CS 240 – Data Structures and Data Management

Module 4: Dictionaries

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Based on lecture notes by many previous cs240 instructors

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Outline

- Dictionaries and Balanced Search Trees
    - Dictionary ADT
    - Review: Binary Search Trees
    - AVL Trees
        - insertion
        - restoring the AVL Property: Rotations
        - full code for insertion
        - deletion

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Dictionary ADT

- A *dictionary* is a collection of *items*, each of which contains
  - a *key*
  - some *data*
- Item is called a *key-value pair* (KVP)
- Keys can be compared and are (typically) unique
  - can extend to handle non-unique keys
- Operations
  - *search*(k)
    - also called *findElement*(k)
  - *insert*(k, v)
    - also called *insertItem*(k, v)
  - *delete*(k)
    - also called *removeElement*(k)
  - optional: *closestKeyBefore*, *join*, *isEmpty*, *size*, *etc.*
- Examples: symbol table, license plate database

Elementary Implementations

- Common assumptions
  - dictionary has n KVPs
  - each KVP uses constant space
    - if not, the “value” could be a pointer
  - keys can be compared in constant time
- **Unordered array or linked list**
  - *search* $\Theta(n)$
  - *insert* $\Theta(1)$
  - *delete* $\Theta(n)$
    - need to search

(7,’Ace’)	(1,’Pot’)	(3,’Top’)	(2,’Dog’)	(0,’Cat’)	(5,’Log’)

- **Ordered array**
  - *search* $\Theta(\log n)$
    - via binary search
  - *insert* $\Theta(n)$
  - *delete* $\Theta(n)$

(0,’Cat’)	(1,’Pot’)	(2,’Dog’)	(3,’Top’)	(5,’Log’)	(7,’Ace’)

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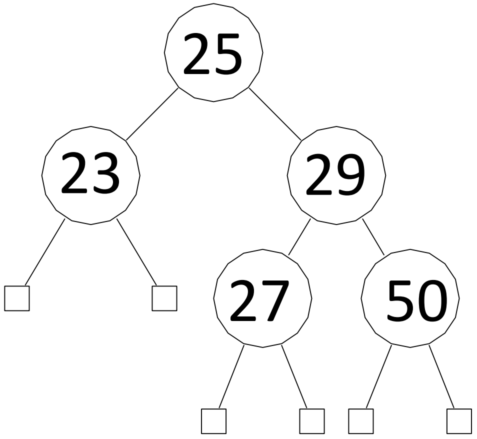
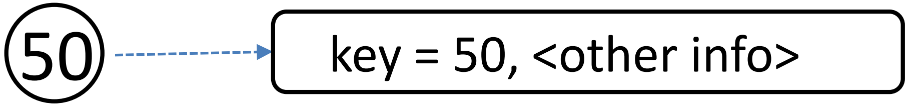
Binary Search Trees (review)



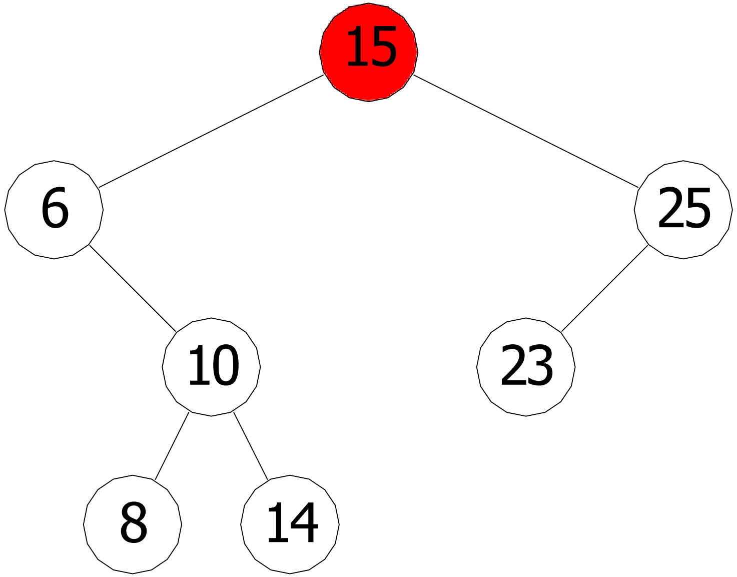
- Structure
  - Binary tree is either empty or consists of nodes
  - All nodes have two (possibly empty) subtrees, *L* (left) and *R* (right)
  - Every node stores a KVP
  - Leaves store empty subtrees
  - Empty subtrees usually not shown
- Ordering
  - every key in the left subtree of node v is less than $v.key$
  - every key the right subtree of node v greater than $v.key$
- Show only keys, directly in the node
- More accurate picture


50

key = 50, <other info>

BST Search

- BST::search(k)
    - start at root, compare k to current node
    - stop if found or subtree is empty, else recurse at subtree
- Example: BST::search(24)



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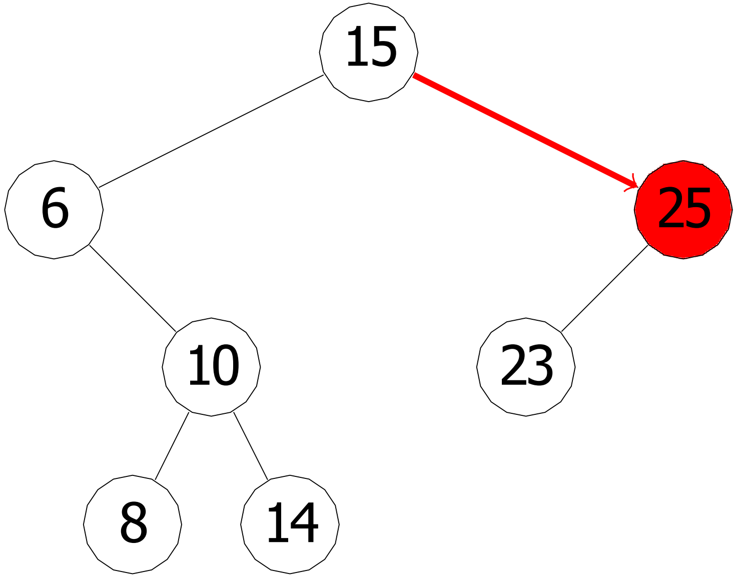

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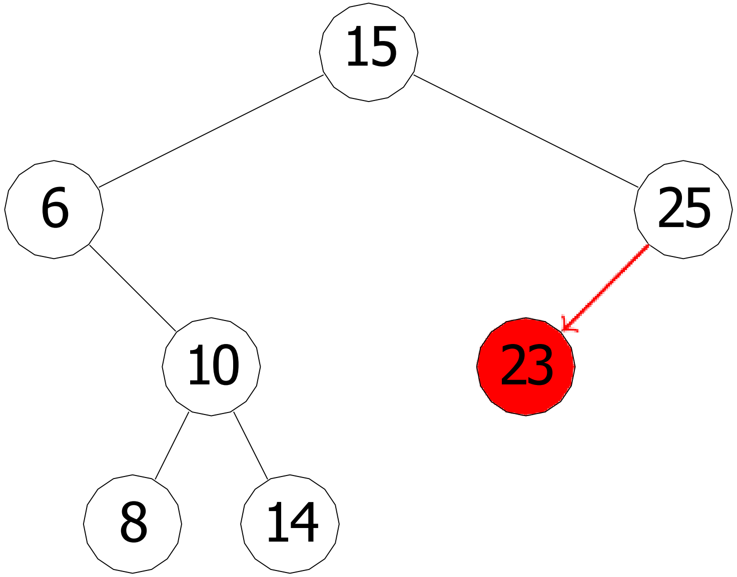

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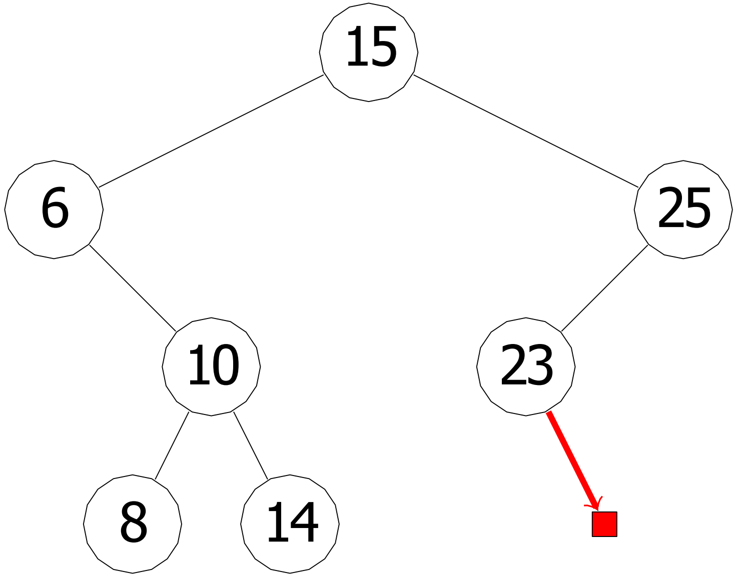
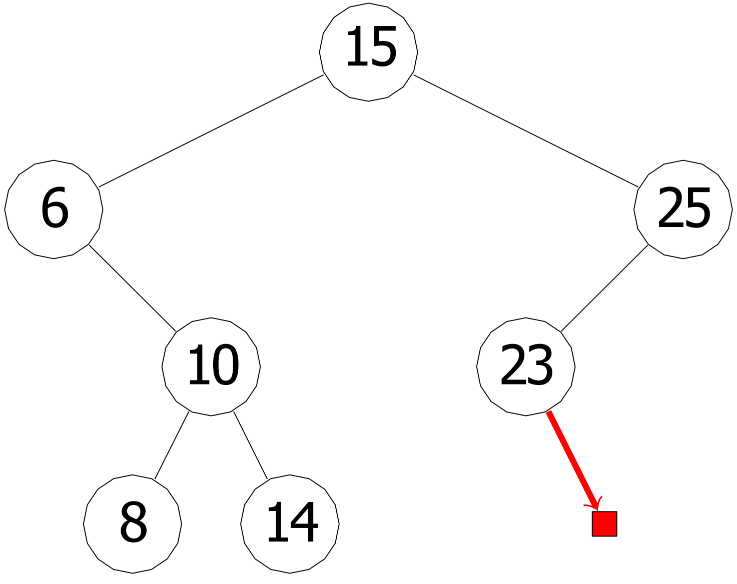
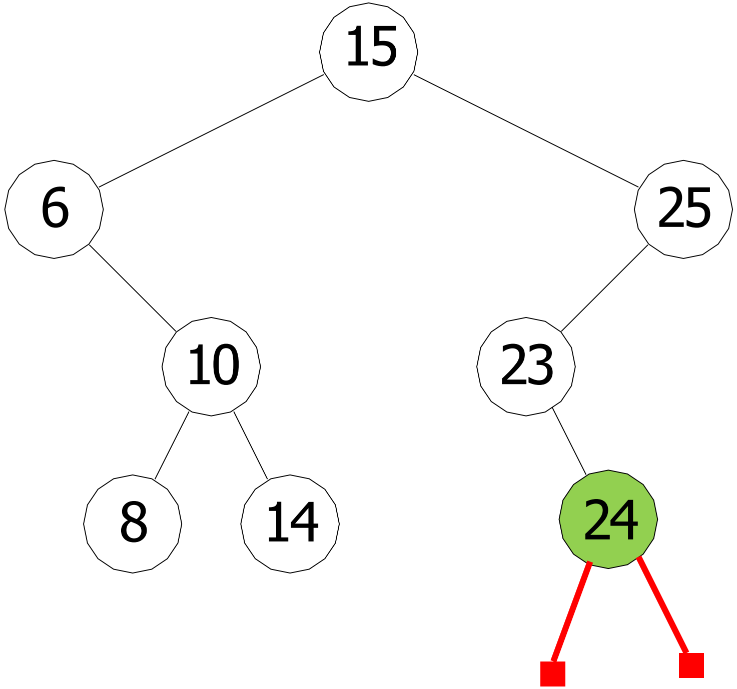


BST Insert

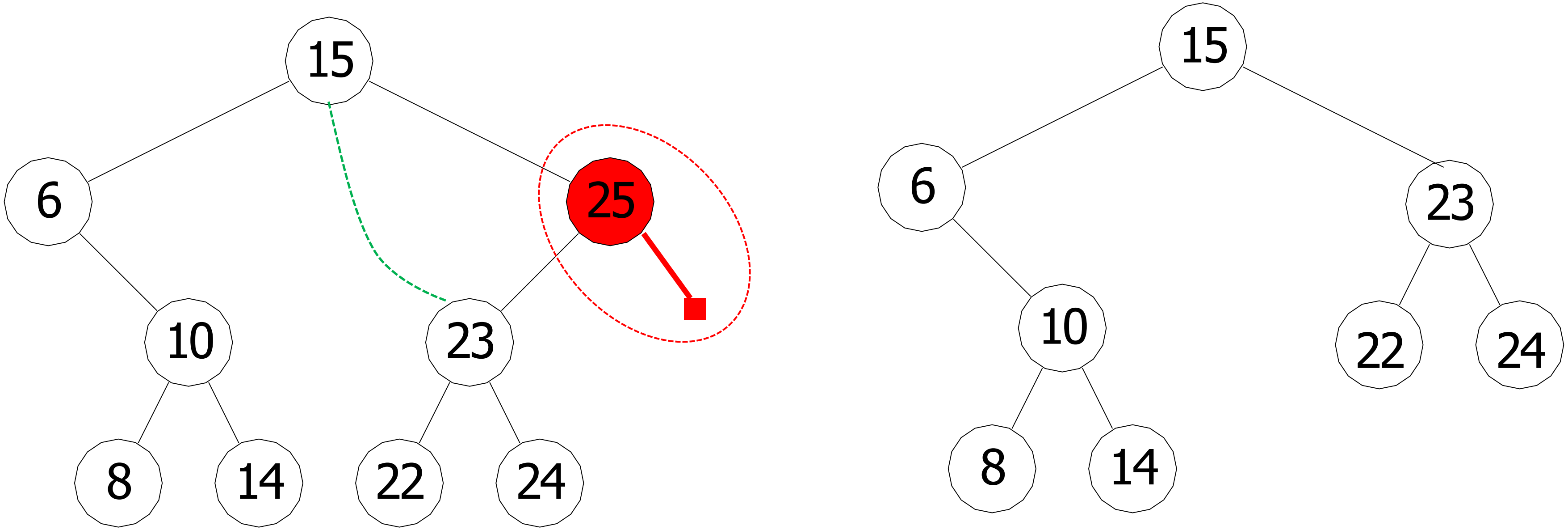
- BST::insert(k, v)
  - search for k, then insert (k, v) as a new node at the empty subtree where search stops “expand at empty”
- Example: *BST::insert*$(24, v)$





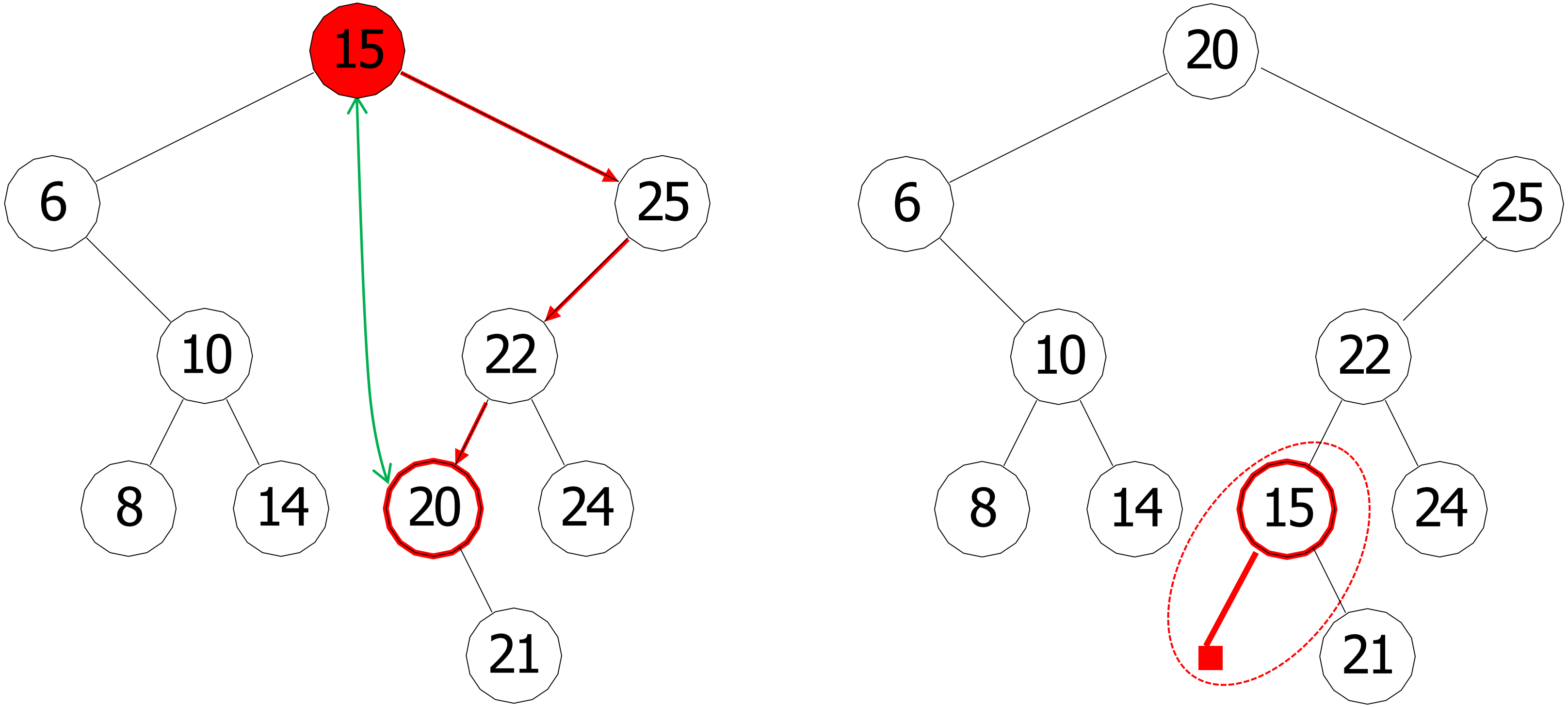
BST Delete, case 1

- First search for the node x that contains the key
    1. If x has at least one empty subtree
        - delete it with the empty subtree
        - If x has a parent, reconnect the other subtree of x to the parent of x
- Example: *BST::delete*(25)



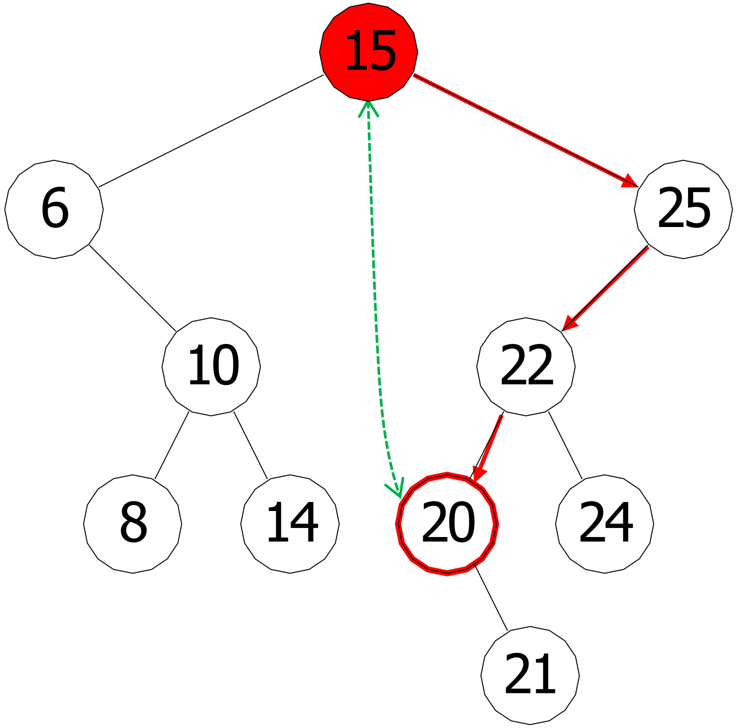
BST Delete, case 2

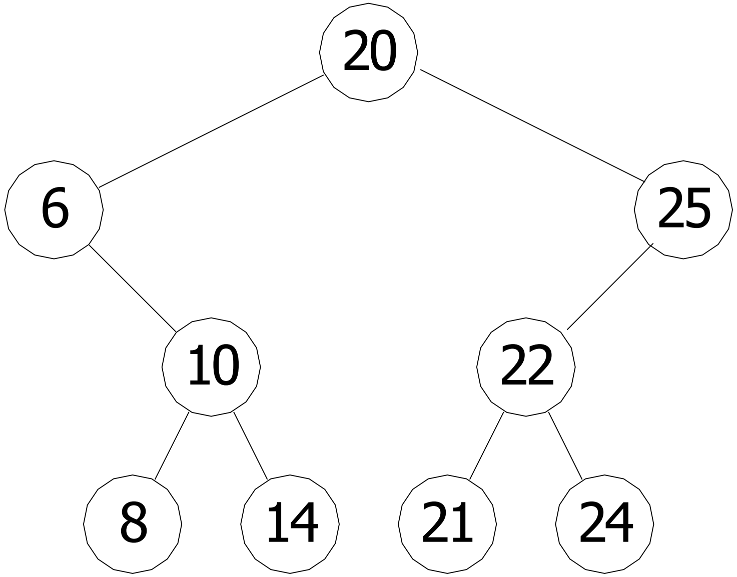
- First search for the node x that contains the key
  2. If *x* has only non-empty subtrees
     - swap *KVP* at x with *KVP* at successor node (or predecessor node)
     - delete successor node (or predecessor node)
       - case 1 applies
- Example: *BST::delete*(15)



