

# CS 240 – Data Structures and Data Management

## Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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# Outline

## 2 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

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# Abstract Data Types

**Abstract Data Type (ADT):** A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various **realizations** of an ADT, which specify:

- How the information is stored (**data structure**)
- How the operations are performed (**algorithms**)

# Stack ADT

**Stack:** an ADT consisting of a collection of items with operations:

- *push*: inserting an item
- *pop*: removing (and typically returning) the most recently inserted item

Items are removed in LIFO (*last-in first-out*) order.

Items enter the stack at the *top* and are removed from the *top*.

We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited web sites, procedure calls

## Realizations of Stack ADT

- using arrays
- using linked lists

# Queue ADT

**Queue:** an ADT consisting of a collection of items with operations:

- *enqueue*: inserting an item
- *dequeue*: removing (and typically returning) the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order.

Items enter the queue at the *rear* and are removed from the *front*.

We can have extra operations: *size*, *isEmpty*, and *front*

Applications: Waiting lines, printer queues

Realizations of Queue ADT

- using (circular) arrays
- using linked lists

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# Priority Queue ADT

**Priority Queue:** An ADT consisting of a collection of items (each having a **priority**) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing and returning the item of *highest* priority

*deleteMax* is also called *extractMax* or *getmax*.

The priority is also called *key*.

The above definition is for a **maximum-oriented** priority queue. A **minimum-oriented** priority queue is defined in the natural way, replacing operation *deleteMax* by *deleteMin*,

Applications: typical “todo” list, simulation systems, sorting

# Using a Priority Queue to Sort

```
PQ-Sort( $A[0..n-1]$ )
1.   initialize PQ to an empty priority queue
2.   for  $i \leftarrow 0$  to  $n-1$  do
3.       PQ.insert( $A[i]$ )
4.   for  $i \leftarrow n-1$  down to  $0$  do
5.        $A[i] \leftarrow PQ.deleteMax()$ 
```

- Note: Run-time depends on how we implement the priority queue.
- Sometimes written as:  $O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$

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- *deleteMax*:  $O(n)$

**Note:** We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes  $O(1)$  extra time.)

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## Realization 2: sorted arrays

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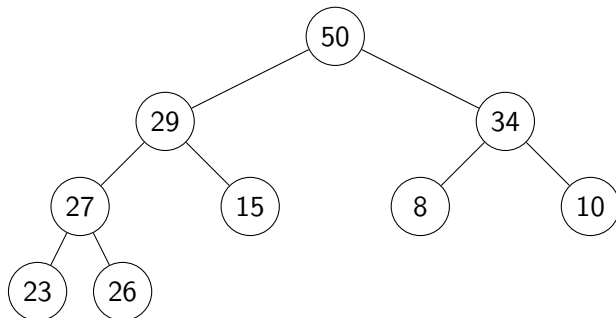
## Realization 3: Heaps

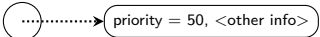
A **(binary) heap** is a certain type of binary tree.

You should know:

- A **binary tree** is either
  - ▶ empty, or
  - ▶ consists of three parts:  
a node and two binary trees (left subtree and right subtree).
- Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
- Any binary tree with  $n$  nodes has height at least  $\log(n + 1) - 1 \in \Omega(\log n)$ .

## Example Heap



( In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be  )

# Heaps – Definition

A **heap** is a binary tree with the following two properties:

- 1 **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
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**Lemma:** The height of a heap with  $n$  nodes is  $\Theta(\log n)$ .

# Storing Heaps in Arrays

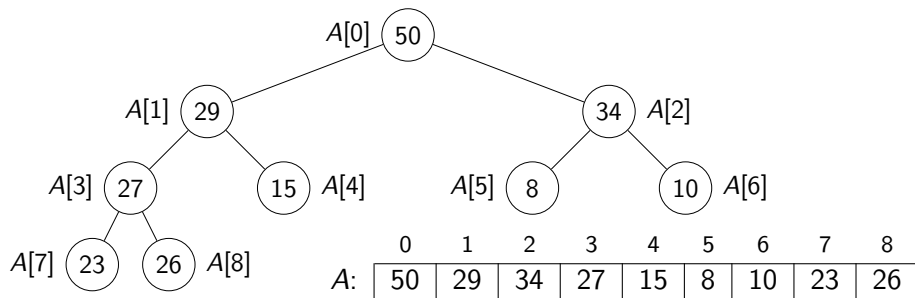
Heaps should *not* be stored as binary trees!

