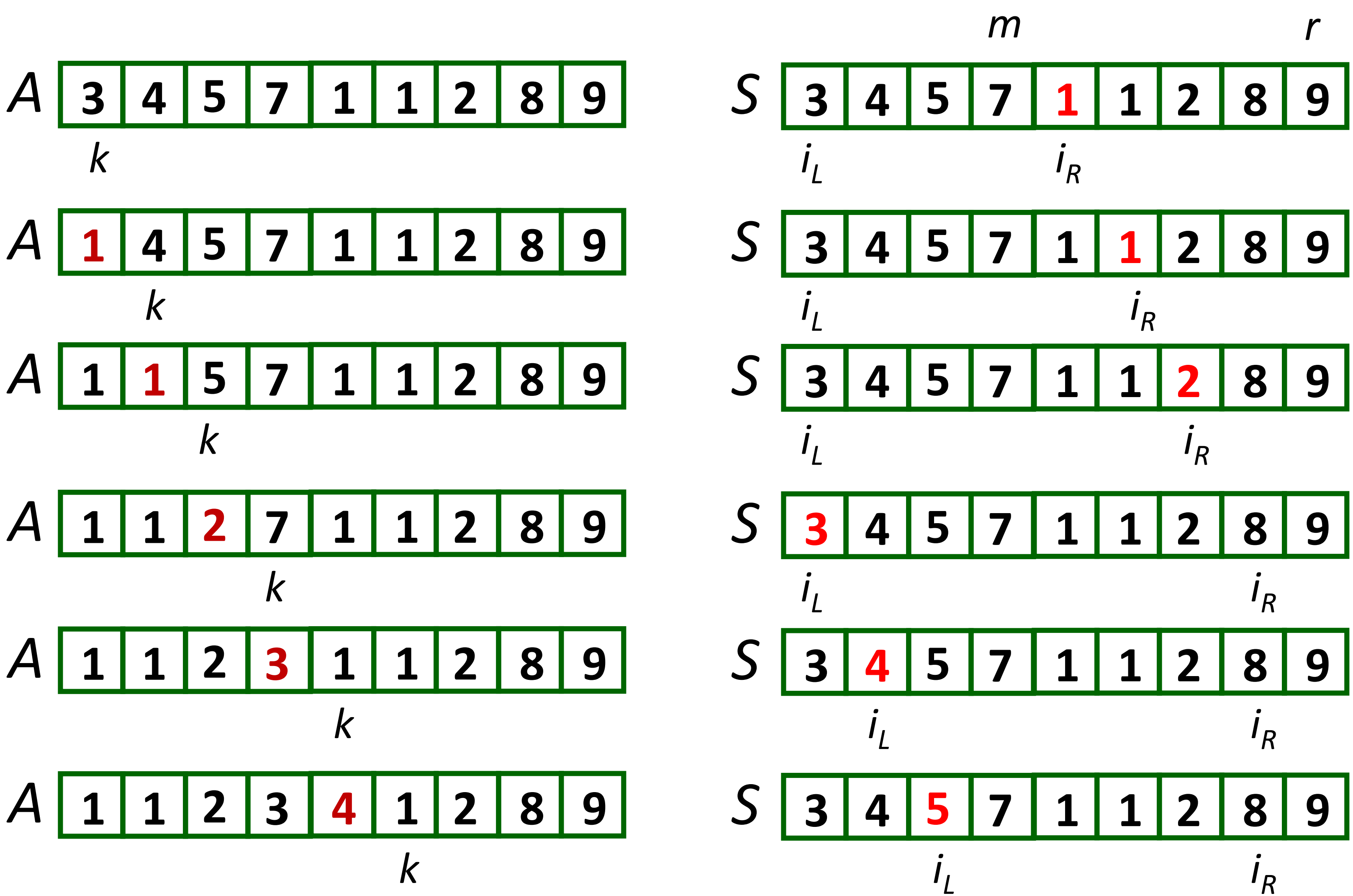
        Merge(A, ℓ, m, r, S)

- Two tricks to avoid copying/initializing too many arrays
  - recursion uses parameters that indicate the range of the array that needs to be sorted
  - array S used for merging is passed along as parameter