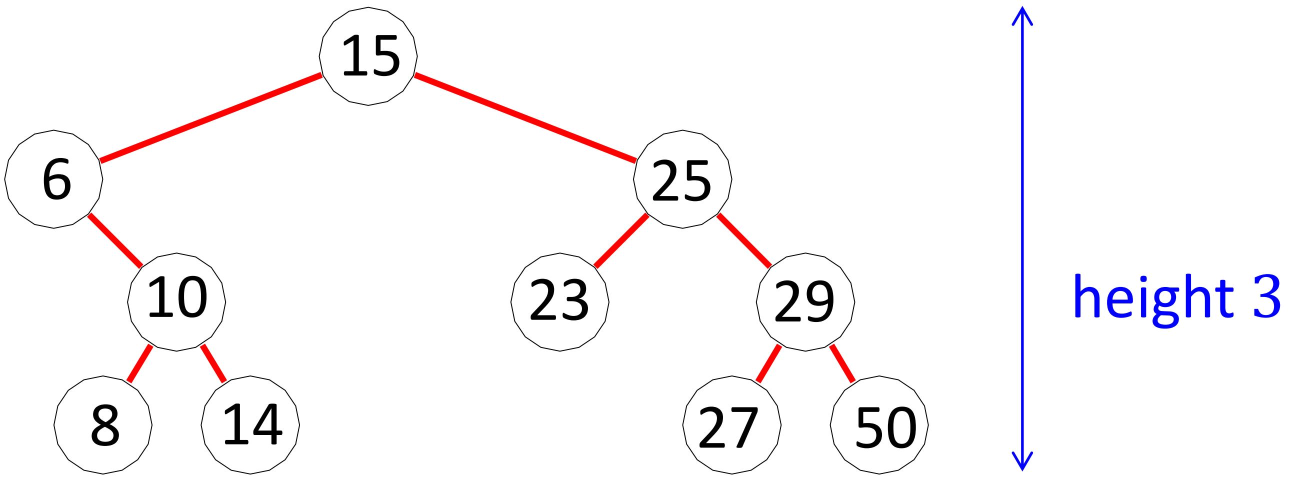
BST Delete, case 2

- First search for the node x that contains the key
    2. If *x* has only non-empty subtrees
        - swap *KVP* at x with *KVP* at successor node (or predecessor node)
        - delete successor node (or predecessor node)
            - case 1 applies
- Example: *BST::delete*(15)





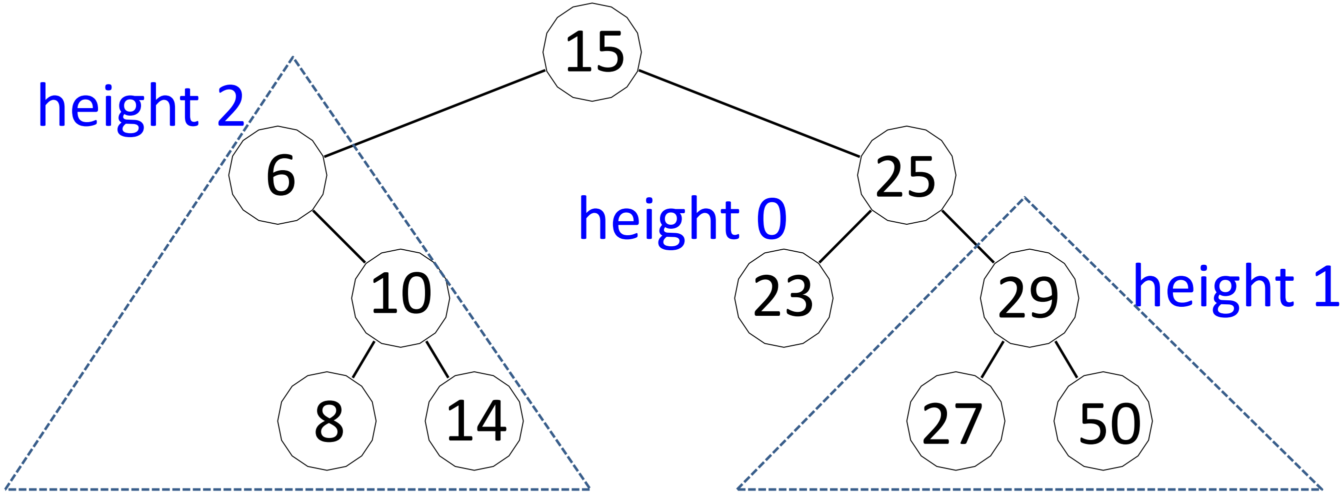
Height of a BST



- *BST::search*, *BST::insert*, *BST::delete* all have cost $\Theta(h)$
    - h = height of the tree = maximum length path from root to a leaf node
    - height of an empty tree is defined to be -1
- If n items are *BST::insert*ed one-at-a-time, how big is h?
    - worst-case is $n - 1 = \Theta(n)$
    - best case is $\Theta(\log n)$
        - binary tree with n nodes has height $\geq \log(n + 1) - 1$
    - can show if insert items in random order then height is $\Theta(\log n)$

Height of a node

- Height of node v is the height of the tree rooted at node v



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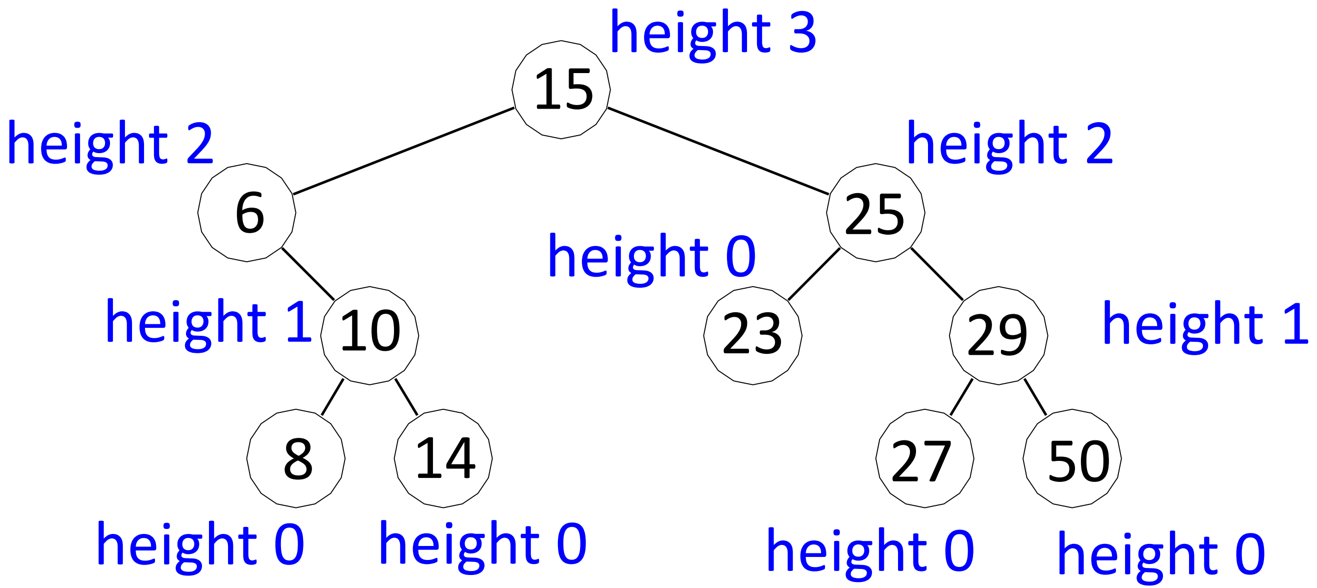

- Can compute heights of all nodes in post order traversal
  - height of a leaf is 0
  - height of any other node v is

$$1 + \max\{\text{height}(v.\text{left}), \text{height}(v.\text{right})\}$$

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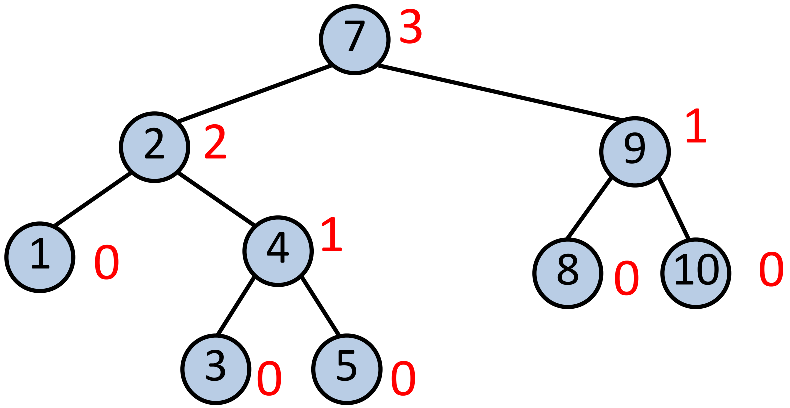
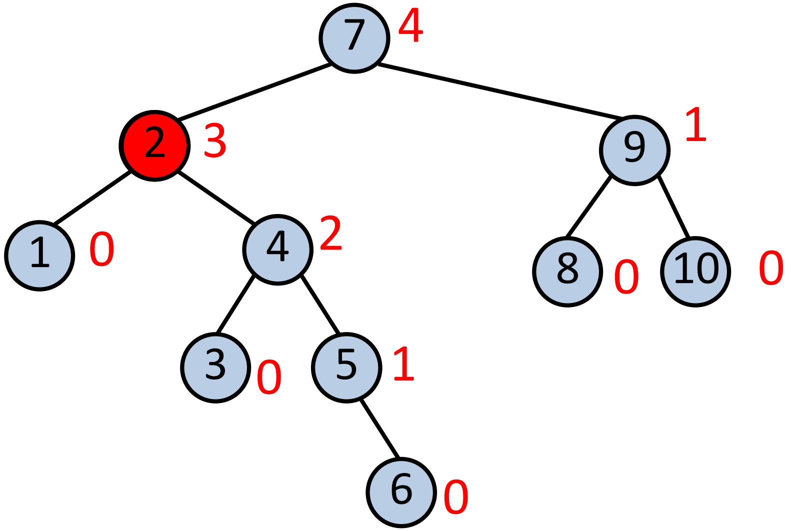
AVL Trees

- **A**delson-**V**elski and **L**andis, 1962
  - "An algorithm for organization of information", *Doklady Akademii Nauk USSR*
- *AVL Tree* is a BST with **height-balance** property
  - for any node v, heights of its left subtree $\boldsymbol{L}$ and right subtree $\boldsymbol{R}$ differ by at most 1

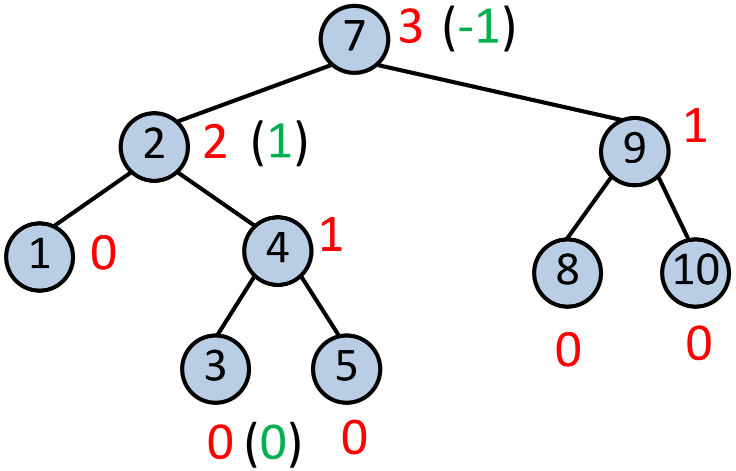


AVL Tree



not AVL Tree

AVL Trees



- *AVL Tree* is a BST with **height-balance** property
  - for any node v, heights of its left subtree $\boldsymbol{L}$ and right subtree $\boldsymbol{R}$ differ by at most 1
  - In other words, $height(v.right) - height(v.left) \in \{-1, 0, 1\}$
    - -1 means v is *left-heavy*
    - 0 means v is *balanced*
    - $+1$ means v is *right-heavy*
- Need to store at each node v its height
  - enough to store **balance factor** $= height(v.right) - height(v.left)$
    - fewer bits
    - but code more complicated, especially for deleting

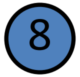
Height of an AVL tree

Theorem: AVL tree on n nodes has $\Theta(\log n)$ height

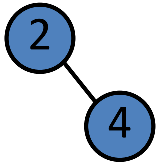
Proof:

- Only need upper bound, as height is $\Omega(\log n)$
- Let $N(h)$ be the *smallest* number of nodes an AVL tree of height h can have
  - any AVL tree of height h has number of nodes $n \geq N(h)$

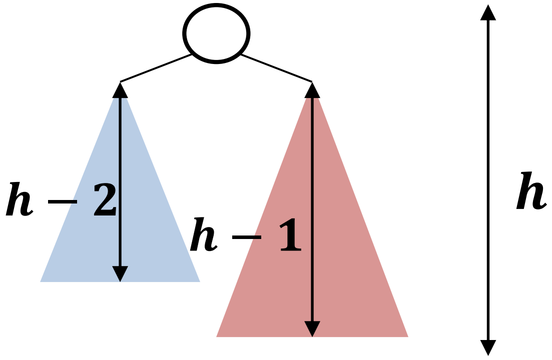
$N(0)$



$N(1)$



$N(h)$



- For $h \geq 2$

$$N(h) = N(h-1) + N(h-2) + 1 \geq N(h-2) + N(h-2) = 2N(h-2)$$

- Thus $N(h) \geq 2N(h-2)$

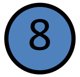
Height of an AVL tree

Theorem: AVL tree on n nodes

Proof:

- Only need upper bound, as heig
- Let $N(h)$ be the *smallest* numb

$N(0)$



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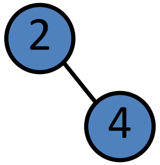


- For $h \geq 2$

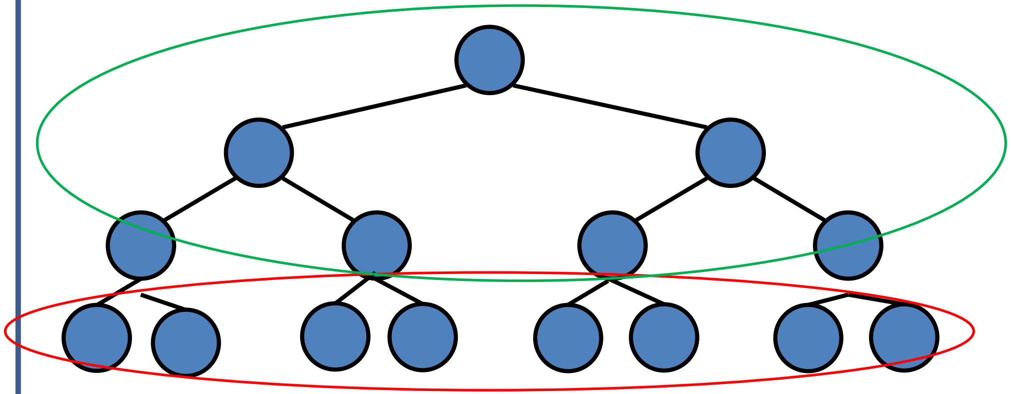
$$N(h) = N(h-1) + N(h-2$$

- Thus $N(h) \geq 2N(h-2)$

Side Note

- Recall heaps
  - add new level i, number of nodes basically doubles

$$2^0 + 2^1 + \cdots + 2^{i-1} = 2^i - 1$$



2^i

- $N(h) \approx 2N(h-1)$
- In AVL tree to add two levels to double number of nodes
  - slower, but also exponential growth

Height of an AVL tree

Proof: (continued)

- $N(h)$ is the *least* number of nodes in height-h AVL tree
    - any AVL tree of height h has number of nodes $n \ge N(h)$
    - $N(0) = 1, N(1) = 2$
- Recurrence inequality for $h \ge 2$ is $N(h) \ge 2N(h-2)$
- Solve by expanding until reach the base case

$$N(h) \ge 2N(h-2) \ge 2^2 N(h-2\cdot 2) \ge 2^3 N(h-2\cdot 3) \ge \cdots \ge 2^i N(h-2\cdot i)$$

Base case for odd h

- expand until $h - 2\cdot i = 1$
- rewriting, $i = (h-1)/2$

$$N(h) \ge 2^{(h-1)/2} N(1) \quad = 2^{\frac{h-1}{2}} \cdot 2$$

- take log

$$\log N(h) \ge \frac{h-1}{2} + 1$$

- rearrange

$$h \le 2\log N(h) - 2 \ \le 2\log n - 2$$

Base case for even h

- expand until $h - 2\cdot i = 0$
- rewriting, $i = h/2$

$$N(h) \ge 2^{h/2} N(0) = 2^{\frac{h}{2}} \cdot 1$$

- take log

$$\log N(h) \ge \frac{h}{2}$$

- rearrange

$$h \le 2\log N(h) \le 2\log n$$

h is $O(\log n)$

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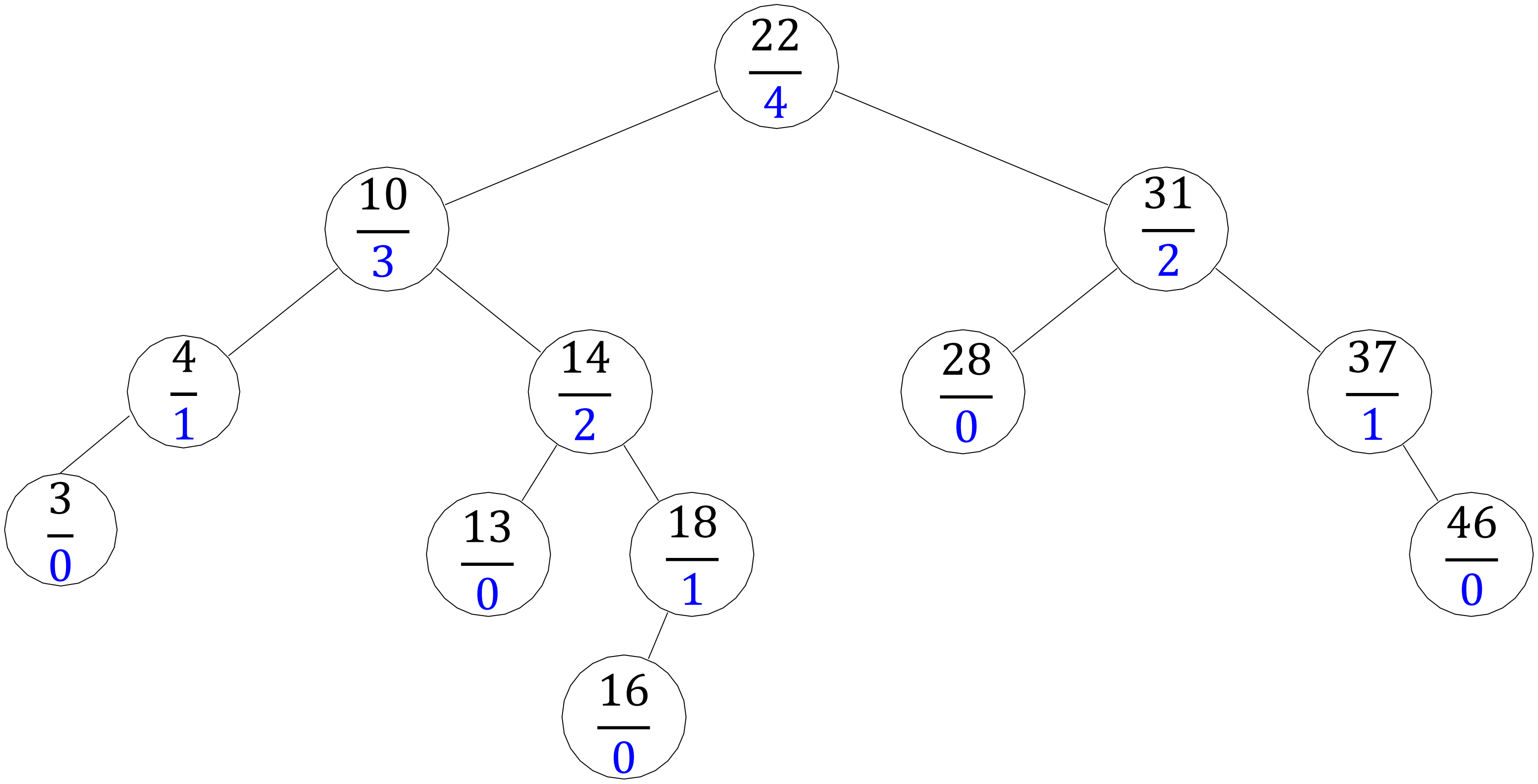
AVL Insertion Example

Example: *AVL::insert*(2)