Let  $H$  be a heap of  $n$  items and let  $A$  be an array of size  $n$ . Store root in  $A[0]$  and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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# Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the *root* node is at index 0  
(We use “node” and “index” interchangeably in this implementation.)
- the *last* node is  $n - 1$  (where  $n$  is the size)
- the *left child* of node  $i$  (if it exists) is node  $2i + 1$
- the *right child* of node  $i$  (if it exists) is node  $2i + 2$
- the *parent* of node  $i$  (if it exists) is node  $\lfloor \frac{i-1}{2} \rfloor$
- these nodes exist if the index falls in the range  $\{0, \dots, n-1\}$

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We should hide implementation details using helper-functions!

- functions *root()*, *last()*, *parent(i)*, etc.

Some of these helper-functions need to know  $n$  (but we omit this in the code for simplicity).

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*fix-up*( $A, i$ )

$i$ : an index corresponding to a node of the heap

1.     **while**  $\text{parent}(i)$  exists **and**  $A[\text{parent}(i)].\text{key} < A[i].\text{key}$  **do**
2.         swap  $A[i]$  and  $A[\text{parent}(i)]$
3.          $i \leftarrow \text{parent}(i)$

The new item “bubbles up” until it reaches its correct place in the heap.

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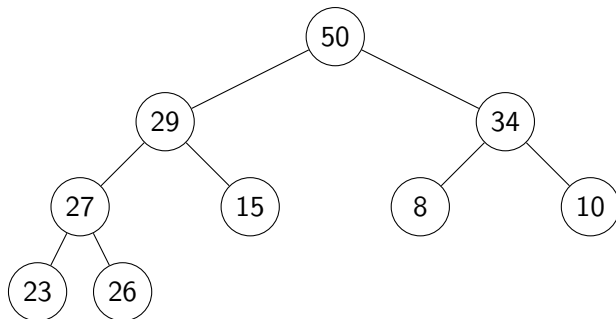
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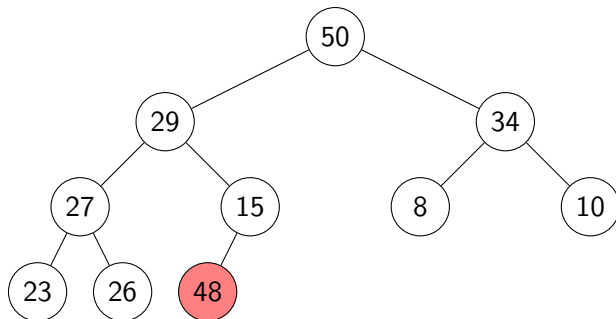
The new item “bubbles up” until it reaches its correct place in the heap.

Time:  $O(\text{height of heap}) = O(\log n)$ .

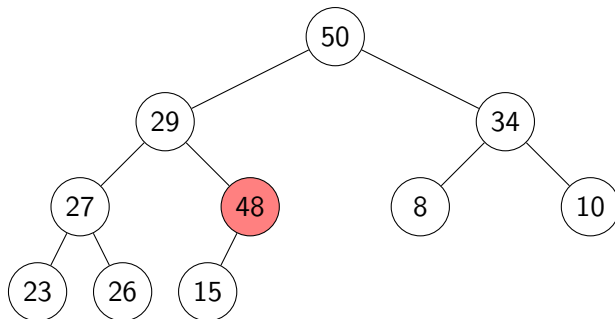
## *fix-up* example



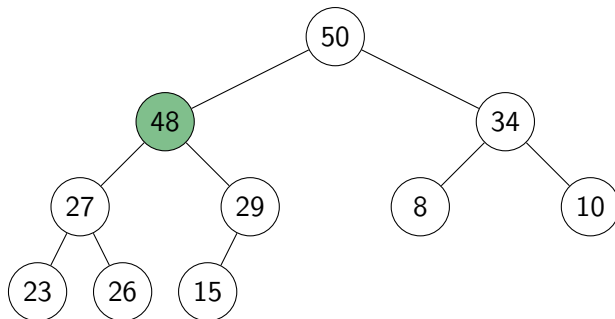
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- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
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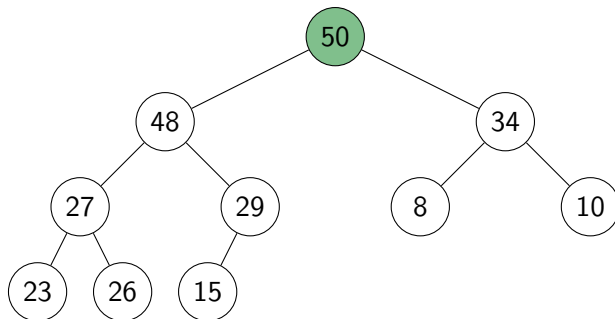
$A$ : an array that stores a heap of size  $n$