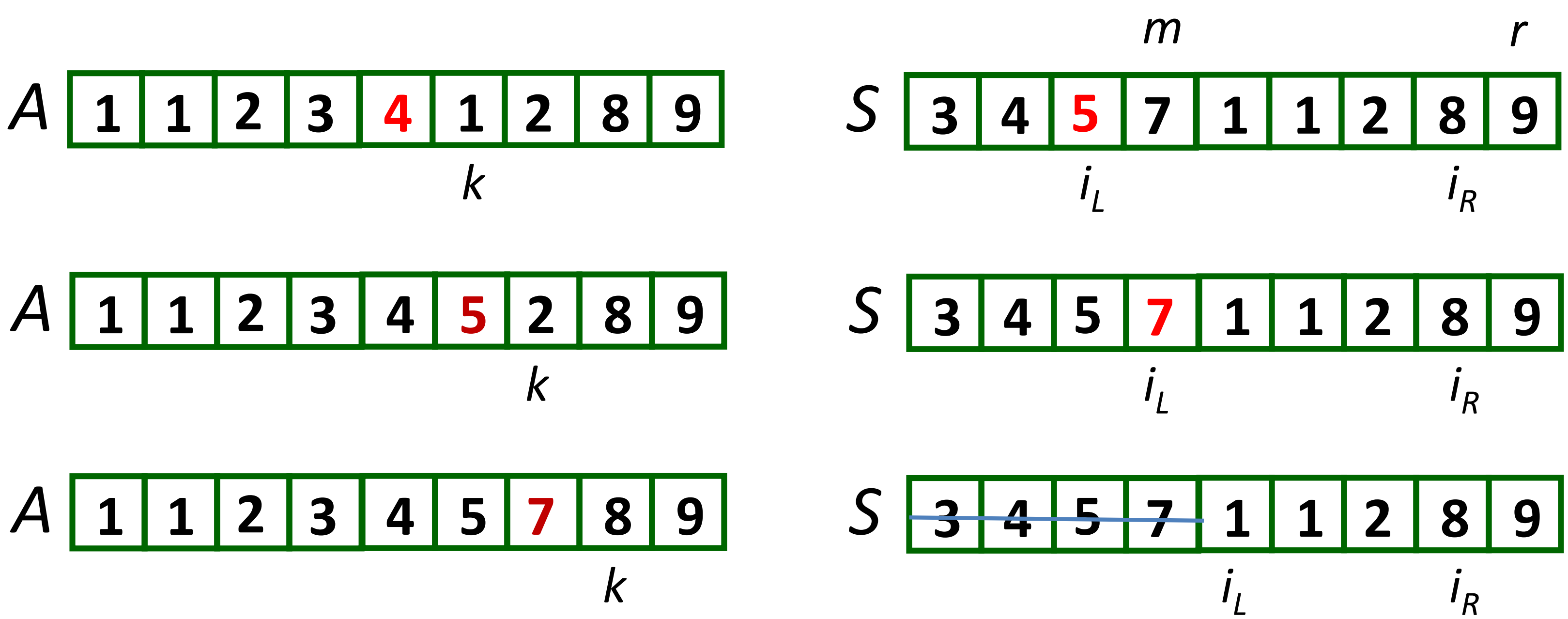
Merging Two Sorted Subarrays: Initialization



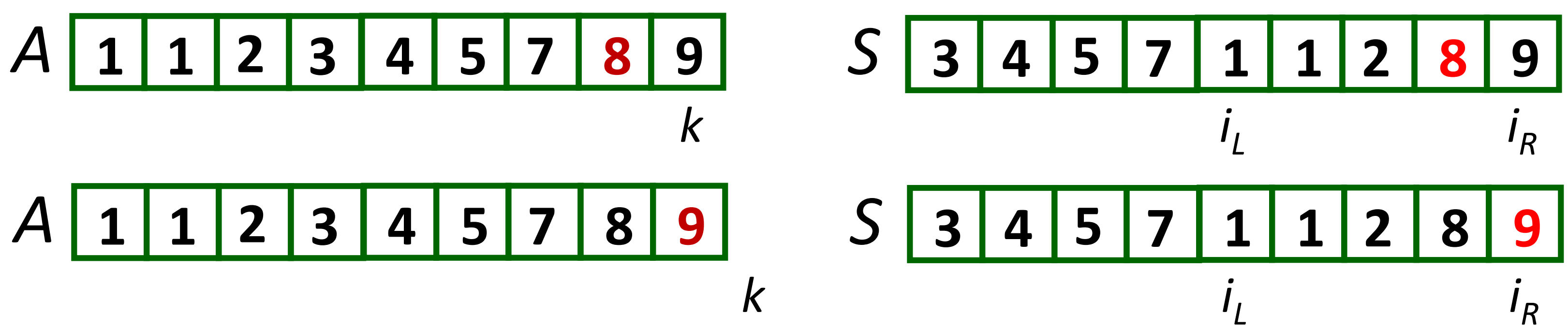
Merging Two Sorted Subarrays: Merging Starts



Merging Two Sorted Subarrays: Merging Cont.