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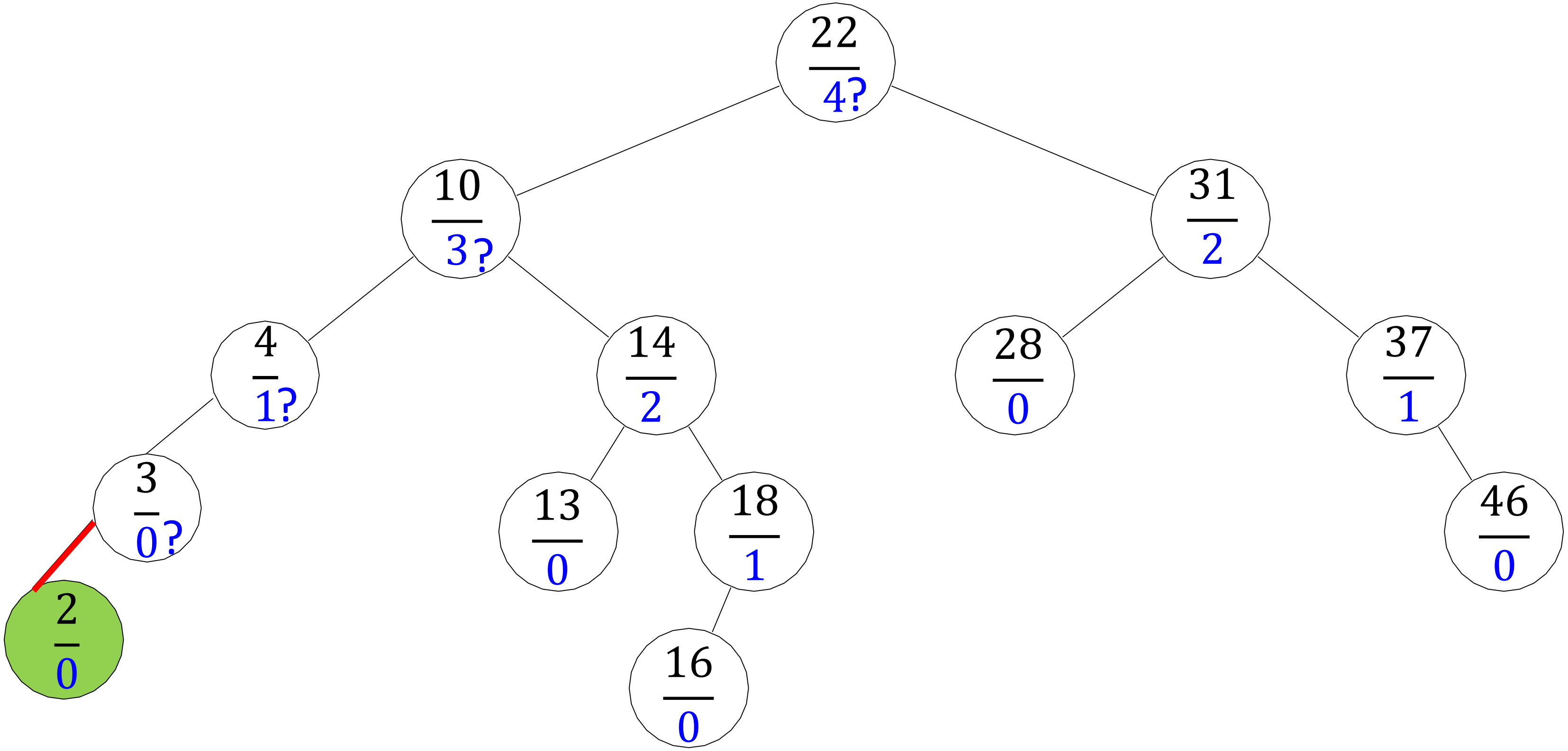

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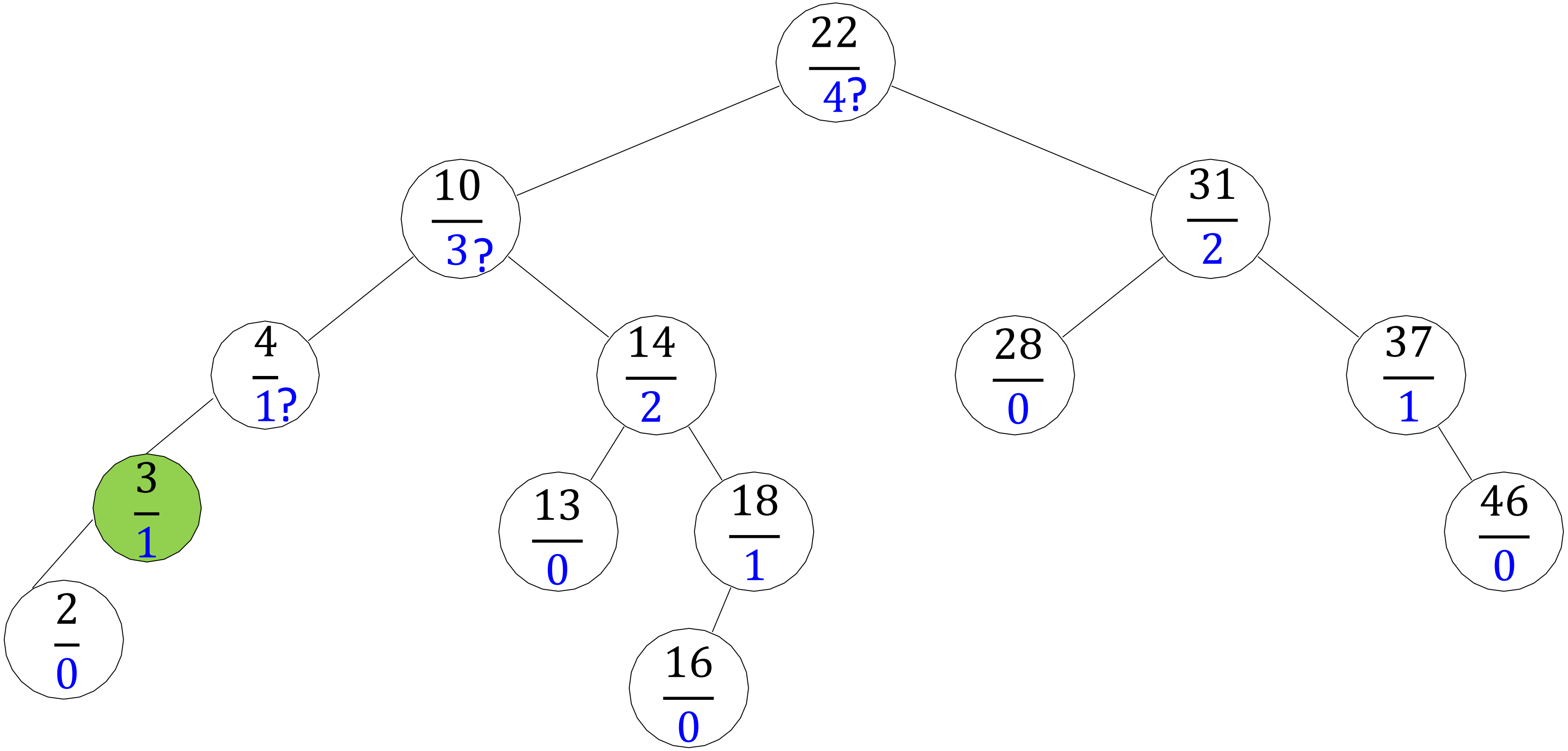

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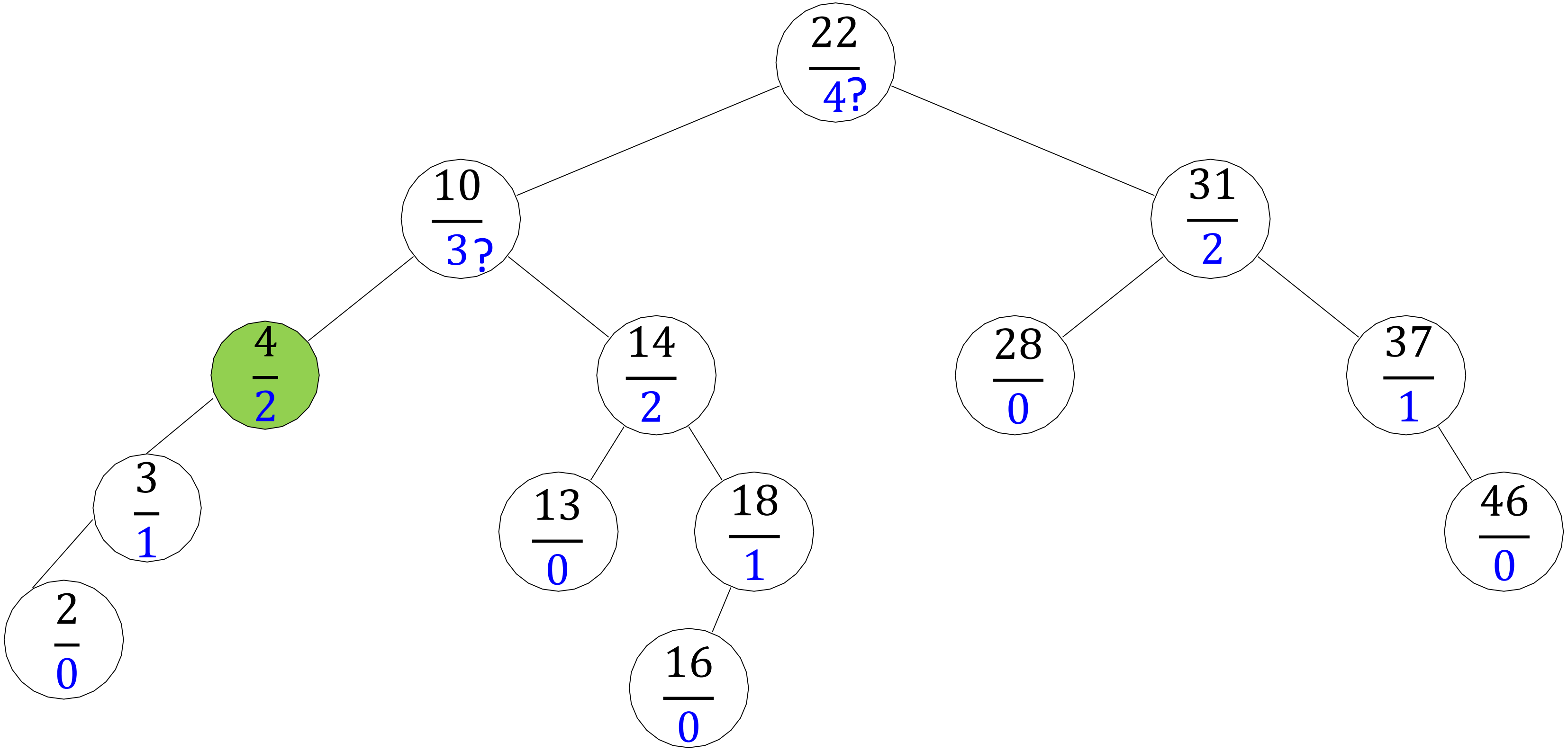

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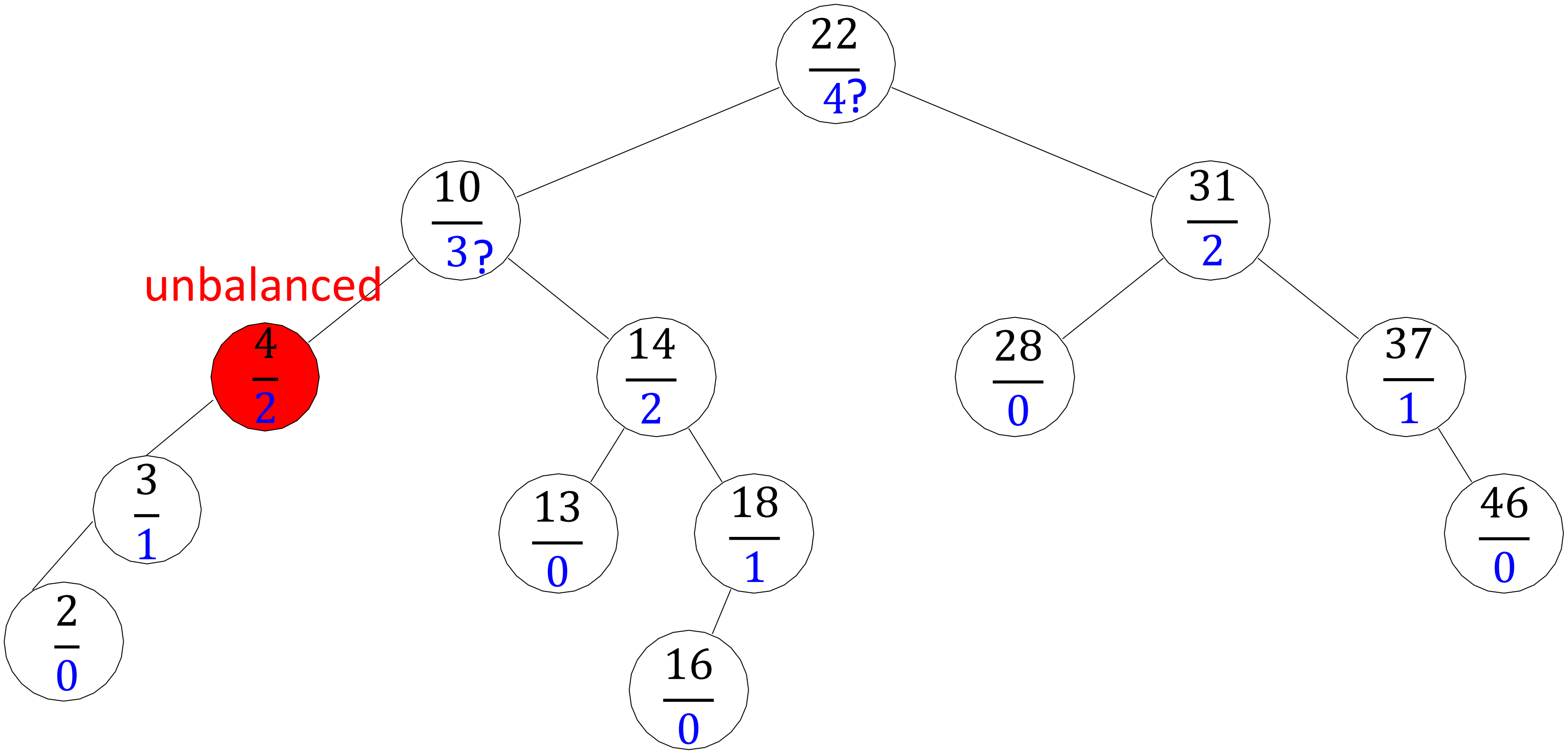

AVL insertion

- *AVL::insert*(T, k, v)
  1. insert (k, v) into T with the usual BST insertion
     - assume this returns the new leaf where the key was inserted
     - heights of nodes on path from this leaf to root may have increased
  2. move up the path from the new leaf to the root, updating heights
  3. if the height difference becomes ± 2 for some node on this path, the node is *unbalanced*
     - must re-structure the tree to restore height-balance property

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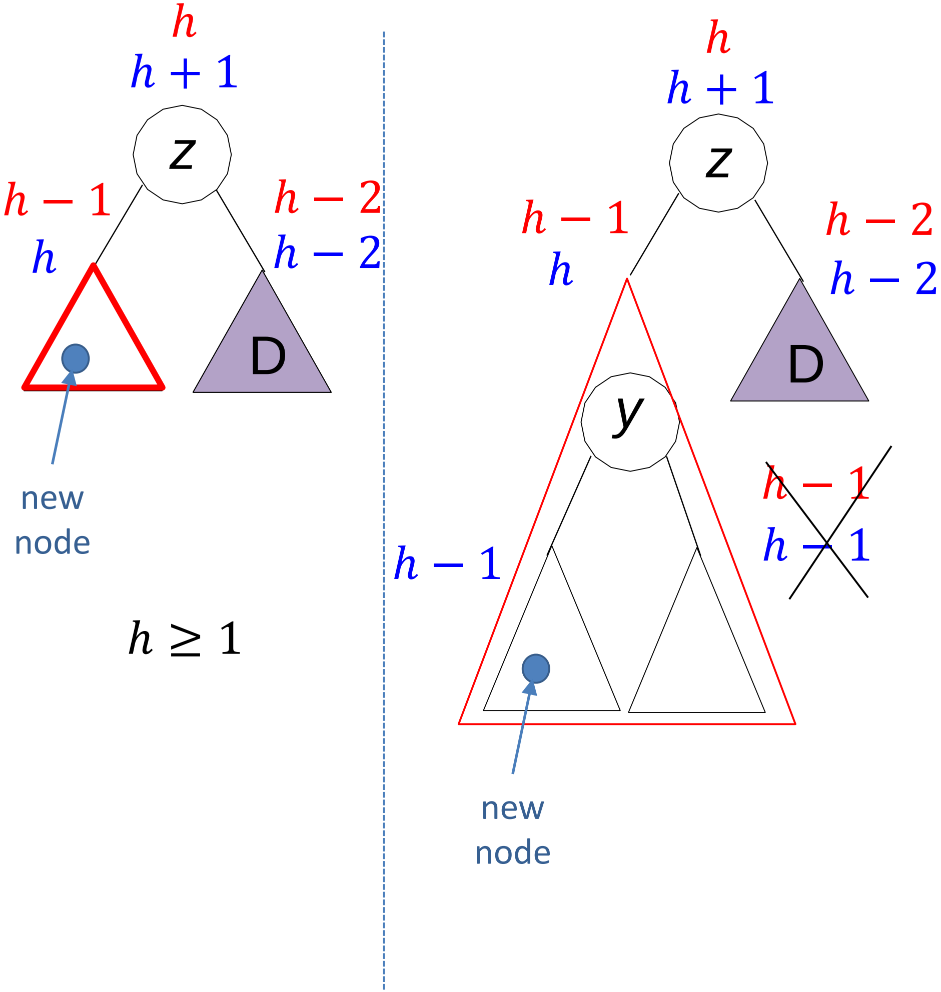
Restoring Height After Insertion

- Let *z* be *the first* unbalanced node on path from inserted node to the root

height after insertion/height before insertion



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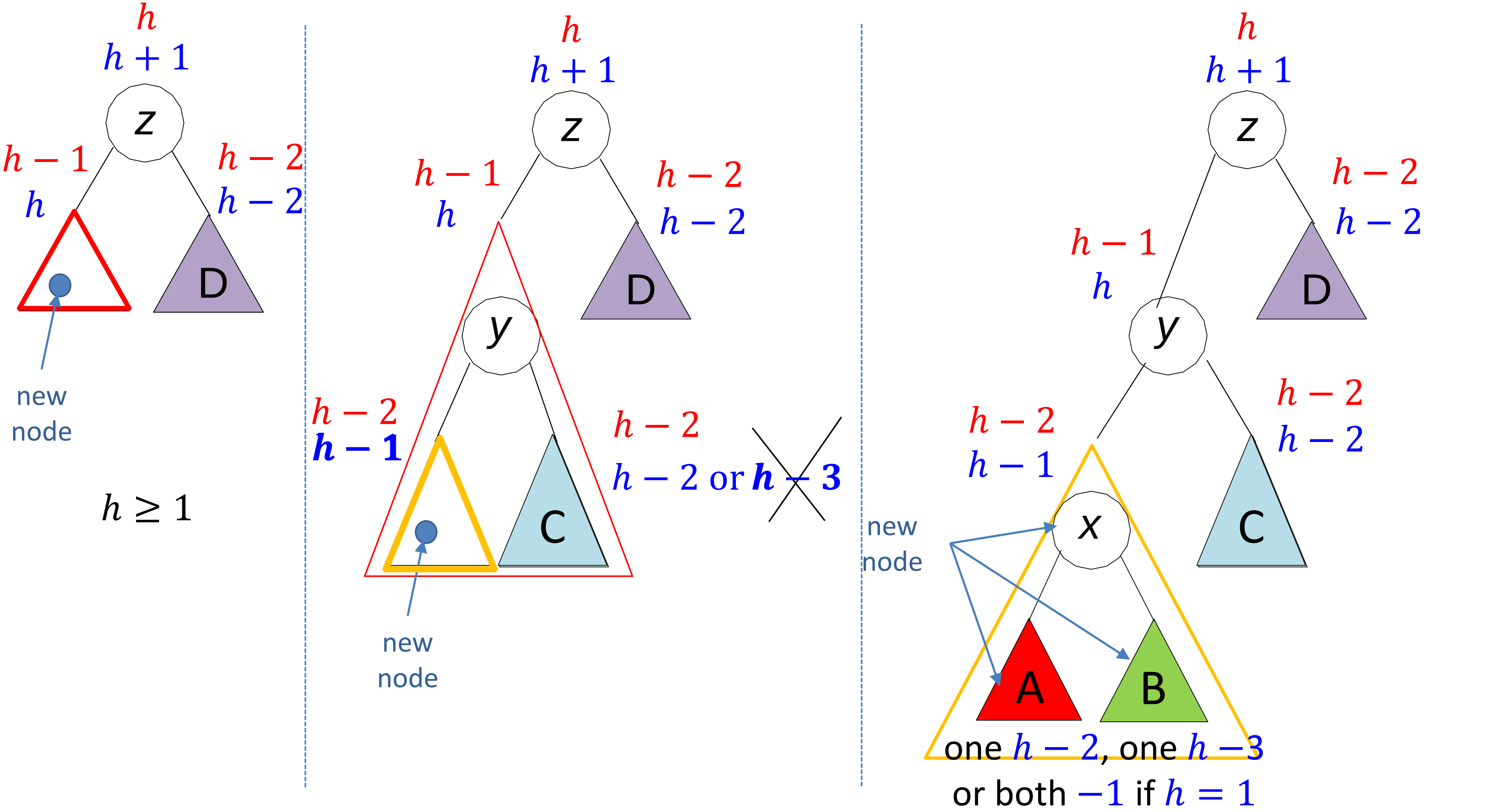
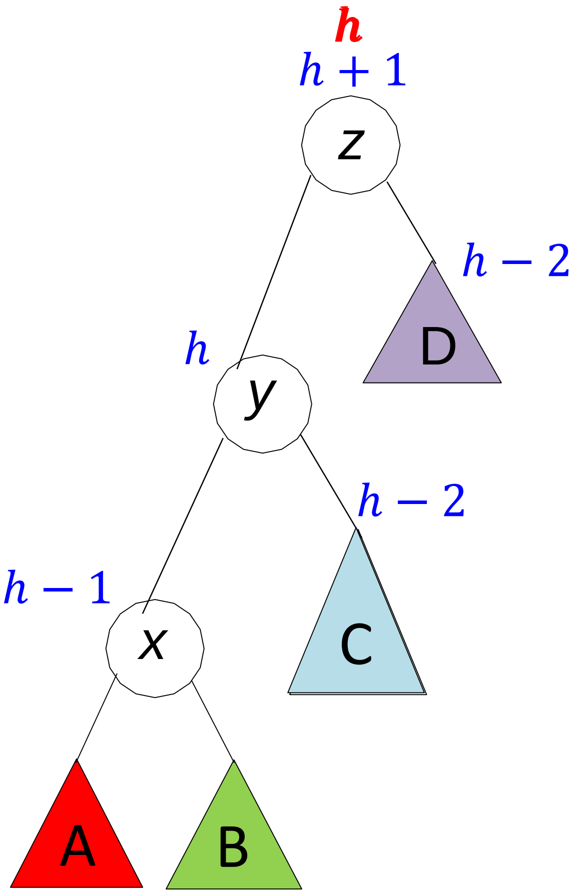
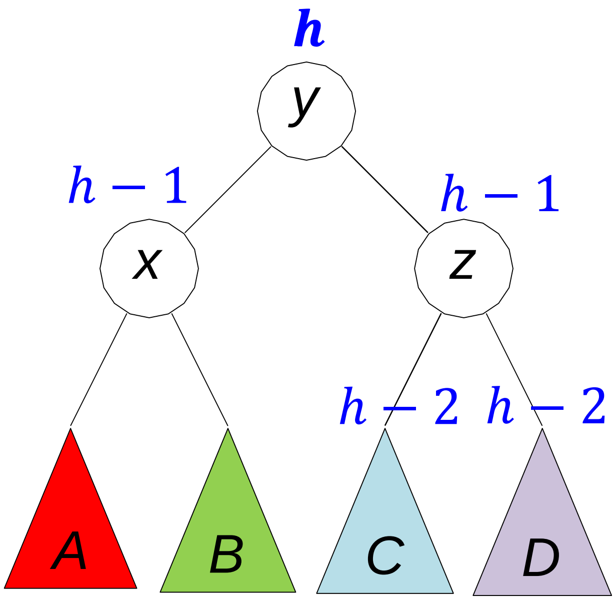