$i$ : an index corresponding to a node of the heap

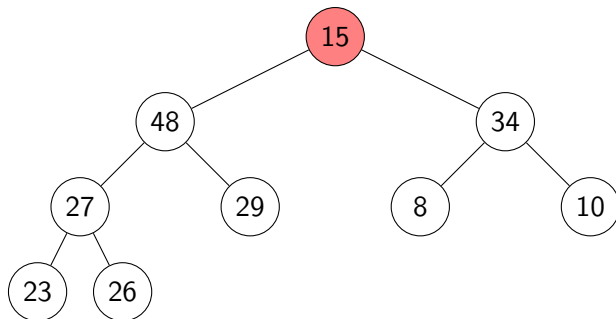
1. **while**  $i$  is not a leaf **do**
2.      $j \leftarrow$  left child of  $i$      // Find the child with the larger key
3.     if ( $i$  has right child and  $A[\text{right child of } i].\text{key} > A[j].\text{key}$ )
4.          $j \leftarrow$  right child of  $i$
5.     **if**  $A[i].\text{key} \geq A[j].\text{key}$  **break**
6.     swap  $A[j]$  and  $A[i]$
7.      $i \leftarrow j$

Time:  $O(\text{height of heap}) = O(\log n)$ .

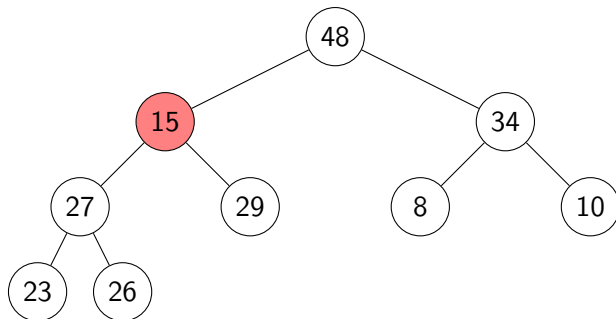
## *deleteMax* example



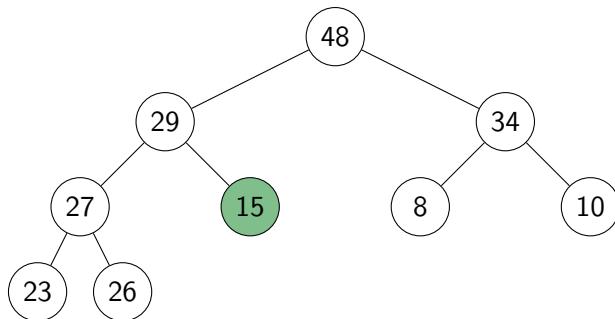
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# Priority Queue Realization Using Heaps

- Store items in array  $A$  and globally keep track of  $size$

*insert*( $x$ )

1. increase  $size$
2.  $\ell \leftarrow last()$
3.  $A[\ell] \leftarrow x$
4. *fix-up*( $A, \ell$ )

*deleteMax*()

1.  $\ell \leftarrow last()$
2. swap  $A[root()]$  and  $A[\ell]$
3. decrease  $size$
4. *fix-down*( $A, root(), size$ )
5. **return**  $A[\ell]$

*insert* and *deleteMax*:  $O(\log n)$  **time**

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## Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$$

- Using the binary-heaps implementation of PQs, we obtain:

*PQsortWithHeaps*(*A*)

1. initialize *H* to an empty heap
2. **for**  $i \leftarrow 0$  **to**  $n - 1$  **do**
3.     *H.insert*(*A*[*i*])
4.     **for**  $i \leftarrow n - 1$  **down to** 0 **do**
5.         *A*[*i*]  $\leftarrow$  *H.deleteMax*()