$i_L > m$, done with the first subarray



Merge

```
Merge(A, ℓ, m, r, S)
A[0..n − 1] is an array, A[ℓ..m] is sorted, A[m + 1..r] is sorted
S[0..n − 1] is an array
      copy A[ℓ..r] into S[ℓ..r]
      (i_L, i_R) ← (ℓ, m + 1);
      for (k ← ℓ; k ≤ r; k++) do
            if (i_L > m) A[k] ← S[i_R++]
            else if (i_R > r) A[k] ← S[i_L++]
            else if (S[i_L] ≤ S[i_R]) A[k] ← S[i_L++]
            else A[k] ← S[i_R++]
```

- Merge takes $\Theta(l - r + 1)$ time
    - this is $\Theta(n)$ time for merging n elements

Analysis of MergeSort

- Let $T(n)$ be time to run *MergeSort* on an array of length n
  - Steps 5 takes $T\left(\left\lceil\frac{n}{2}\right\rceil\right)$
  - Steps 6 takes $T\left(\left\lfloor\frac{n}{2}\right\rfloor\right)$
  - Step 7 takes $\Theta(n)$
- The recurrence relation for *MergeSort*

$$T(n) = \begin{cases} T\left(\left\lceil\frac{n}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + cn & if\ \ n > 1 \\ c & if\ \ n = 1 \end{cases}$$

MergeSort$(A, \ell \leftarrow 0, r \leftarrow n-1)$
A: array of size n, $0 \le \ell \le r \le n-1$

```
1.   if (r ≤ ℓ) then
2.        return
3.   else
4.        m = (r + ℓ)/2
5.        MergeSort(A, ℓ, m)
6.        MergeSort(A, m + 1, r)
7.        Merge(A, ℓ, m, r)
```

Analysis of MergeSort

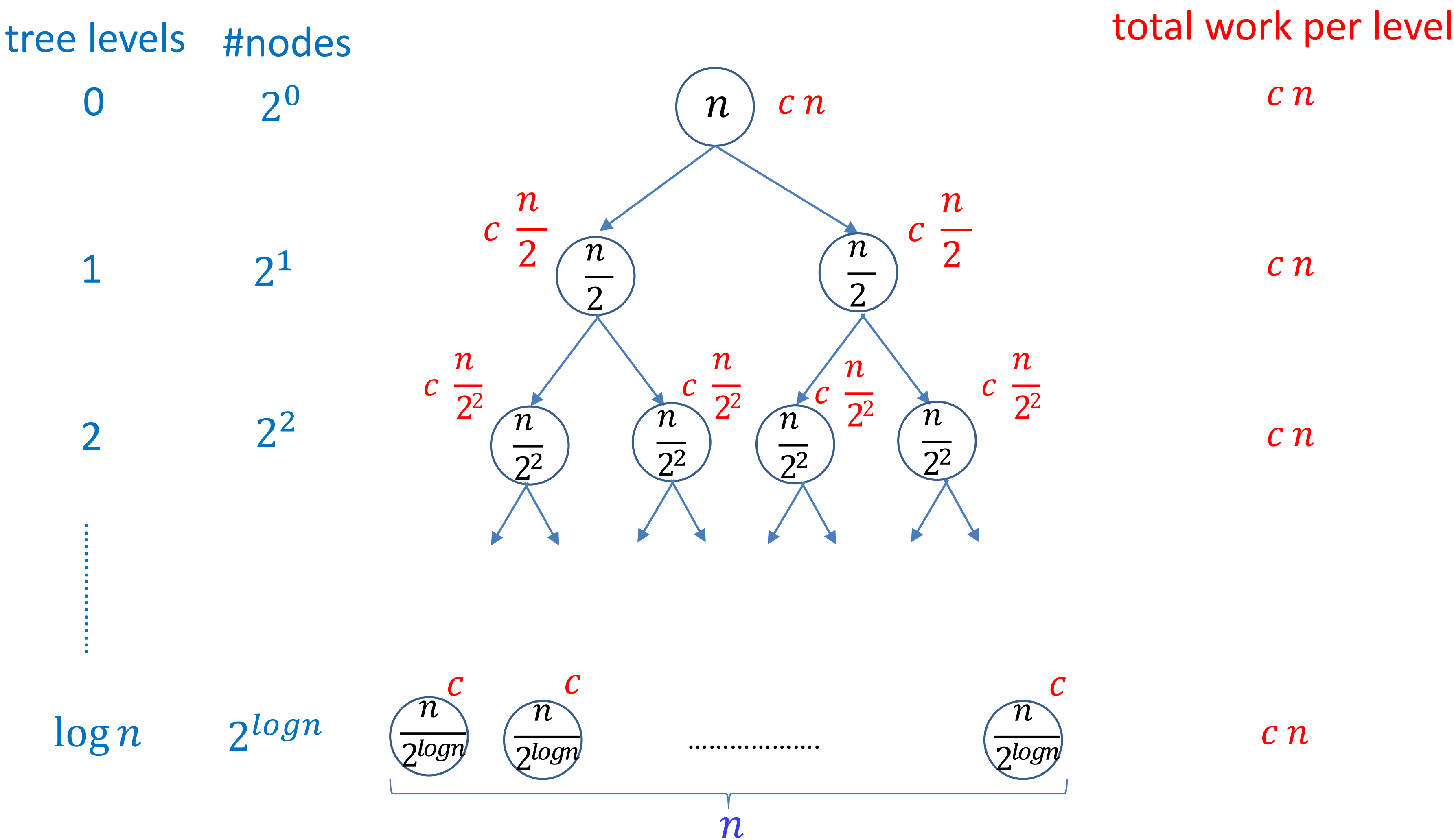
- *Sloppy recurrence* with floors and ceilings removed