Restoring Height: Right Rotation

- Let *z* be the first unbalanced node on path from inserted node to the root
- Right rotation is used for left-left imbalance (taller left child and grandchild)

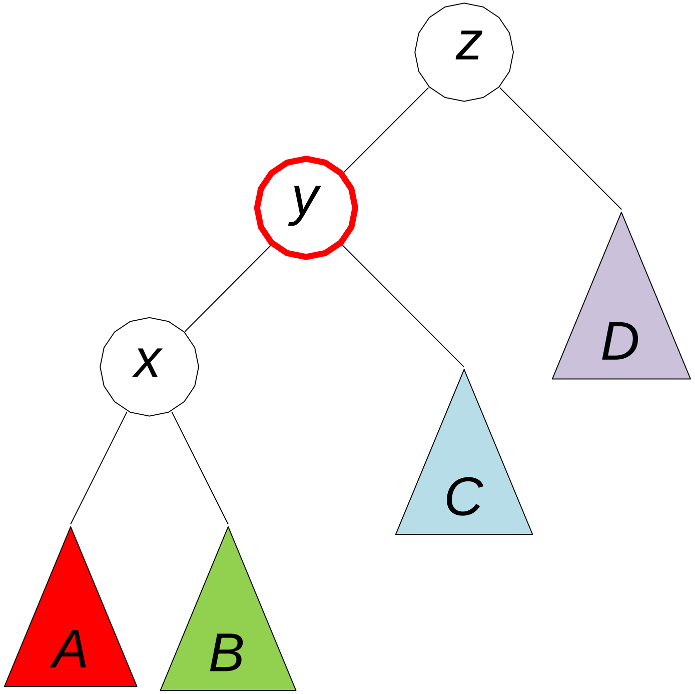




- BST order is preserved
- Balanced
- Same height as before insertion

Right Rotation Pseudocode

- Right rotation on node z



```
rotate-right($z$)
        y ← z.left, z.left ← y.right, y.right ← z
        setHeightFromChildren($z$), setHeightFromChildren($y$)
        return $y$    // returns new root of subtree
```

Right Rotation Pseudocode

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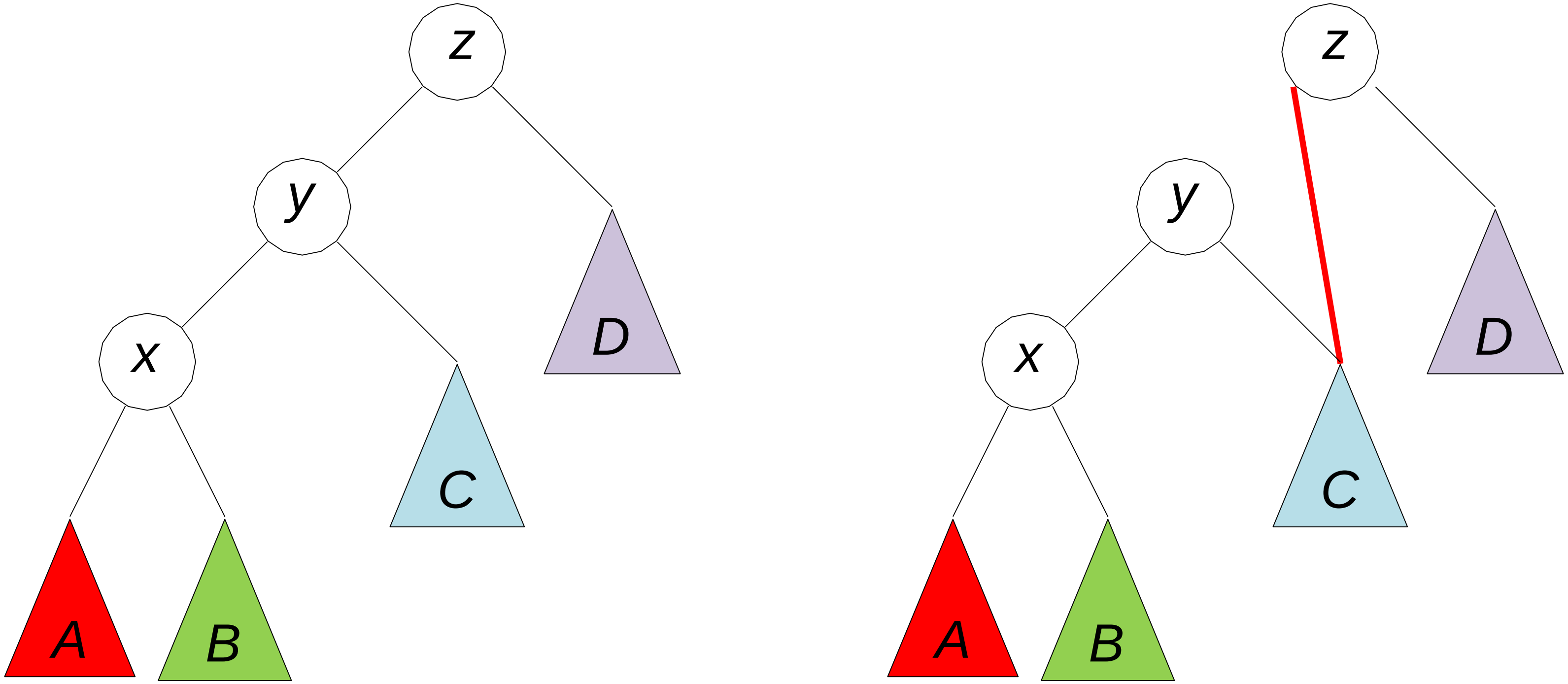

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rotate-right($z$)
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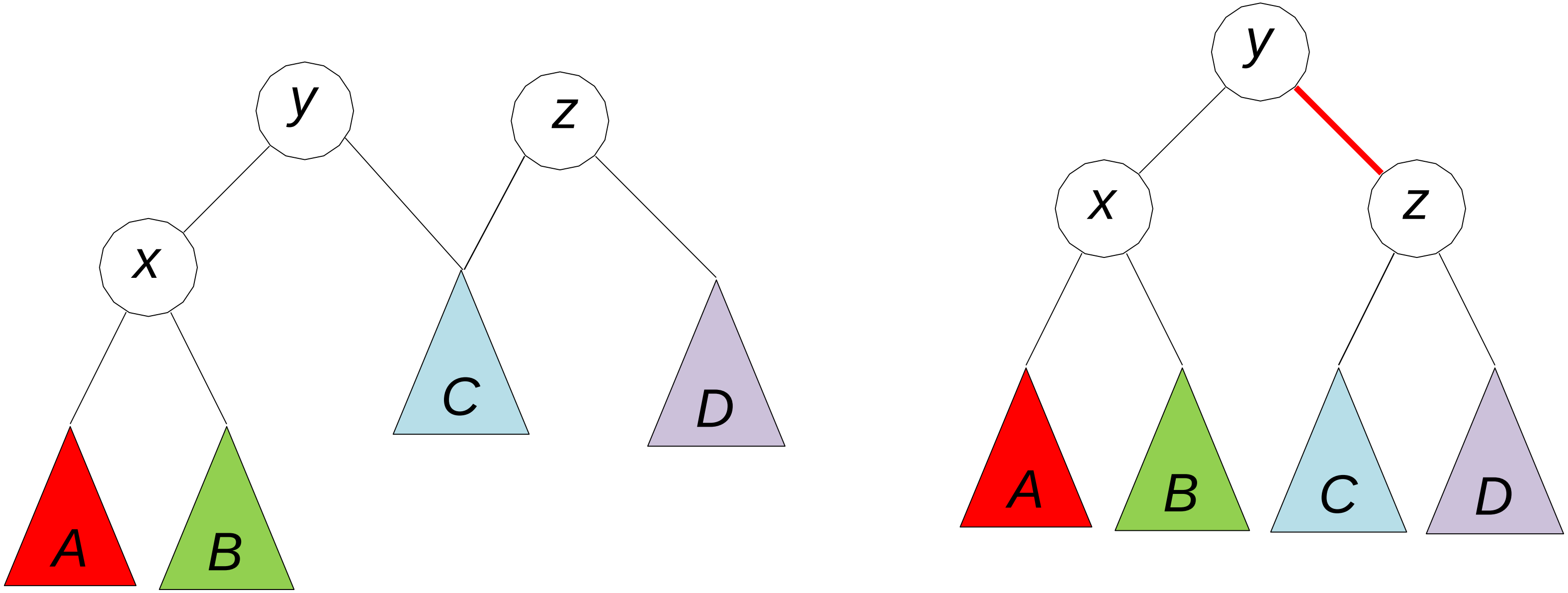

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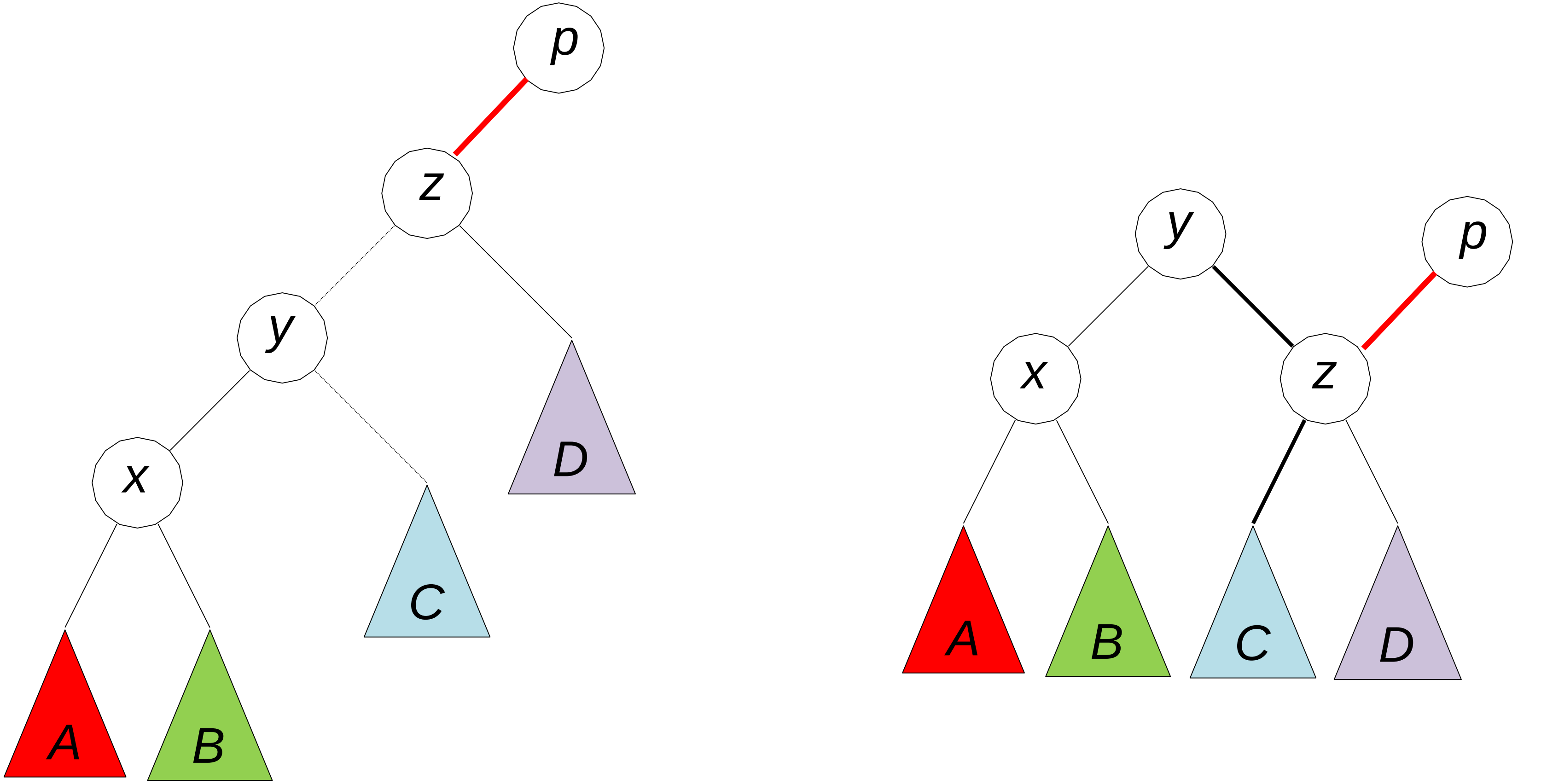
After Rotation:

- If z had a parent p, need to set y as the new child of p



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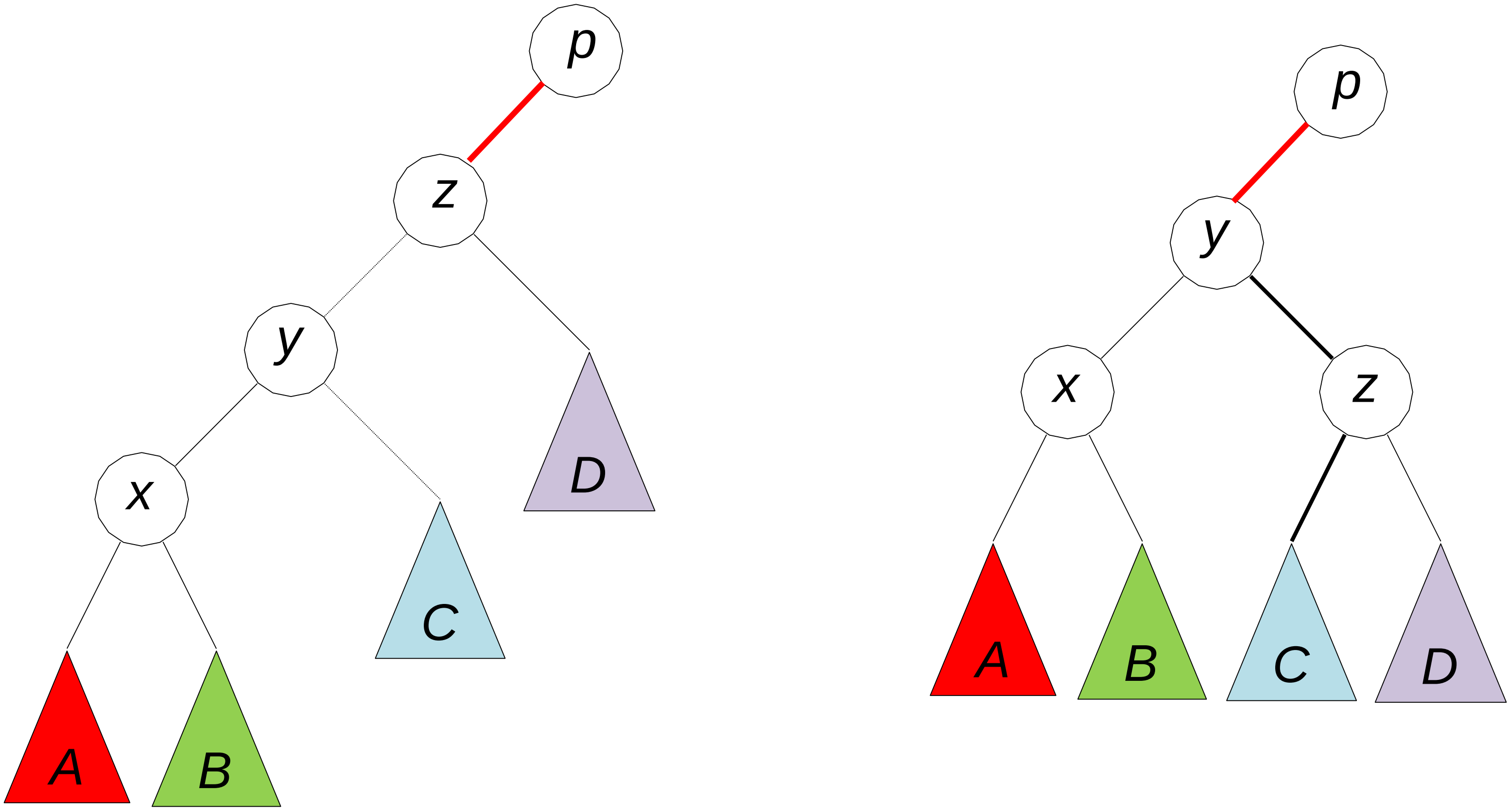
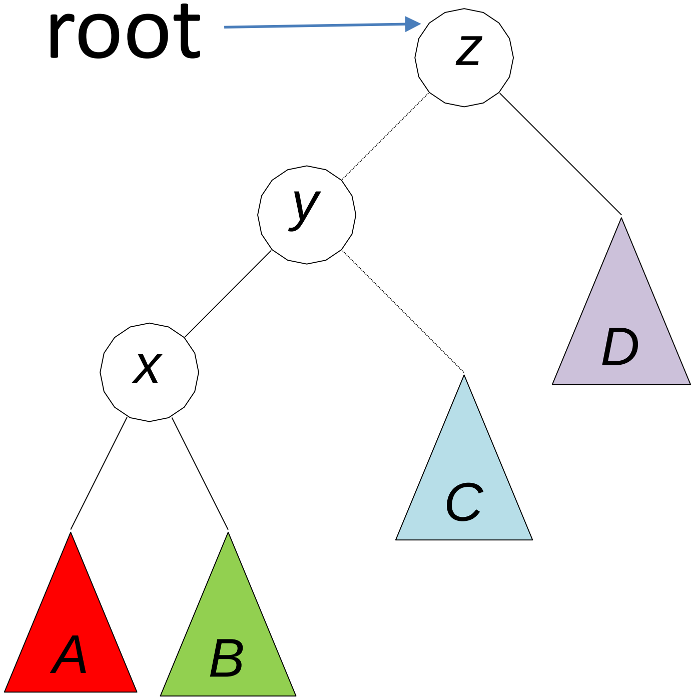
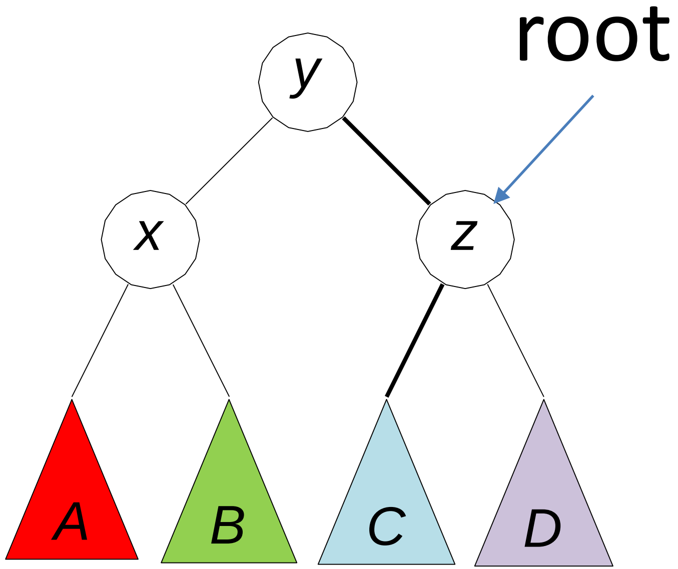