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```
PQsortWithHeaps(A)
1.  initialize  $H$  to an empty heap
2.  for  $i \leftarrow 0$  to  $n - 1$  do
3.       $H.\textit{insert}(A[i])$ 
4.  for  $i \leftarrow n - 1$  down to  $0$  do
5.       $A[i] \leftarrow H.\textit{deleteMax}()$ 
```

- both operations run in  $O(\log n)$  time for heaps

~> *PQ-Sort* using heaps takes  $O(n \log n)$  time.

- Can improve this with two simple tricks → **Heapsort**

- Heaps can be built faster if we know all input in advance.
- Can use the same array for input and heap. ~>  $O(1)$  *auxiliary space!*

# Building Heaps with Fix-up

**Problem:** Given  $n$  items all at once (in  $A[0 \cdots n - 1]$ ) build a heap containing all of them.

## Building Heaps with Fix-up

**Problem:** Given  $n$  items all at once (in  $A[0 \cdots n - 1]$ ) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

*simpleHeapBuilding*( $A$ )

$A$ : an array

1. initialize  $H$  as an empty heap
2. **for**  $i \leftarrow 0$  **to**  $A.\text{size}() - 1$  **do**
3.      $H.\text{insert}(A[i])$

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This corresponds to doing *fix-ups*

Worst-case running time:  $\Theta(n \log n)$ .

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**Problem:** Given  $n$  items all at once (in  $A[0 \cdots n - 1]$ ) build a heap containing all of them.

**Solution 2:** Using *fix-downs* instead:

```
heapify(A)
```

```
A: an array
```

1.  $n \leftarrow A.size()$
2. **for**  $i \leftarrow \text{parent}(\text{last}())$  **downto**  $\text{root}()$  **do**
3.     *fix-down*(A,  $i$ ,  $n$ )

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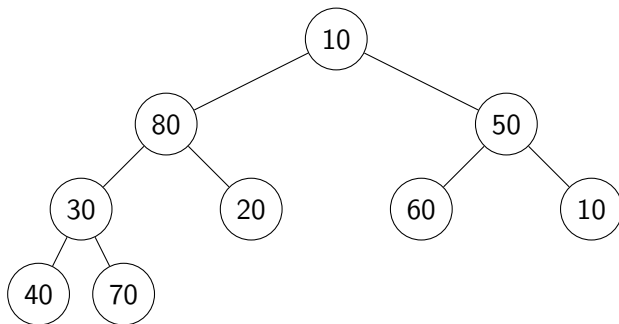
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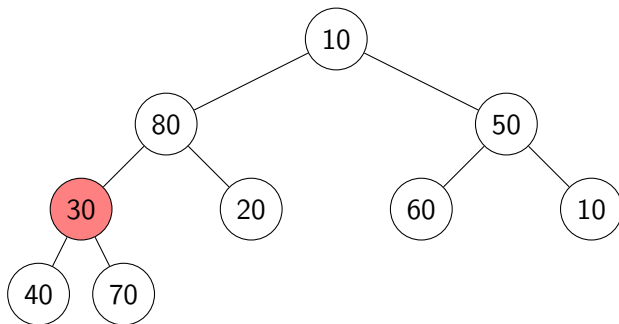
A careful analysis yields a worst-case complexity of  $\Theta(n)$ .

A heap can be built in linear time.

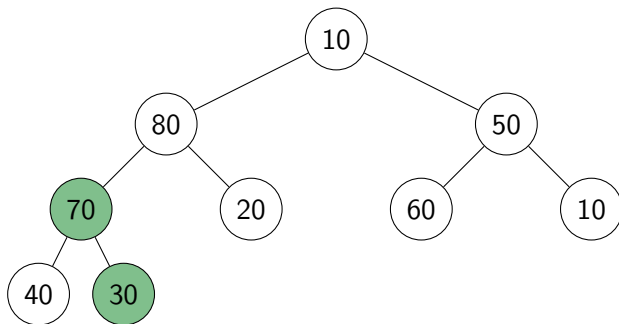
## heapify example



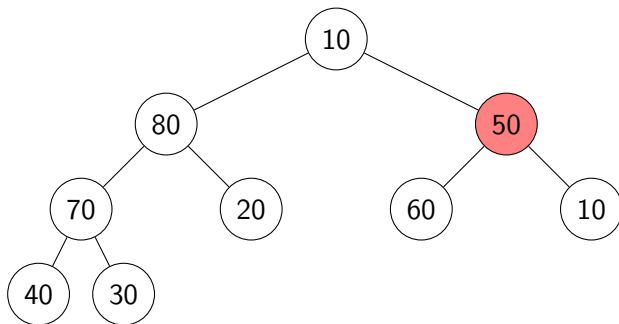
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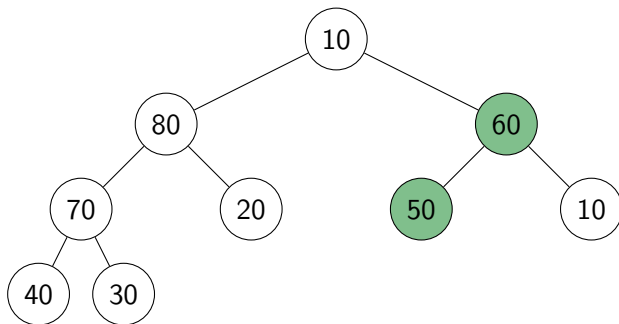
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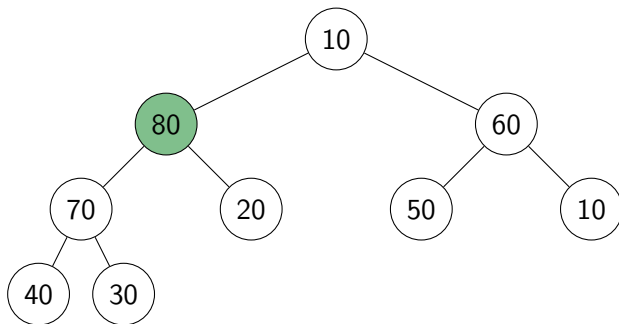