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1 \end{cases}$$

- Exact and sloppy recurrences are *identical* when n is a power of 2
- Recurrence easily solved when $n = 2^j$

Visual proof via Recursion Tree

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + c\,n & \text{if } n > 1 \\ c & \text{if } n = 1 \end{cases}$$



- cn operations on each tree level, $\log n$ levels, total work is $cn \log n \in \Theta(n \log n)$

Analysis of MergeSort

- Can show $T(n) \in \Theta(n \log n)$ for all n by analyzing exact recurrence
    - for smallest m s.t. $2^{m-1} \leq n$
        - $T(2^{m-1}) \leq T(n) \leq T(2^m)$
        - $T(2^{m-1}), T(2^m) \in \Theta(n \log n)$

Some Recurrence Relations

Recursion	resolves to	example
$T(n) = T(n/2) + \Theta(1)$	$T(n) \in \Theta(\log n)$	Binary search
$T(n) = 2T(n/2) + \Theta(n)$	$T(n) \in \Theta(n \log n)$	Mergesort
$T(n) = 2T(n/2) + \Theta(\log n)$	$T(n) \in \Theta(n)$	Heapify ($\rightarrow$ later)
$T(n) = T(cn) + \Theta(n)$ for some $0 < c < 1$	$T(n) \in \Theta(n)$	Selection ($\rightarrow$ later)
$T(n) = 2T(n/4) + \Theta(1)$	$T(n) \in \Theta(\sqrt{n})$	Range Search ($\rightarrow$ later)
$T(n) = T(\sqrt{n}) + \Theta(1)$	$T(n) \in \Theta(\log \log n)$	Interpolation Search ($\rightarrow$ later)

- Once you know the result, it is (usually) easy to prove by induction
- You can use these facts without a proof, unless asked otherwise
- Many more recursions, and some methods to solve, in cs341

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Order Notation Summary

- O-notation $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$
- Ω-notation $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$
- Θ-notation $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ s.t. $c_1|g(n)| \leq |f(n)| \leq c_2|g(n)|$ for all $n \geq n_0$
- o-notation

  $f(n) \in o(g(n))$ if **for all constants** $c > 0$, there exists a constant $n_0 \geq 0$ s.t. $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$
- ω-notation

  $f(n) \in \omega(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ s.t. $0 \leq c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$

Useful Sums

- Arithmetic $\sum_{i=0}^{n-1}(a+di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2)$
- Geometric $\sum_{i=0}^{n-1} ar^i = \begin{cases} a\frac{r^n-1}{r-1} \in \Theta(r^{n-1}) & \text{if } r > 1 \\ na \in \Theta(n) & \text{if } r = 1 \\ a\frac{1-r^n}{1-r} \in \Theta(1) & \text{if } 0 < r < 1 \end{cases}$
- Harmonic $\sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$
- A few more $\sum_{i=1}^{n} \frac{1}{i^2} \in \Theta(1)$ $\quad \sum_{i=1}^{n} i^k \in \Theta\left(n^{k+1}\right)$ for $k \geq 0$

  $\sum_{i=0}^{\infty} ip(1-p)^{i-1} = \frac{1}{p}$ for $0 < p < 1$
- You can use these facts without a proof, unless asked otherwise

Useful Math Facts

- **Logarithms:**
  - $c = \log_b(a)$ means $b^c = a$. E.g. $n = 2^{\log n}$.
  - $\log(a)$ (in this course) means $\log_2(a)$
  - $\log(a \cdot c) = \log(a) + \log(c)$, $\log(a^c) = c \log(a)$,
  - $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$.
  - $a^{\log_b c} = c^{\log_b a}$
  - $\ln(x) = \text{natural log} = \log_e(x)$, $\frac{d}{dx} \ln x = \frac{1}{x}$
- **Factorial:**
  - $n! := n(n-1)(n-2) \cdots 2 \cdot 1 = \#$ ways to permute n elements
  - $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$
- **Probability and moments:**
  - $E[aX] = aE[X]$, $E[X+Y] = E[X] + E[Y]$ (linearity of expectation)