After Rotation:

- If node z was the tree root, then y becomes new tree root





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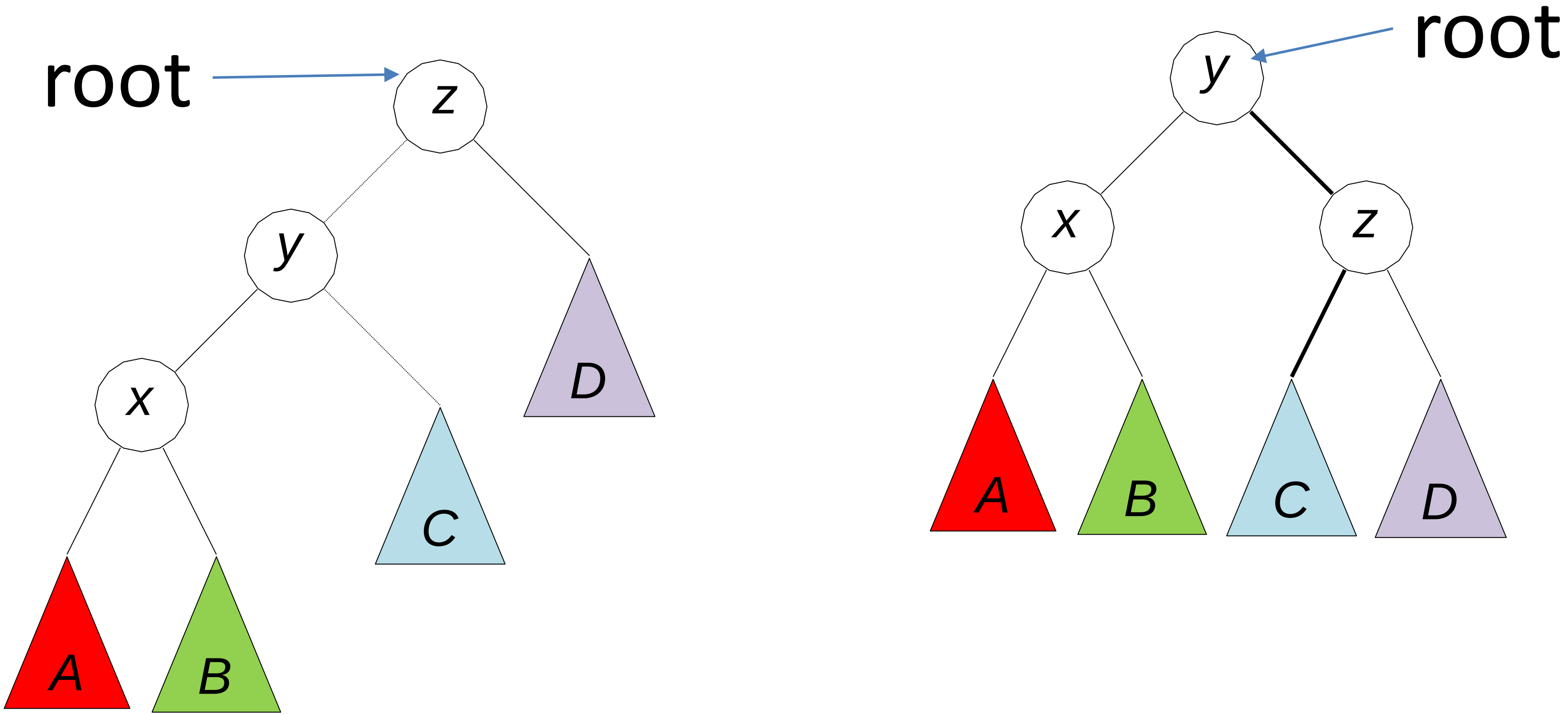
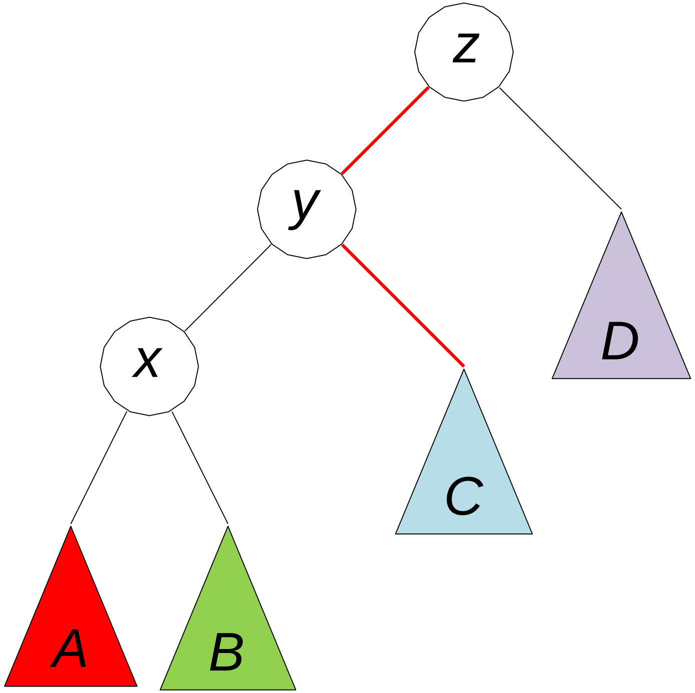


Why do we call this a rotation?



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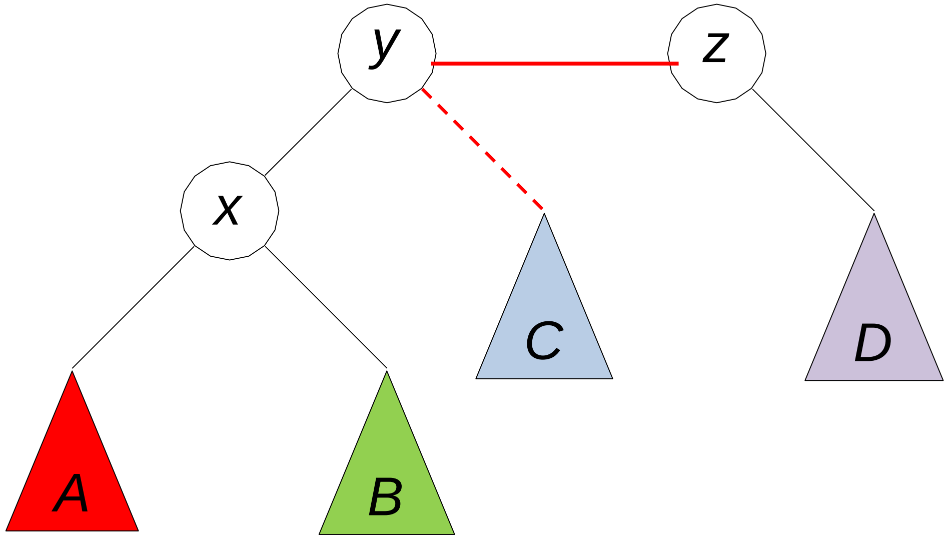
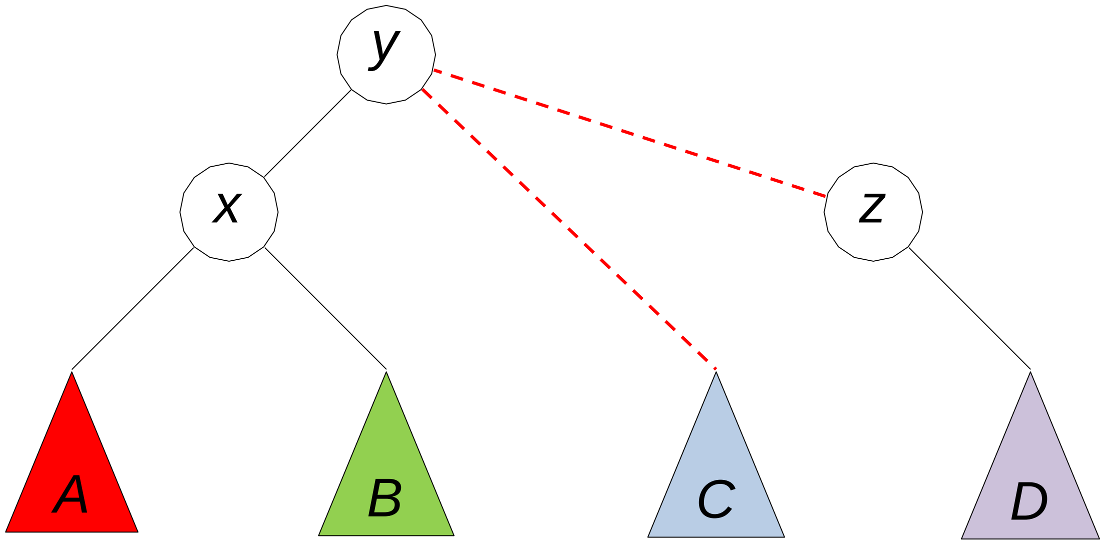
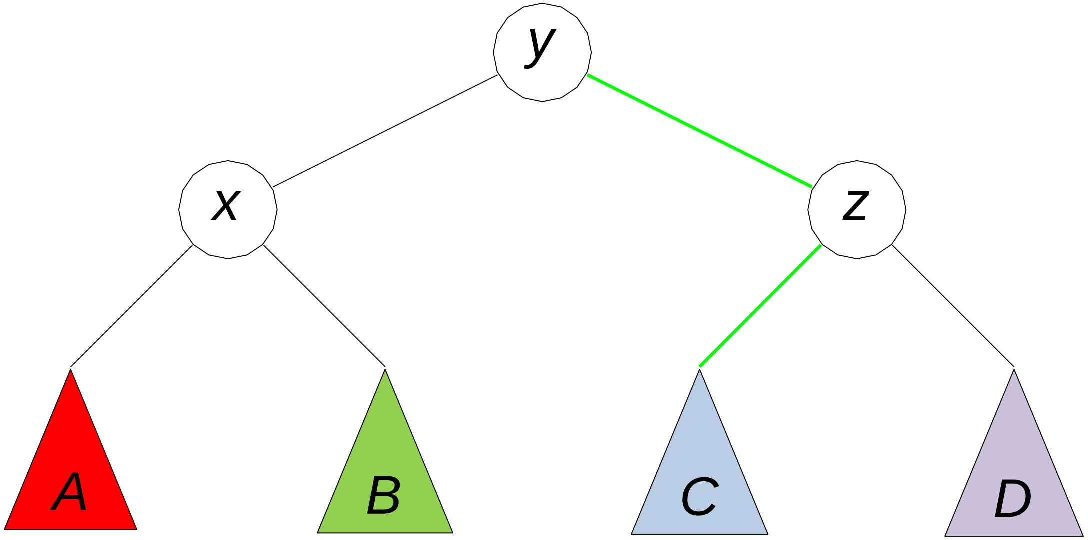


Why do we call this a rotation?

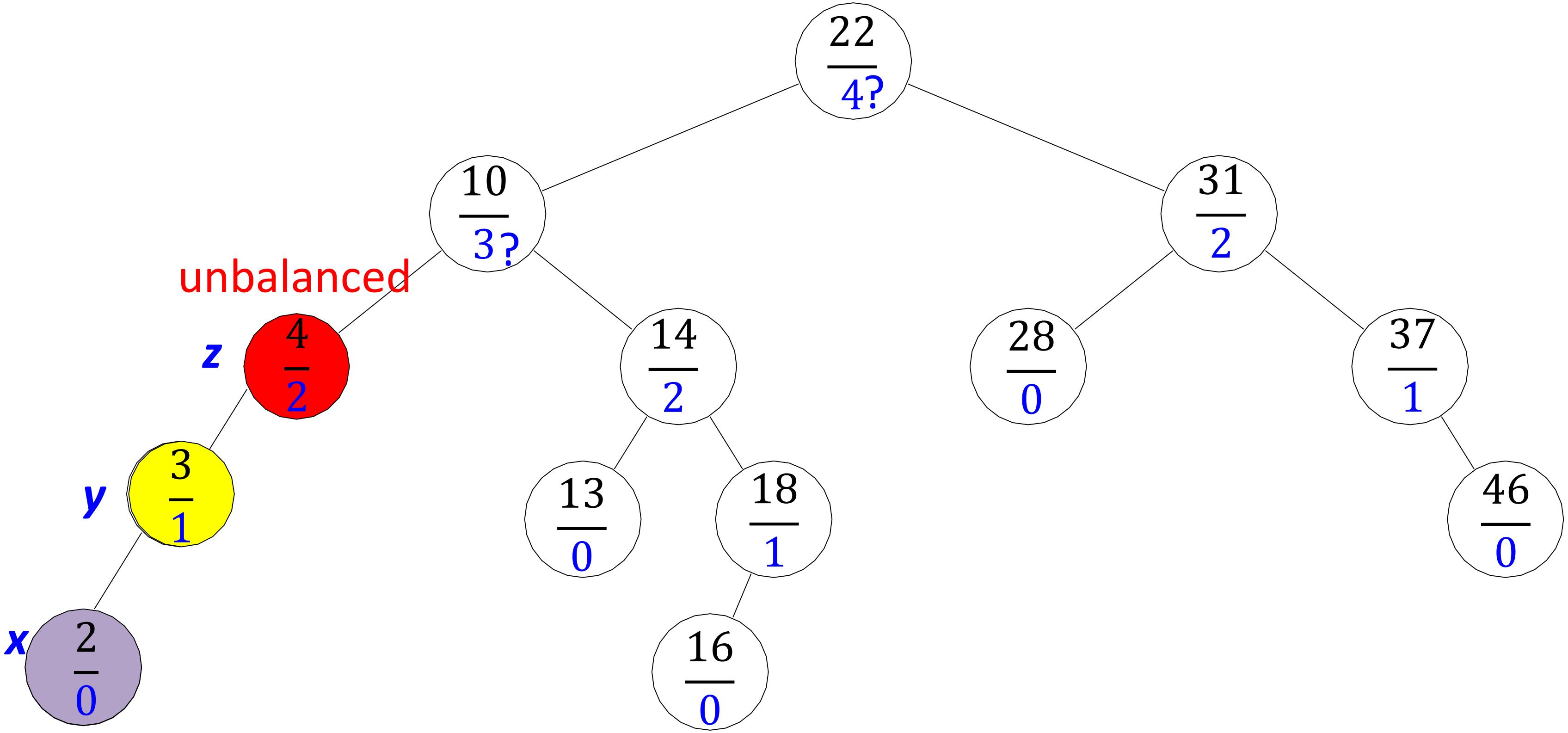


Why do we call this a rotation?



AVL Insertion Example

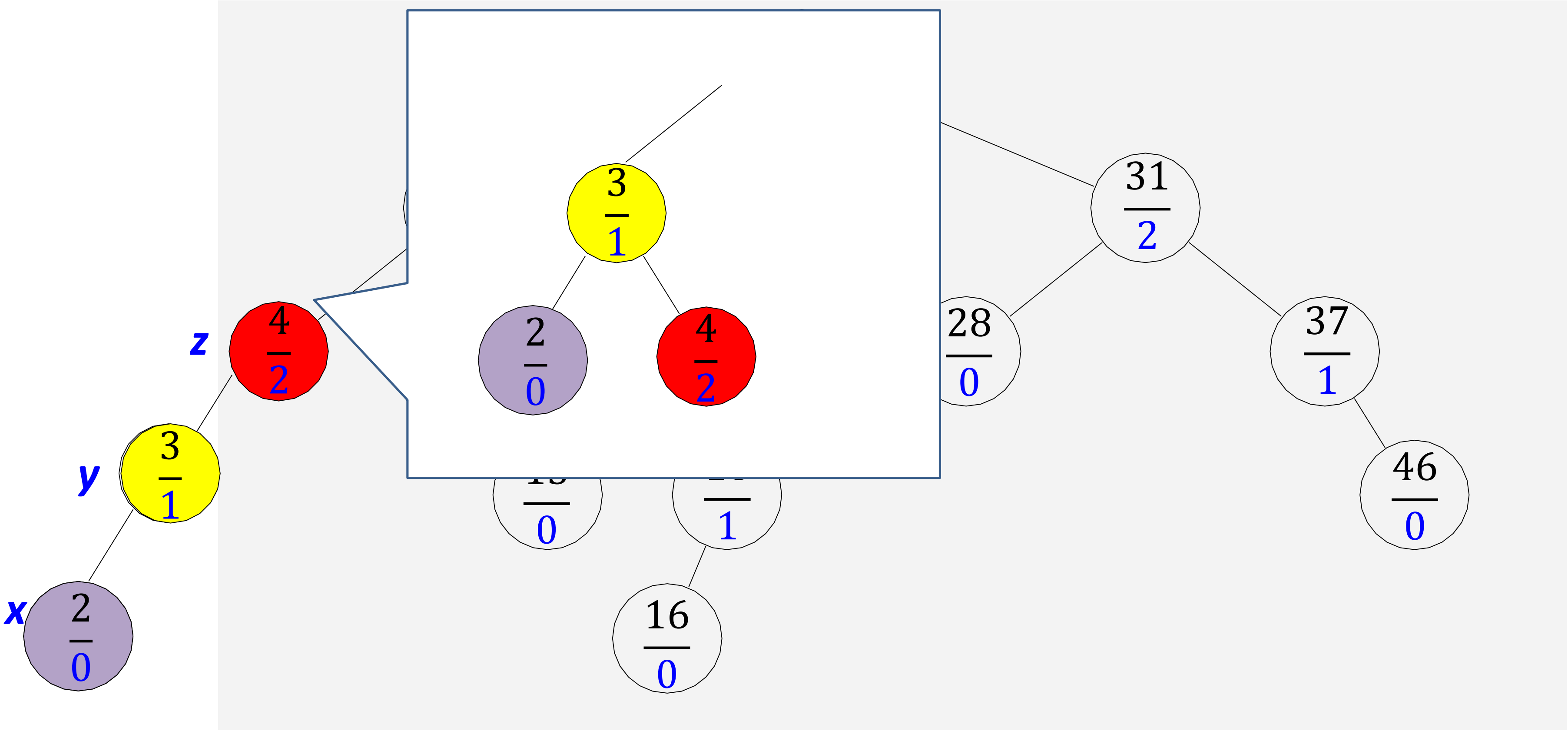
Example: *AVL::insert*(2)



- Fix with right rotation on node ***z***

AVL Insertion Example

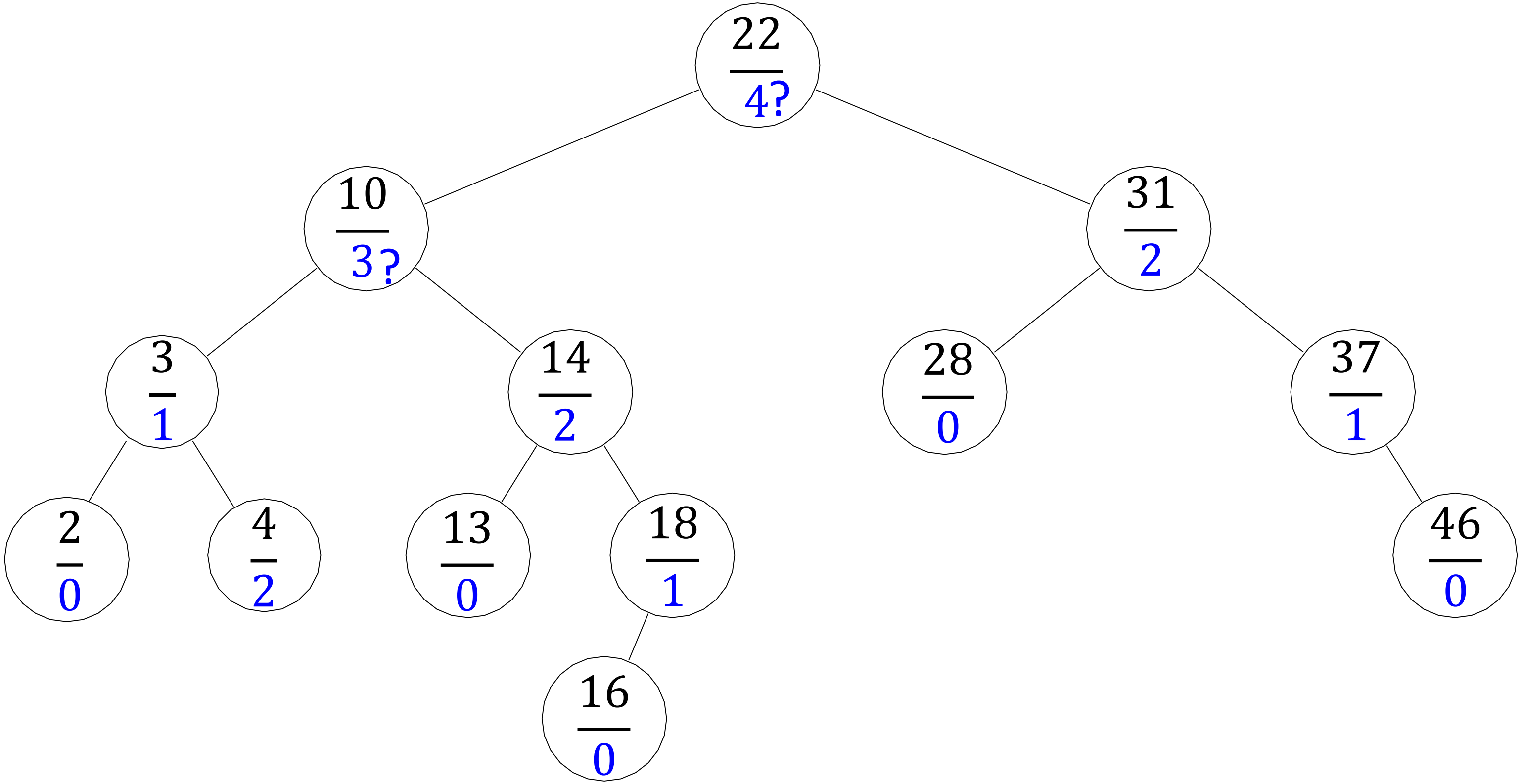
Example: *AVL::insert*(2)