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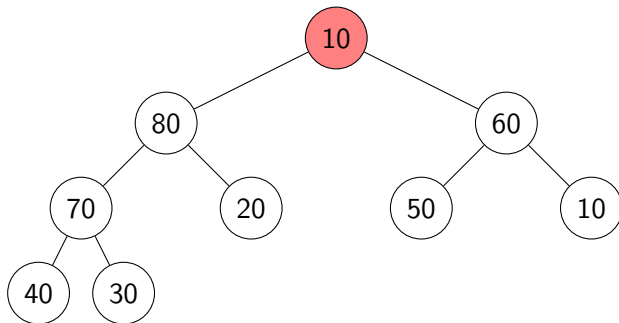
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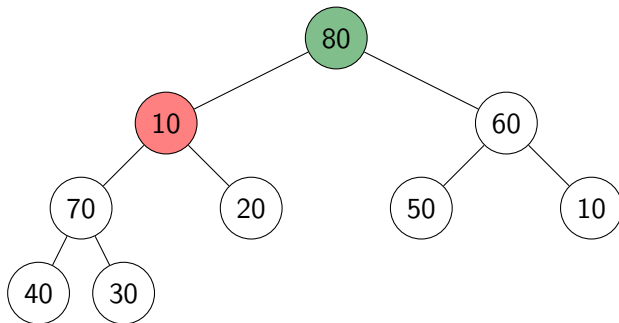
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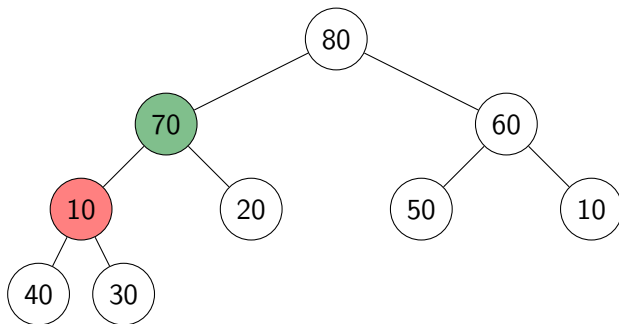
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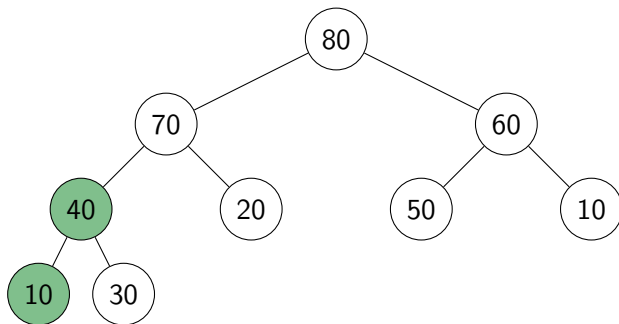
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# Efficient sorting with heaps

- Idea: *PQ-sort* with heaps.
- $O(1)$  auxiliary space: Use same input-array  $A$  for storing heap.

*HeapSort*( $A, n$ )

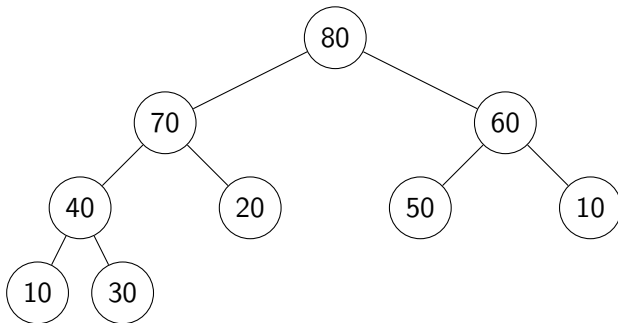
```
1.  // heapify
2.   $n \leftarrow A.size()$ 
3.  for  $i \leftarrow \text{parent}(\text{last}())$  downto 0 do
4.      fix-down( $A, i, n$ )

5.  // repeatedly find maximum
6.  while  $n > 1$ 
7.      // 'delete' maximum by moving to end and decreasing  $n$ 
8.      swap items at  $A[\text{root}()]$  and  $A[\text{last}()]$ 
9.      decrease  $n$ 
10.     fix-down( $A, \text{root}(), n$ )
```

The for-loop takes  $\Theta(n)$  time and the while-loop takes  $O(n \log n)$  time.

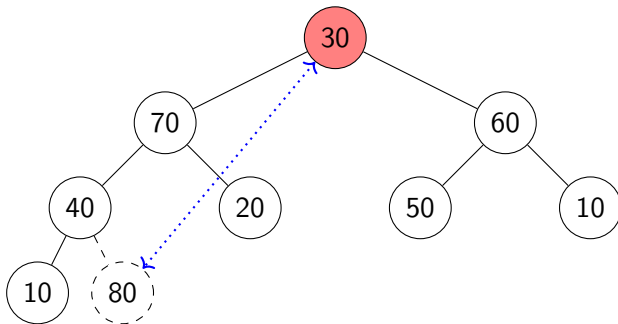
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Continue with the example from heapify:



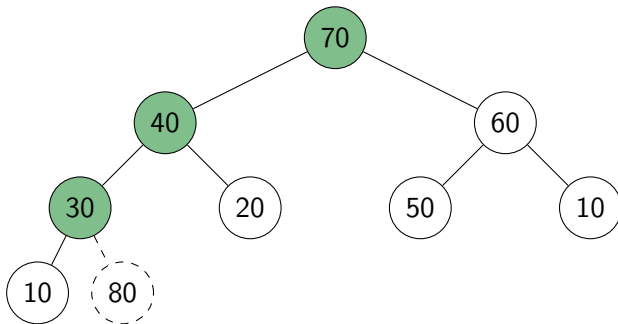
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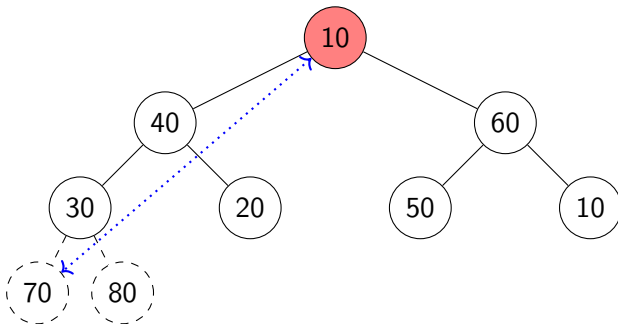
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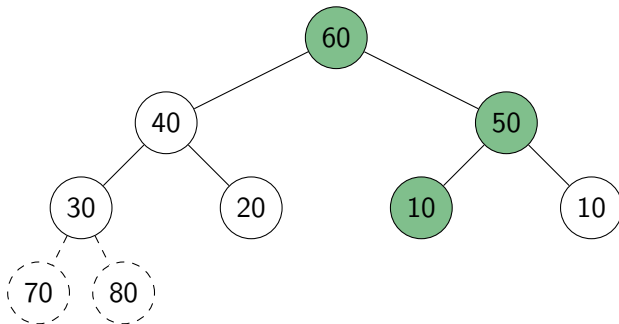
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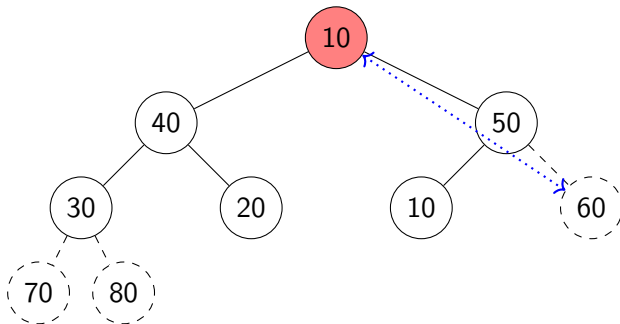
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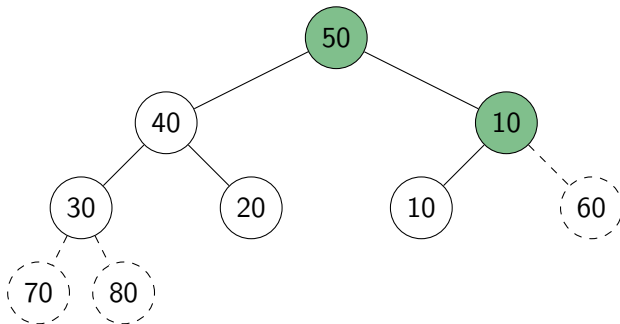
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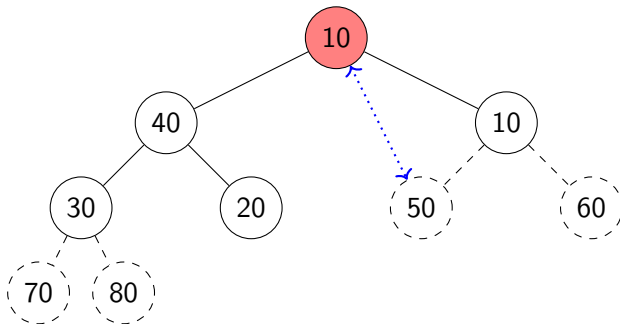
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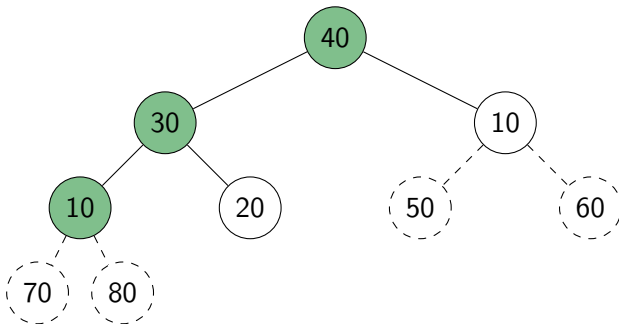
## Heapsort example

Continue with the example from heapify:



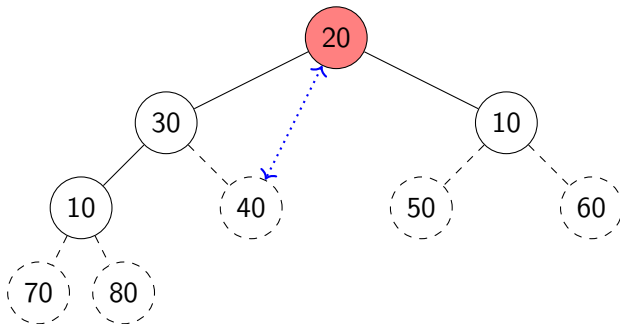
## Heapsort example

Continue with the example from heapify:



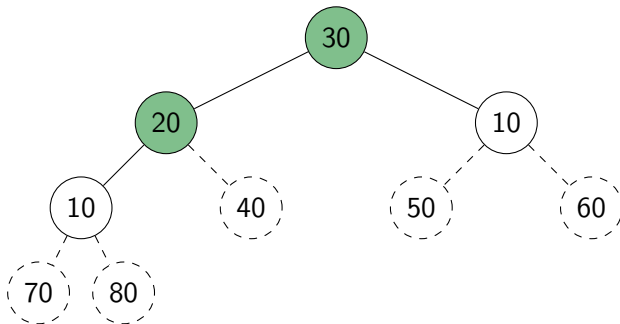
## Heapsort example

Continue with the example from heapify:



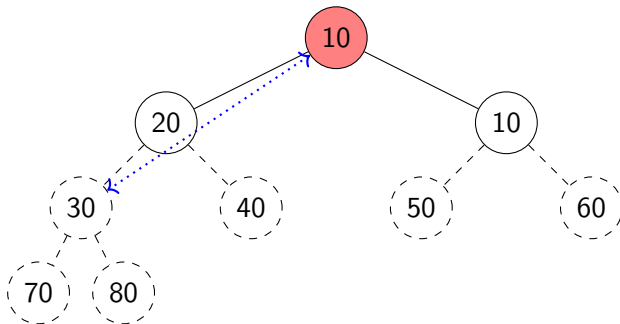
## Heapsort example

Continue with the example from heapify:



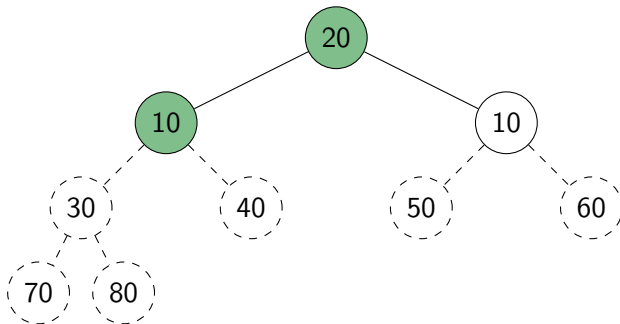
## Heapsort example

Continue with the example from heapify:



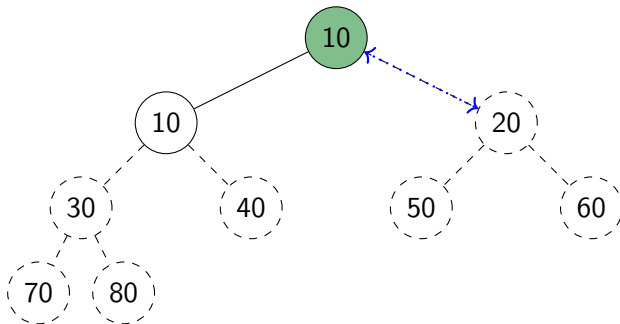
## Heapsort example

Continue with the example from heapify:



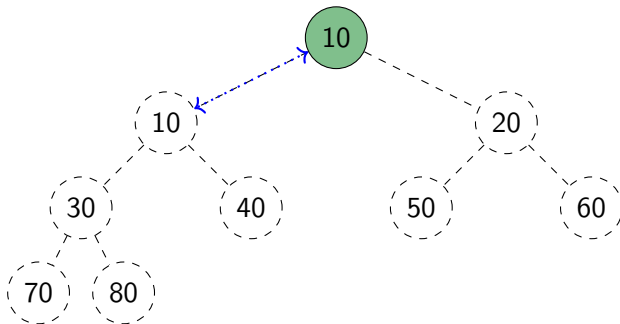
## Heapsort example

Continue with the example from heapify:



## Heapsort example

Continue with the example from heapify:



The array (i.e., the heap in level-by-level order) is now in sorted order.

# Heap summary

- **Binary heap**: A binary tree that satisfies structural property and heap-order property.
- Heaps are one possible realization of ADT PriorityQueue:
  - ▶ *insert* takes time  $O(\log n)$
  - ▶ *deleteMax* takes time  $O(\log n)$
  - ▶ Also supports *findMax* in time  $O(1)$
- A binary heap can be built in linear time.
- *PQ-sort* with binary heaps leads to a sorting algorithm with  $O(n \log n)$  worst-case run-time ( $\rightsquigarrow$  *HeapSort*)
- We have seen here the *max-oriented version* of heaps (the maximum priority is at the root).
- There exists a symmetric *min-oriented version* that supports *insert* and *deleteMin* with the same run-times.

# Outline

## 2 Priority Queues

- Abstract Data Types
- ADT Priority Queue
- Binary Heaps
- Operations in Binary Heaps
- *PQ-sort* and *Heapsort*
- Towards the Selection Problem

## Finding the smallest items

**Problem:** Find the *kth smallest item* in an array  $A$  of  $n$  distinct numbers.

**Solution 1:** Make  $k$  passes through the array, deleting the minimum number each time.

Complexity:  $\Theta(kn)$ .

**Solution 2:** Sort  $A$ , then return  $A[k]$ .

Complexity:  $\Theta(n \log n)$ .

**Solution 3:** Scan the array and maintain the  $k$  smallest numbers seen so far in a max-heap

Complexity:  $\Theta(n \log k)$ .

**Solution 4:** Create a min-heap with *heapify*( $A$ ). Call *deleteMin*( $A$ )  $k$  times.

Complexity:  $\Theta(n + k \log n)$ .