- Fix with right rotation on node z

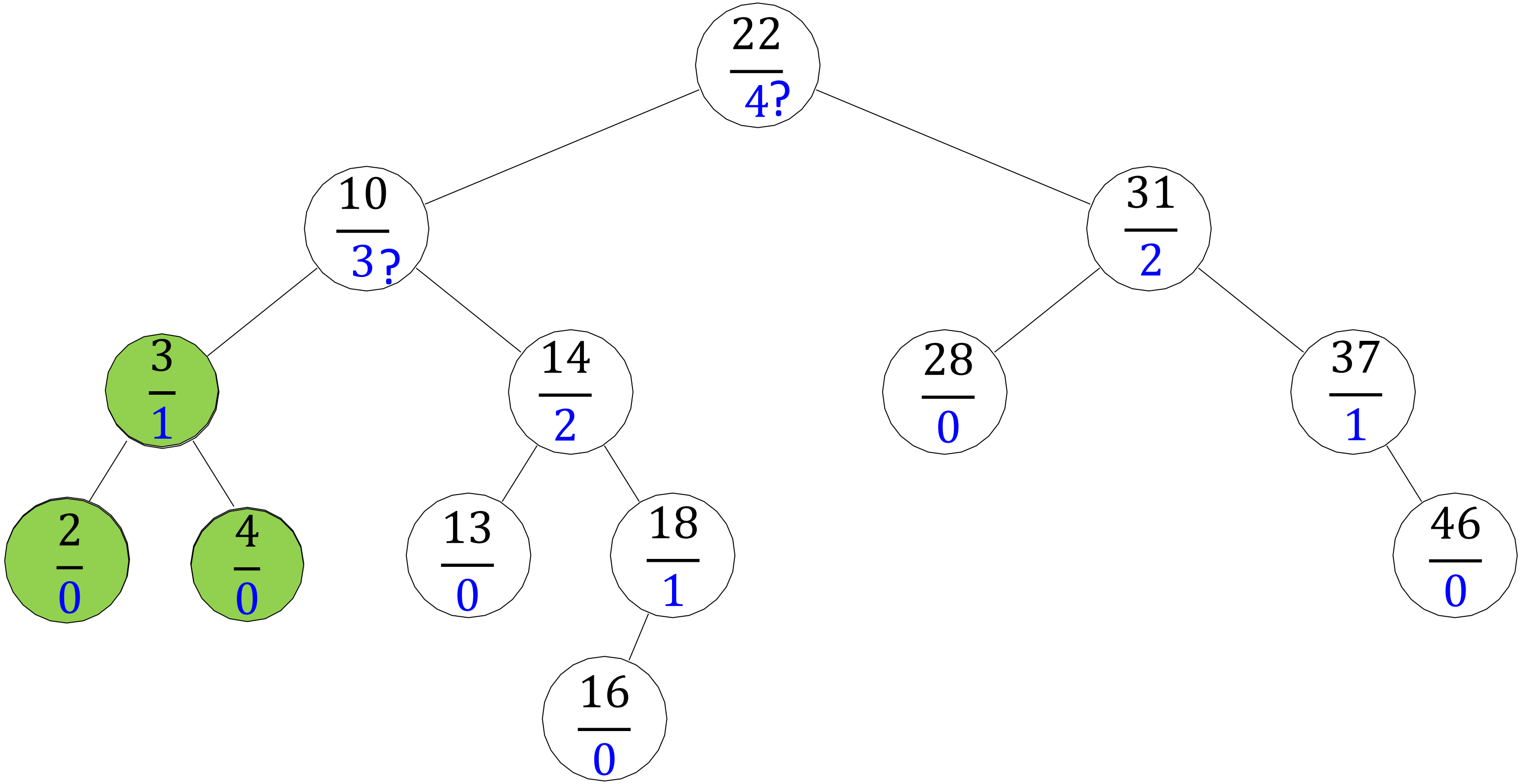
AVL Insertion Example

Example: *AVL::insert*(2)



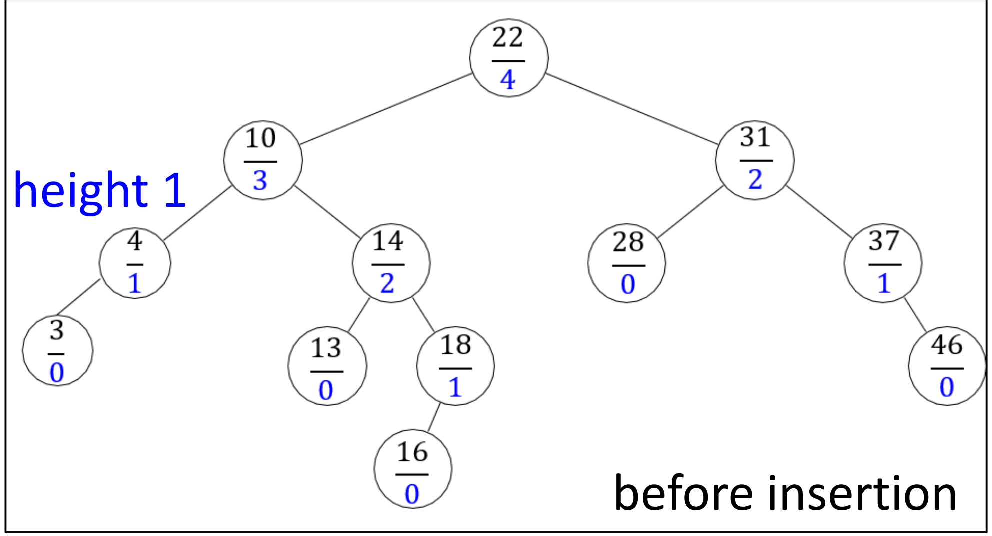
AVL Insertion Example

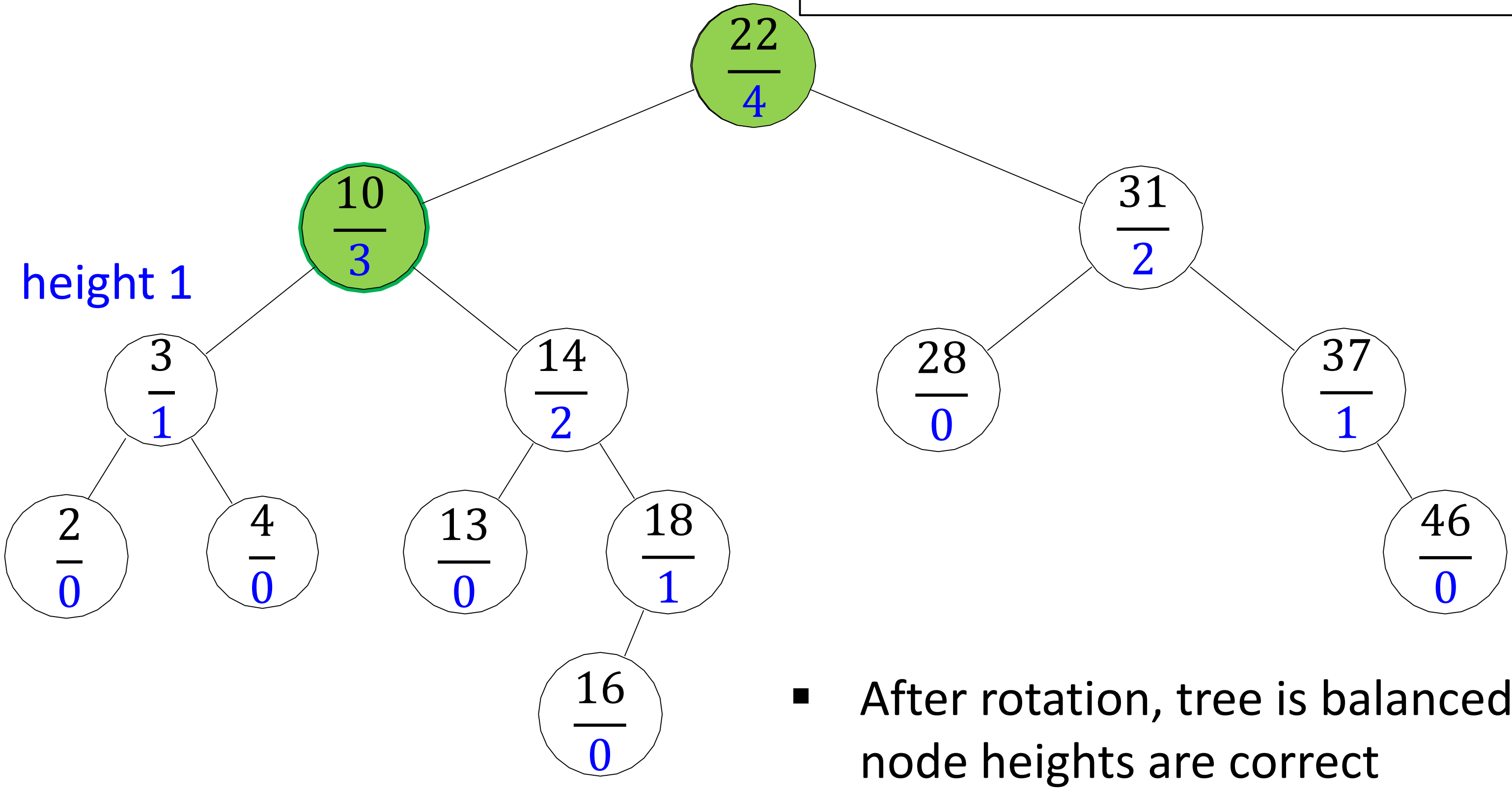
Example: *AVL::insert*(2)



AVL Insertion Example

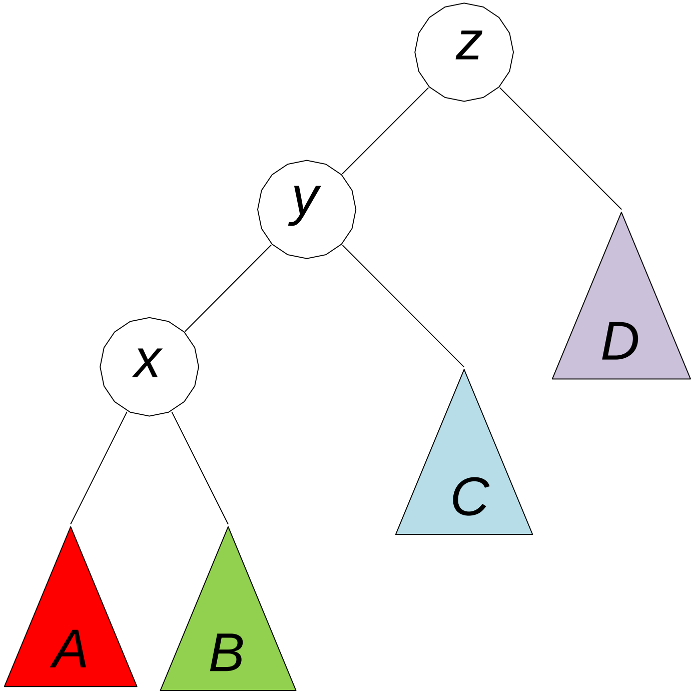
Example: *AVL::insert*(2)



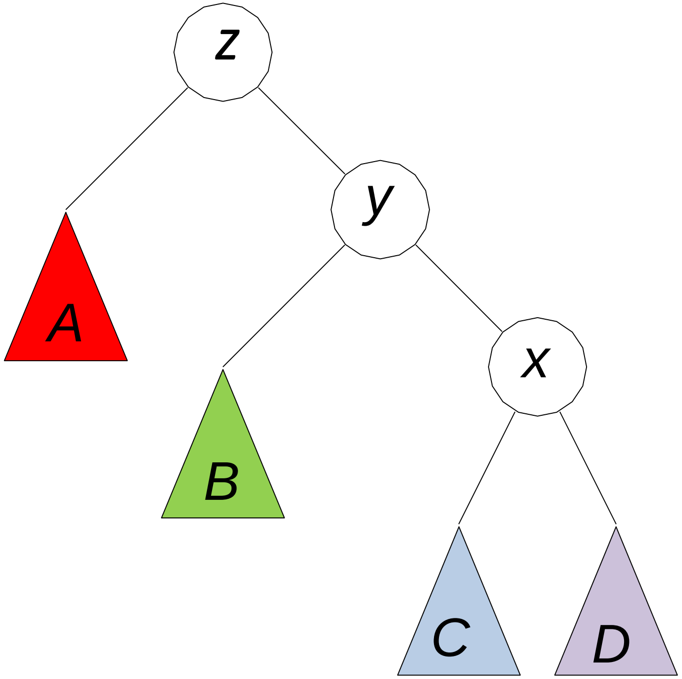


- After rotation, tree is balanced and all node heights are correct

Restoring Height Balance, Case 2



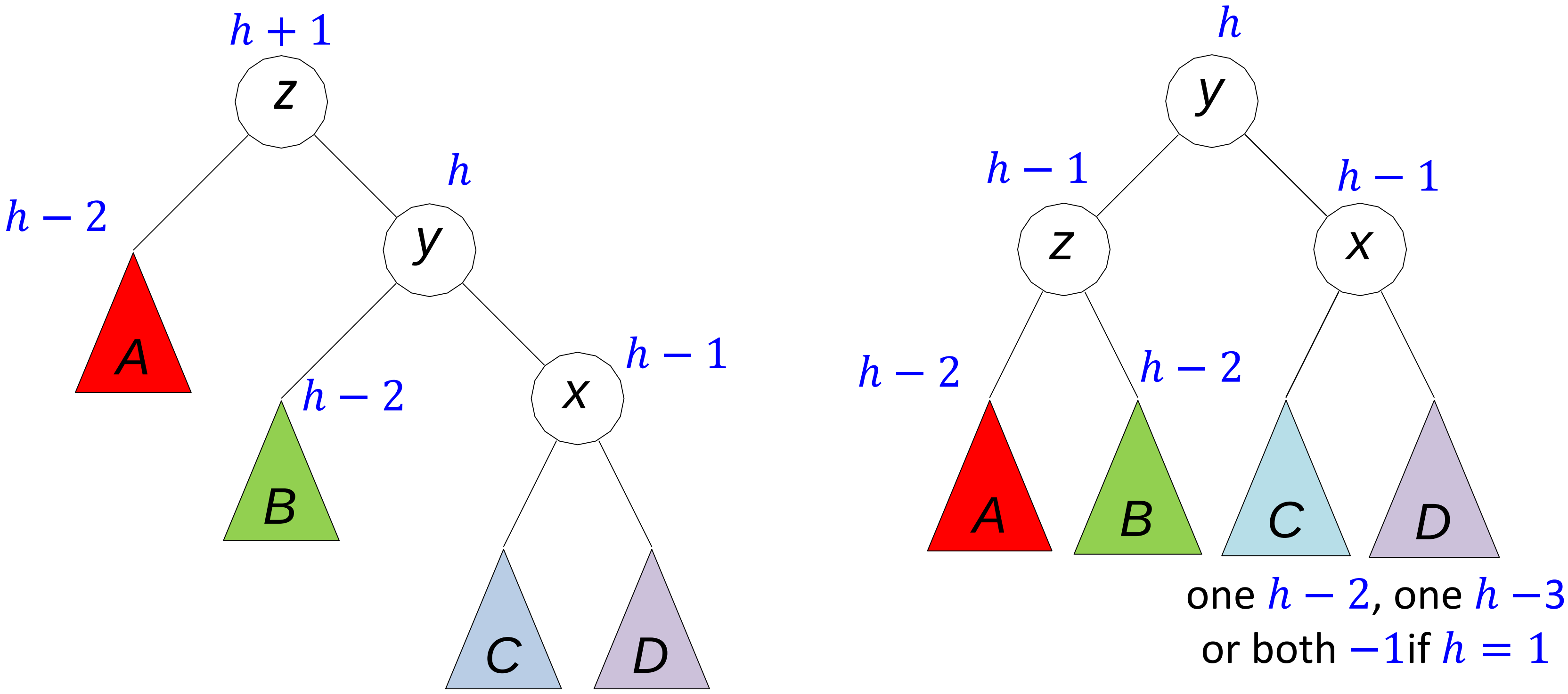
Case 1: Fixed with right rotation



Case 2: Fixed with left rotation

Left Rotation

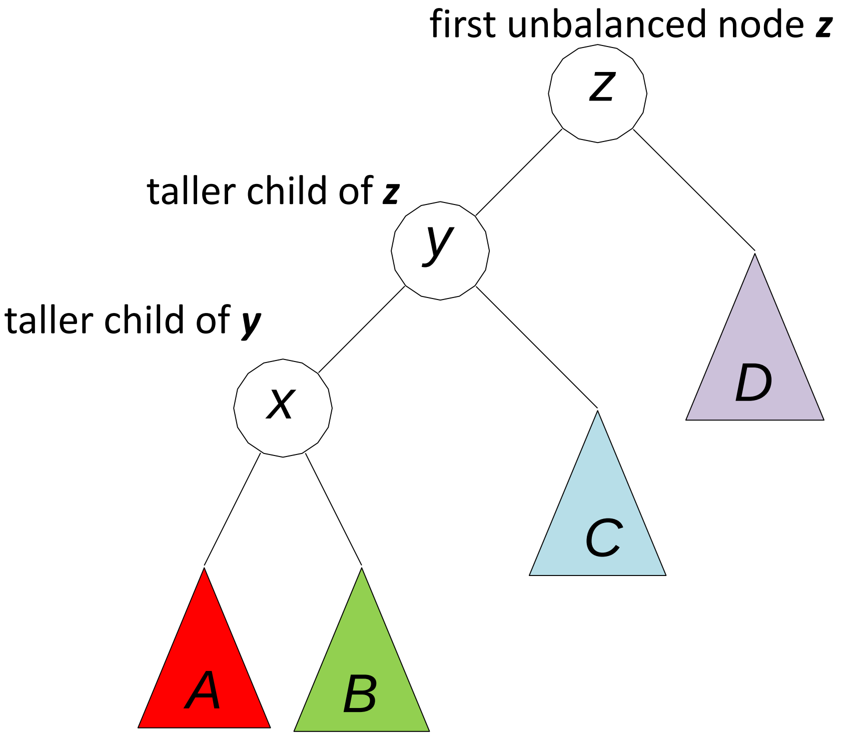
- Symmetrically, this is a *left rotation* on node *z*
- Useful to fix right-right imbalance



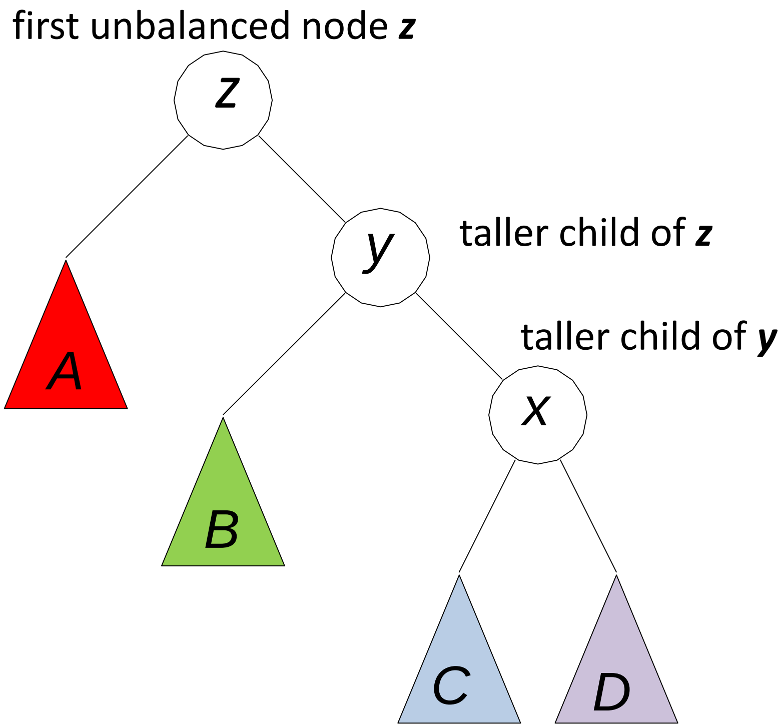
one $h-2$, one $h-3$
or both -1if $h=1$

- BST order is preserved
- Balanced
- Same height as before insertion

Distinguishing between Case 1 and Case 2



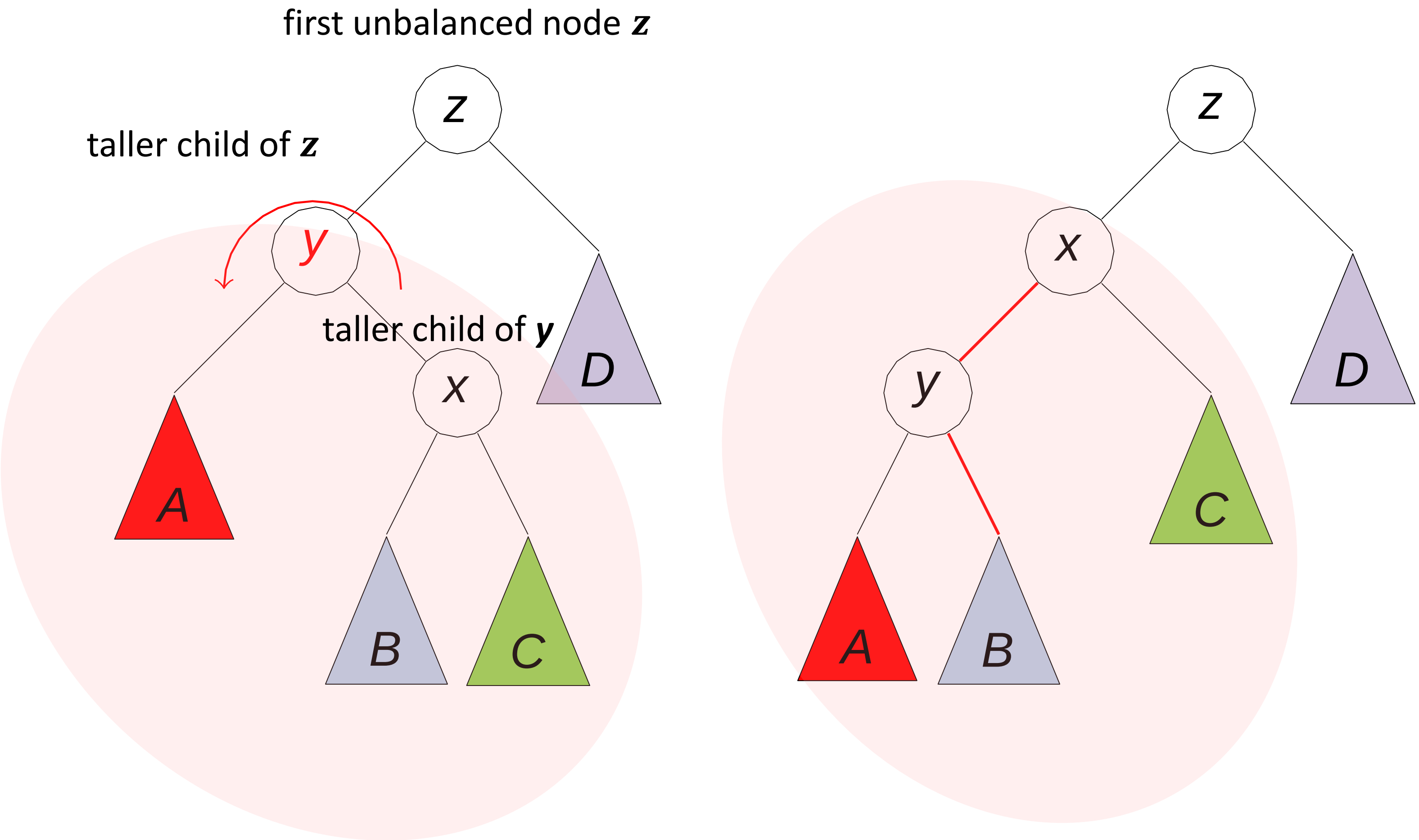
Case 1: Fixed with right rotation



Case 2: Fixed with left rotation

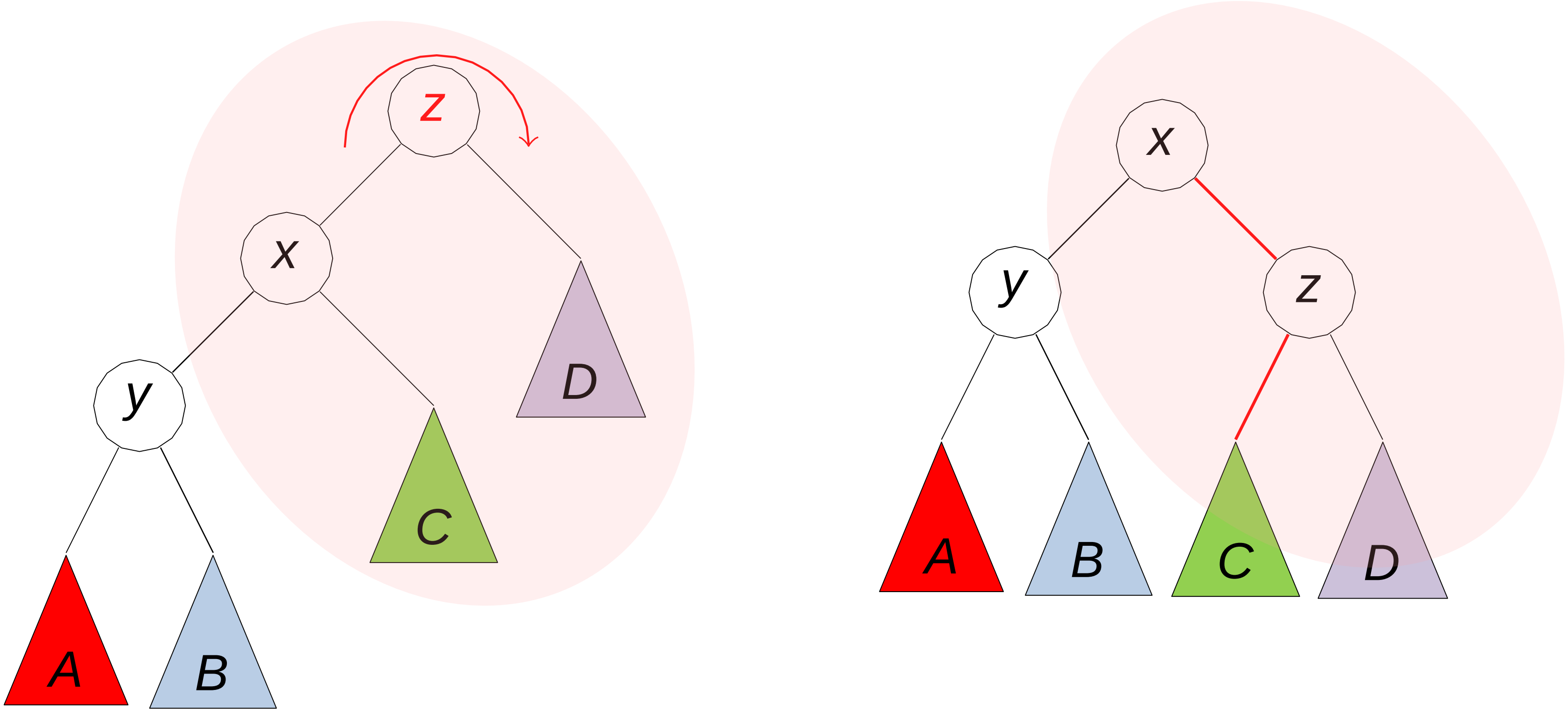
- *z* ← the first unbalanced node on path from inserted node to the root
- *y* ← taller child of *z*
- *x* ← taller child of *y*

Case 3



- Fix with double rotation on node z
  - **first, left rotation at y**

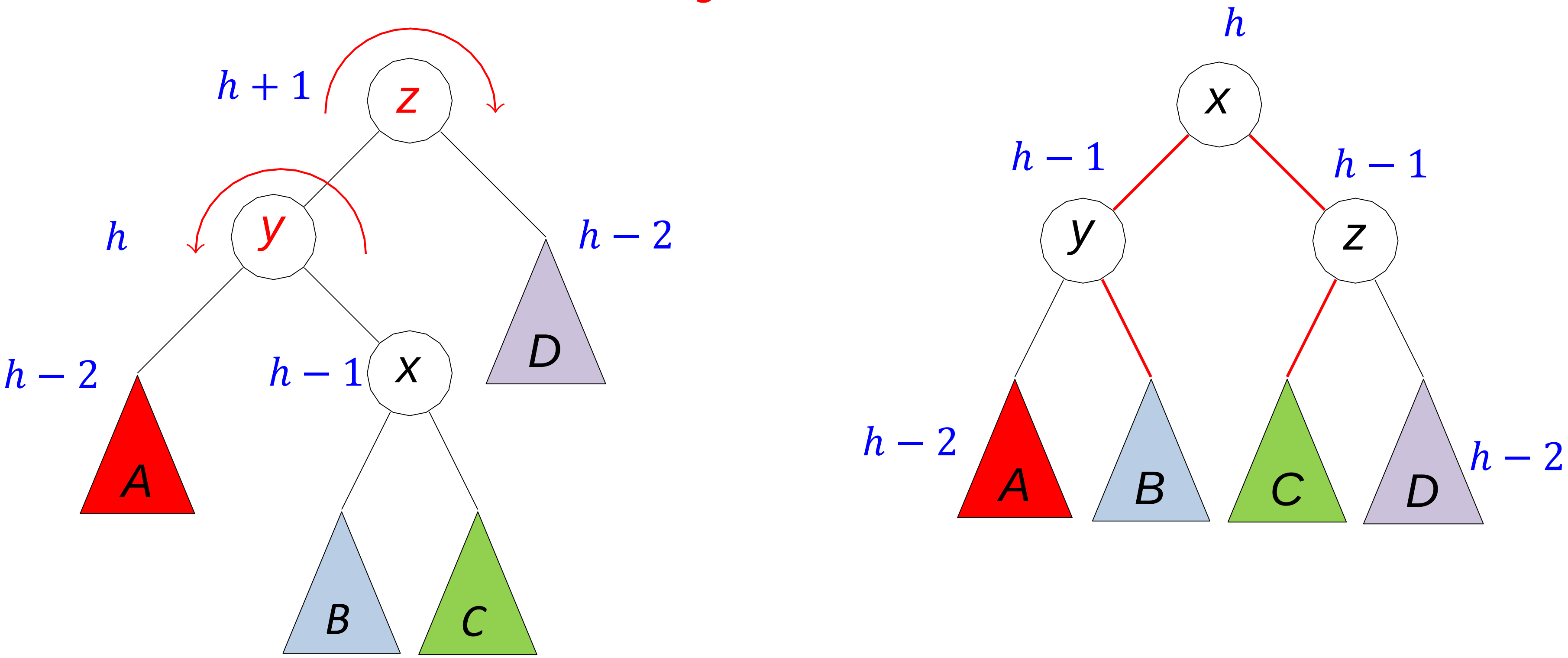
Case 3



- Fix with double rotation on node z
    - first, left rotation at y
    - **second, right rotation at z**

Case 3

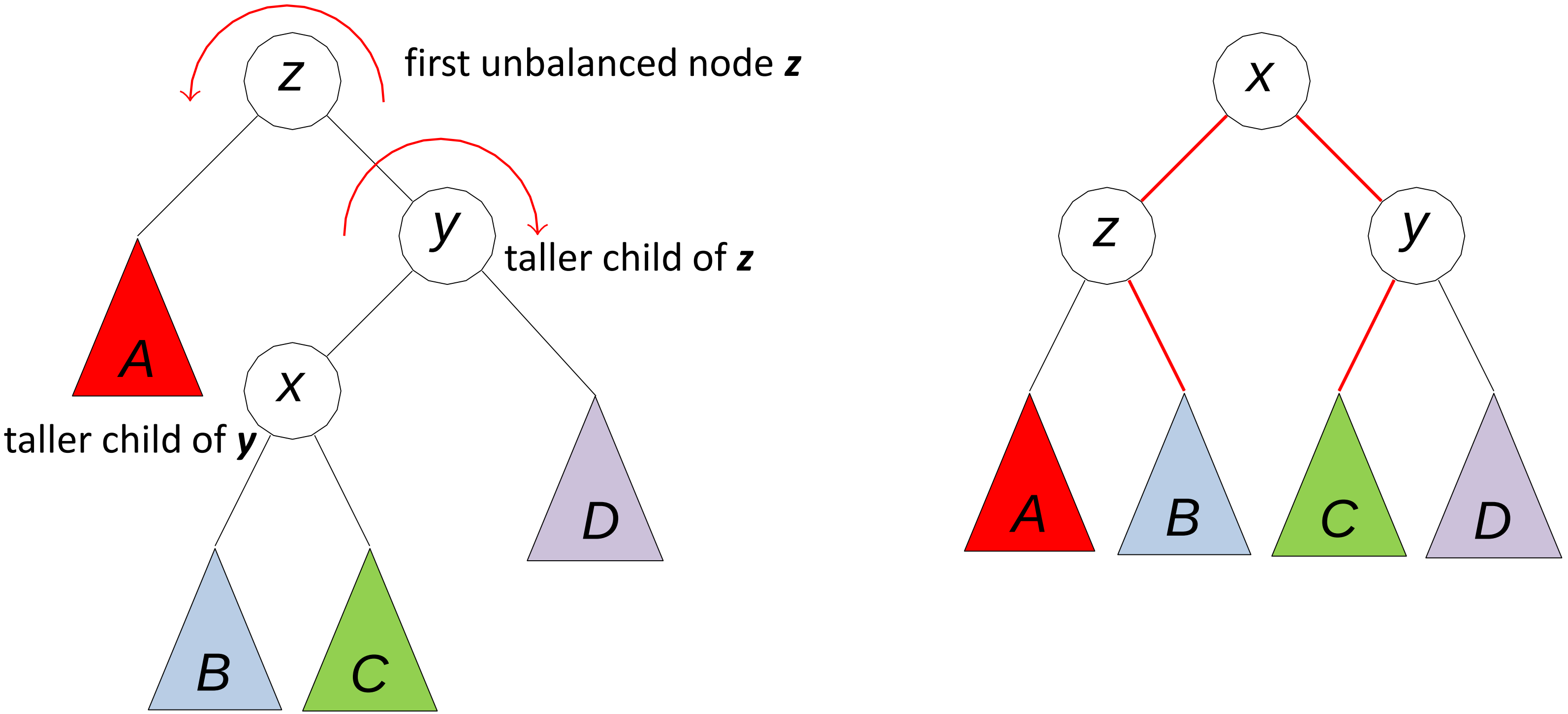
- Cumulative result of *double right rotation* on node z



- First, left rotation at y, second, right rotation at z
- BST order is preserved
- Useful for left-right imbalance
  - can argue height balance property restored as before

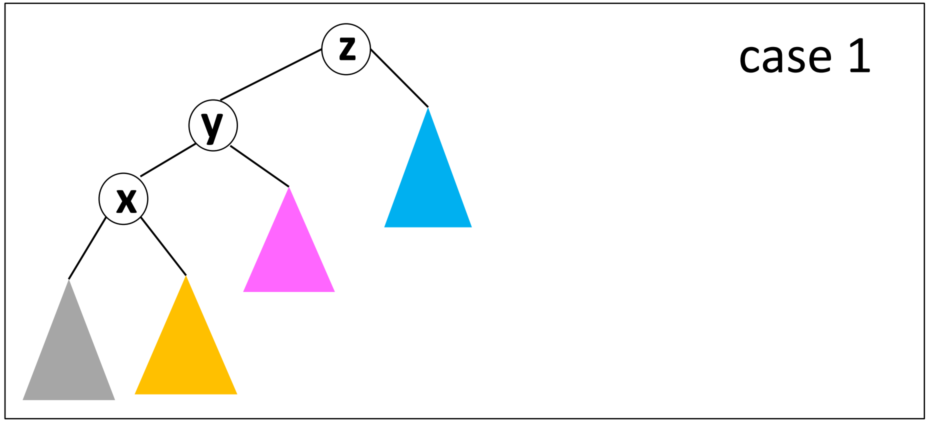
Case 4

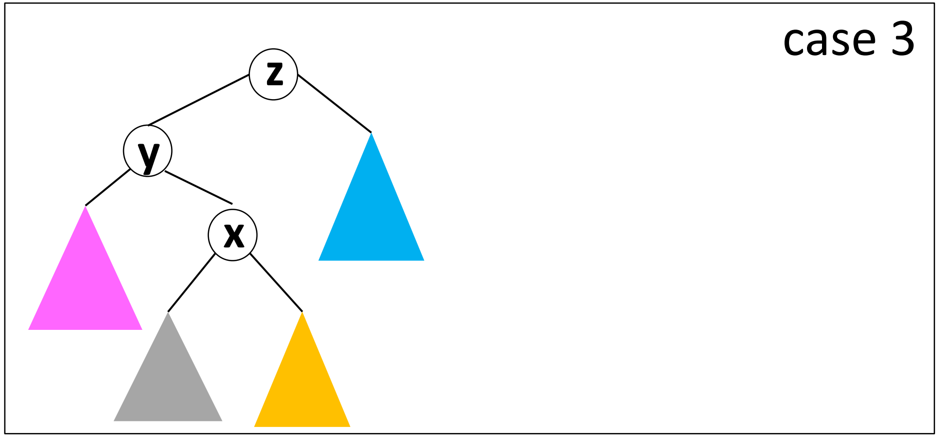
- Symmetrically, there is a *double left rotation* on node z

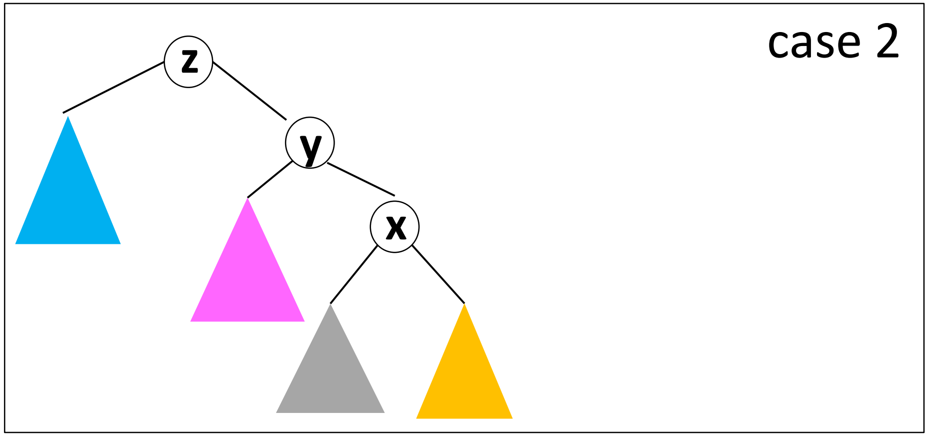


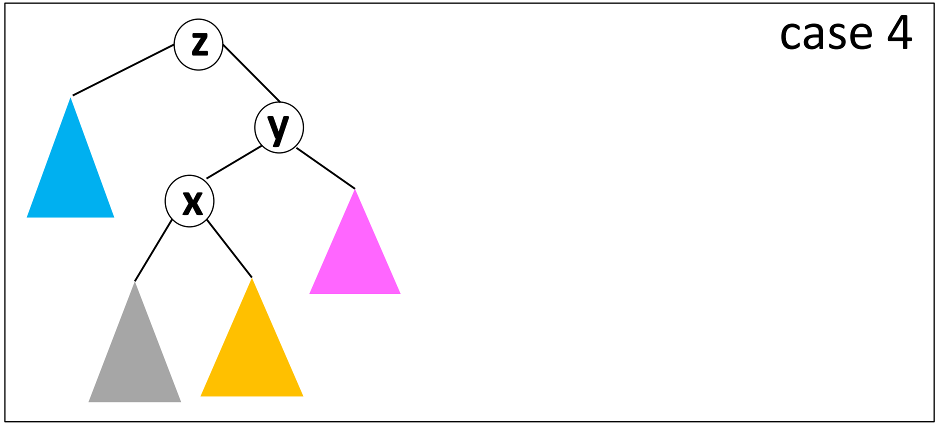
- First, a right rotation at y, second, a left rotation at z
- BST order is preserved
- Useful for right-left imbalance
  - can argue height balance property restored as before

Unbalanced Node z: all 4 cases







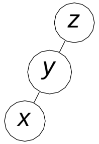


- z is the first unbalanced node on the path from inserted node to the root
- y is the taller child of z
  - z is guaranteed to have one child taller than the other
- x is the taller child of y
  - y is guaranteed to have one child taller than the other

Fixing Unbalanced AVL tree

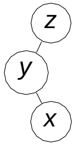
```
restructure$(x, y, z)$
$x$ : node of BST that has an unbalanced grandparent,
$y$ and $z$: the parent and grandparent of $x$
    case
```

case 1



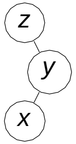
```
      : // Right rotation
        return rotate-right$(z)$
```

case 3



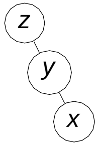
```
      : // Double-right rotation
        z.left ← rotate-left$(y)$
        return rotate-right$(z)$
```

case 4



```
      : // Double-left rotation
        z.right ← rotate-right$(y)$
        return rotate-left$(z)$
```

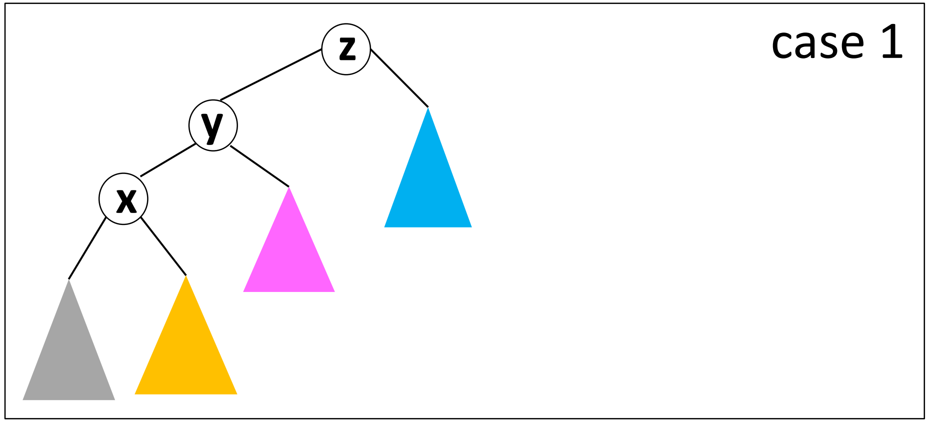
case 2

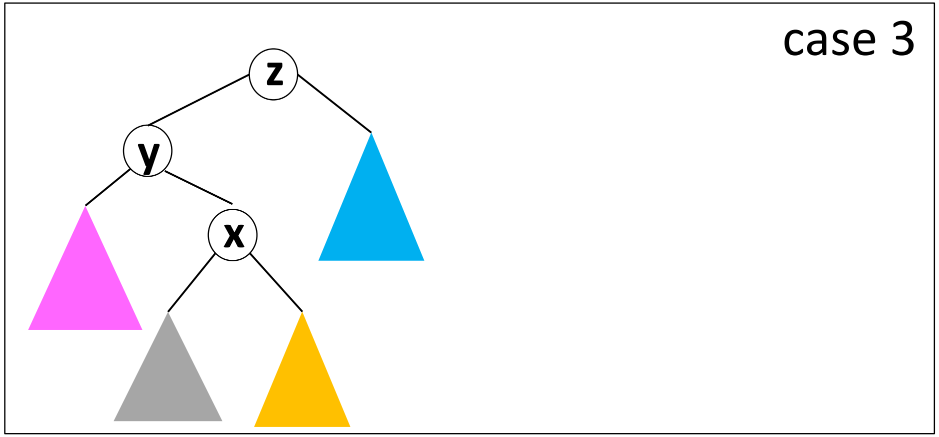


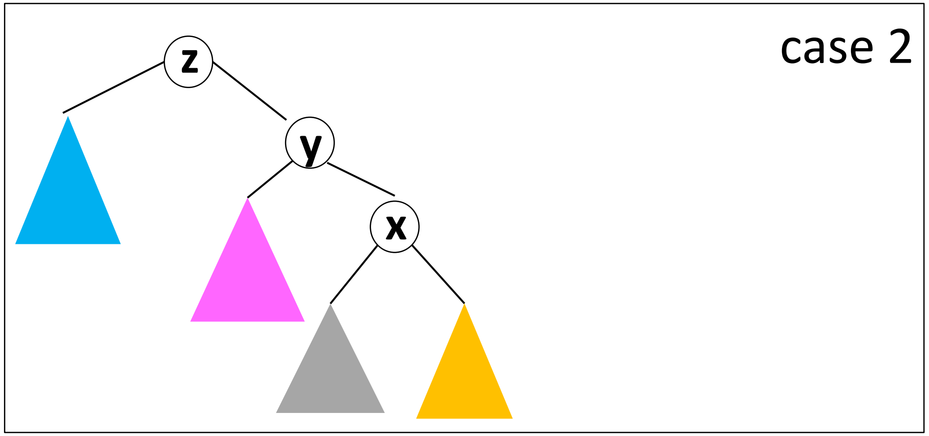
```
      : // Left rotation
        return rotate-left$(z)$
```

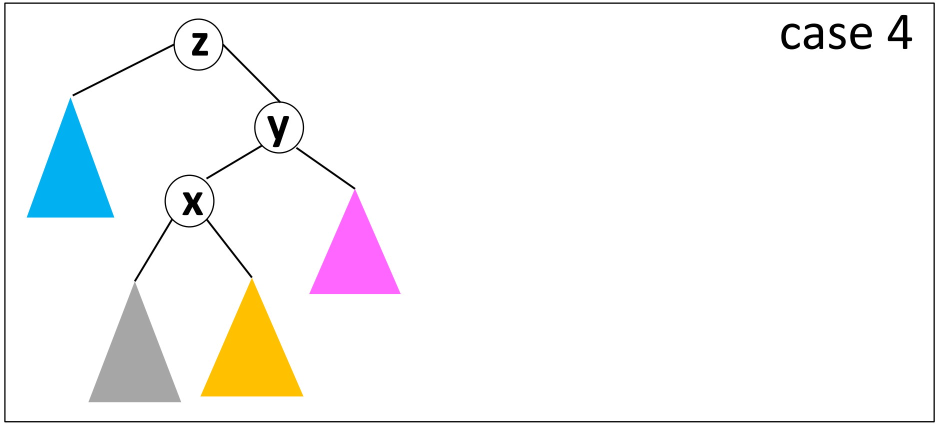
- In each case, the middle key of x, y, z becomes the new root

Tri-Node Restructuring



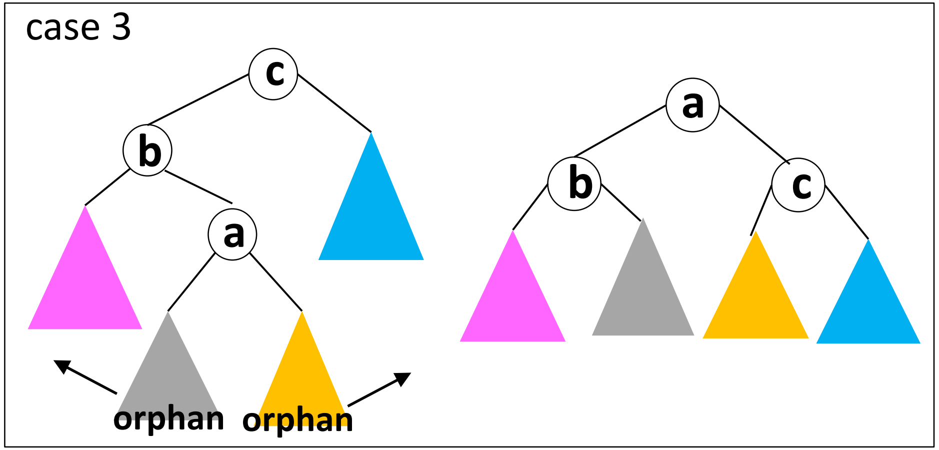


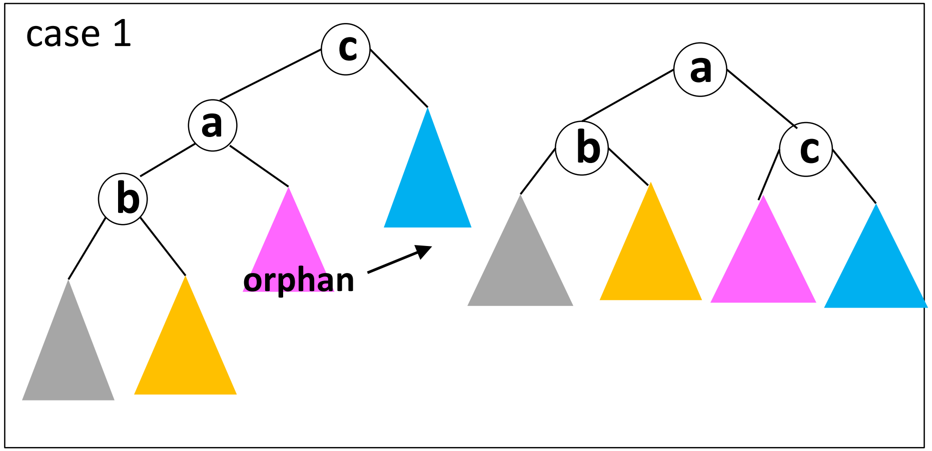




- All four cases can be handled with one method, Tri-Node restructuring

Tri-Node Restructuring





- New names
  - ***a*** = node with middle key
  - ***b*** = node with smallest key
  - ***c*** = node with largest key
- Restructure
  - **a** becomes new subtree parent
  - **b** becomes left child of **a**
  - **c** becomes right child of **a**
  - one or two subtrees of **a** get "orphaned"
    - left subtree, if orphan, becomes right child of **b**
    - right subtree, if orphan, becomes left child of **c**

Outline

- Dictionaries and Balanced Search Trees
    - Dictionary ADT
    - Review: Binary Search Trees
    - AVL Trees
        - insertion
        - restoring the AVL Property: Rotations
        - full code for insertion
        - deletion

AVL insertion

```
AVL::insert$(k, v)$
        $z \leftarrow$ BST::insert$(k, v)$
        $z.height \leftarrow 0$
        while ($z$ is not NIL)
            $z \leftarrow$ parent of $z$
            if ($|z.left.height - z.right.height| > 1$) then
                let $y$ be tallest child of $z$
                let $x$ be tallest child of $y$
                $z \leftarrow$ restructure$(x, y, z)$
                break                    // done after one restructure
            setHeightFromSubtrees$(z)$
```

```
setHeightFromSubtrees$(u)$
    if $u$ is not an empty subtree
        $u.height \leftarrow 1 + \max\{u.left.height, u.right.height\}$
```

Outline

- Dictionaries and Balanced Search Trees
  - Dictionary ADT
  - Review: Binary Search Trees
  - AVL Trees
    - insertion
    - restoring the AVL Property: Rotations
    - full code for insertion
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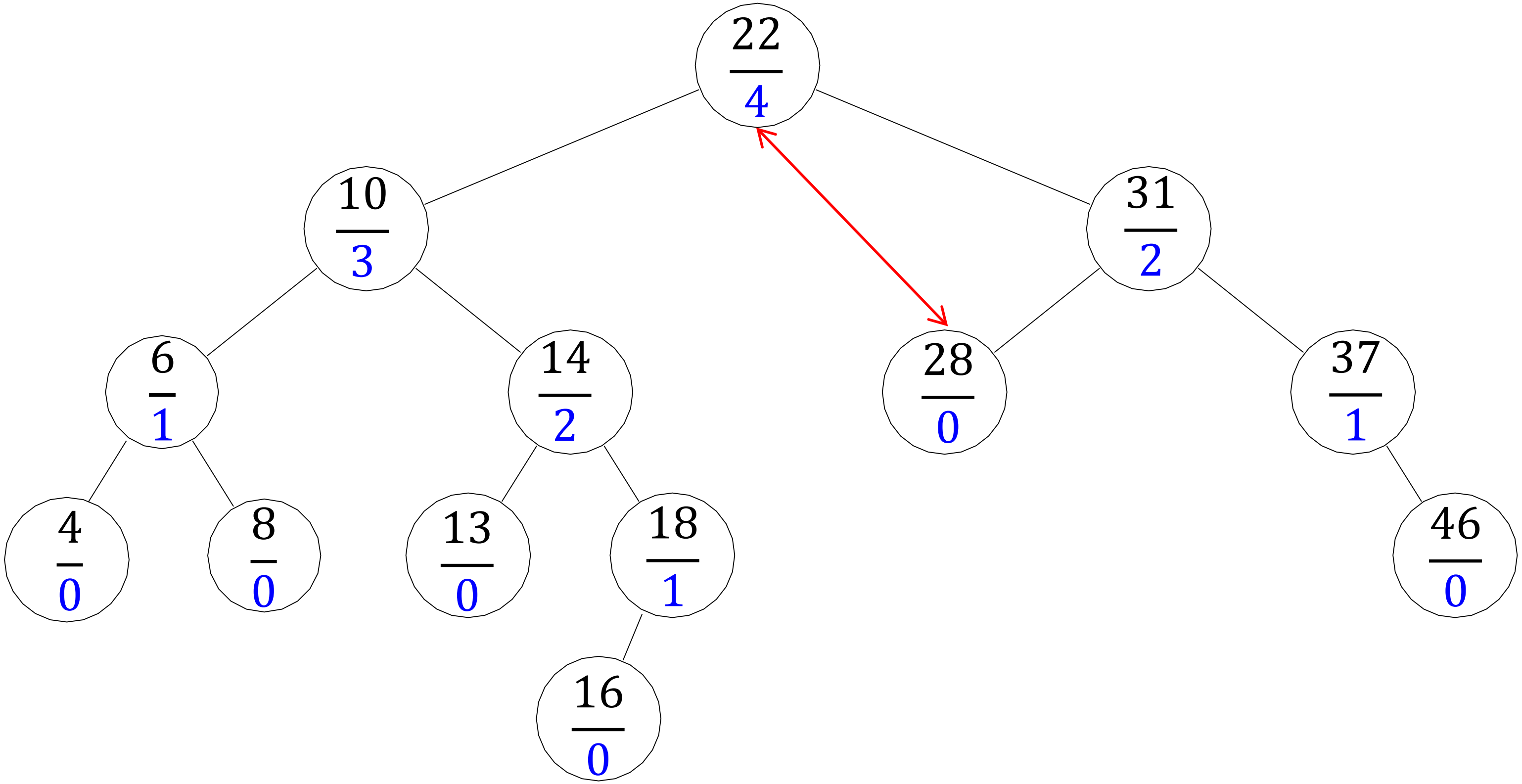
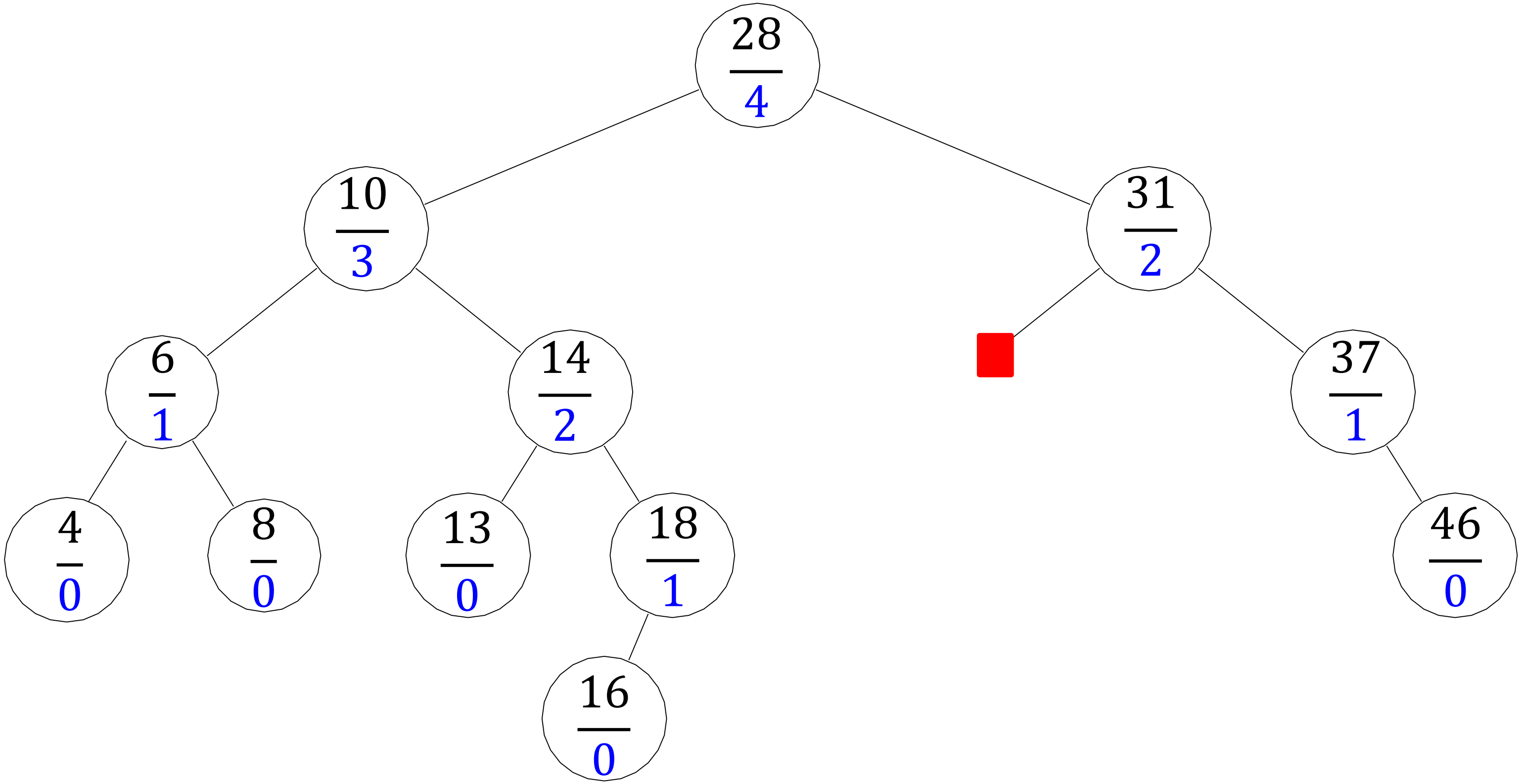
AVL Deletion Example

Example: *AVL::delete*(22)



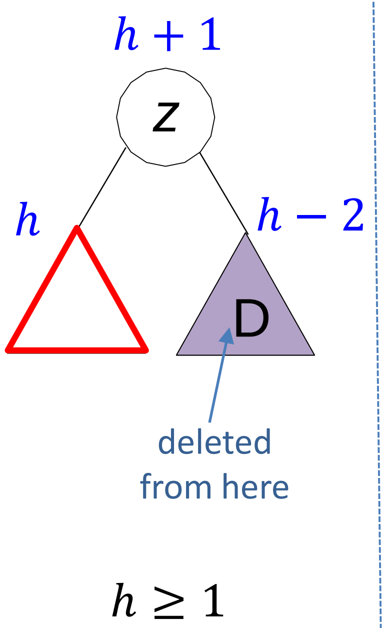
AVL Deletion Example

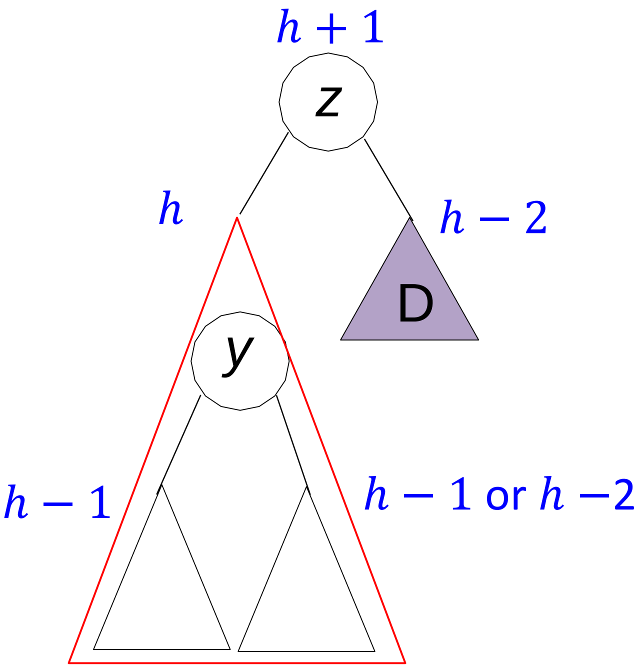
Example: *AVL::delete*(22)

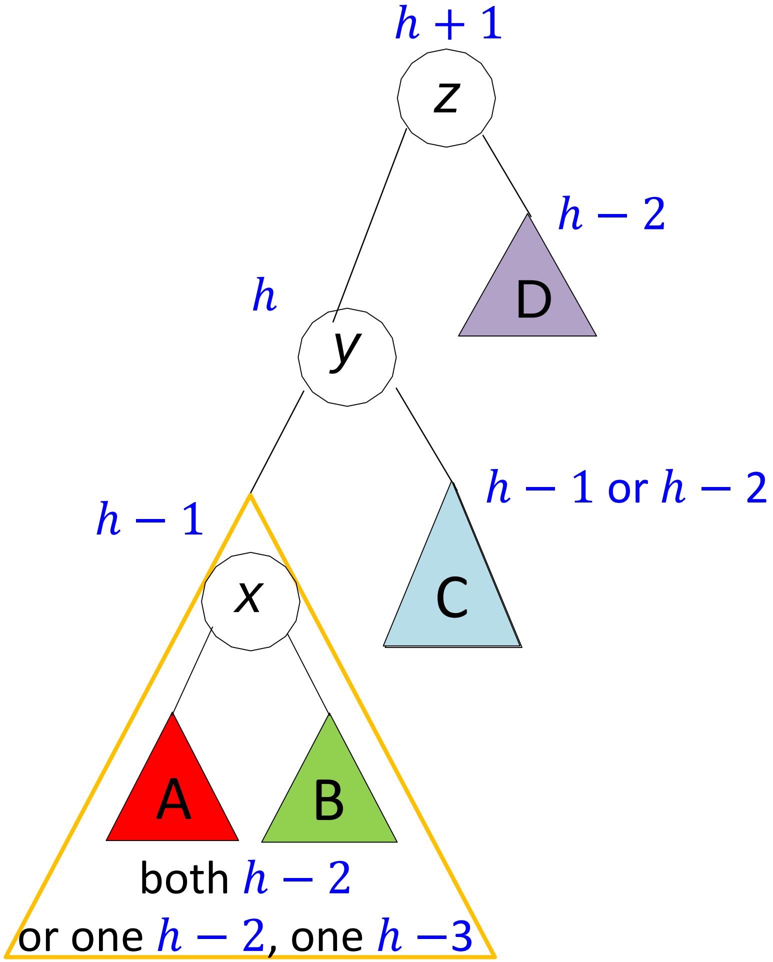


Restoring Height After Deletion: Case 1

- Let *z* be *the first* unbalanced node on path from the parent of deleted node to the root height after deletion



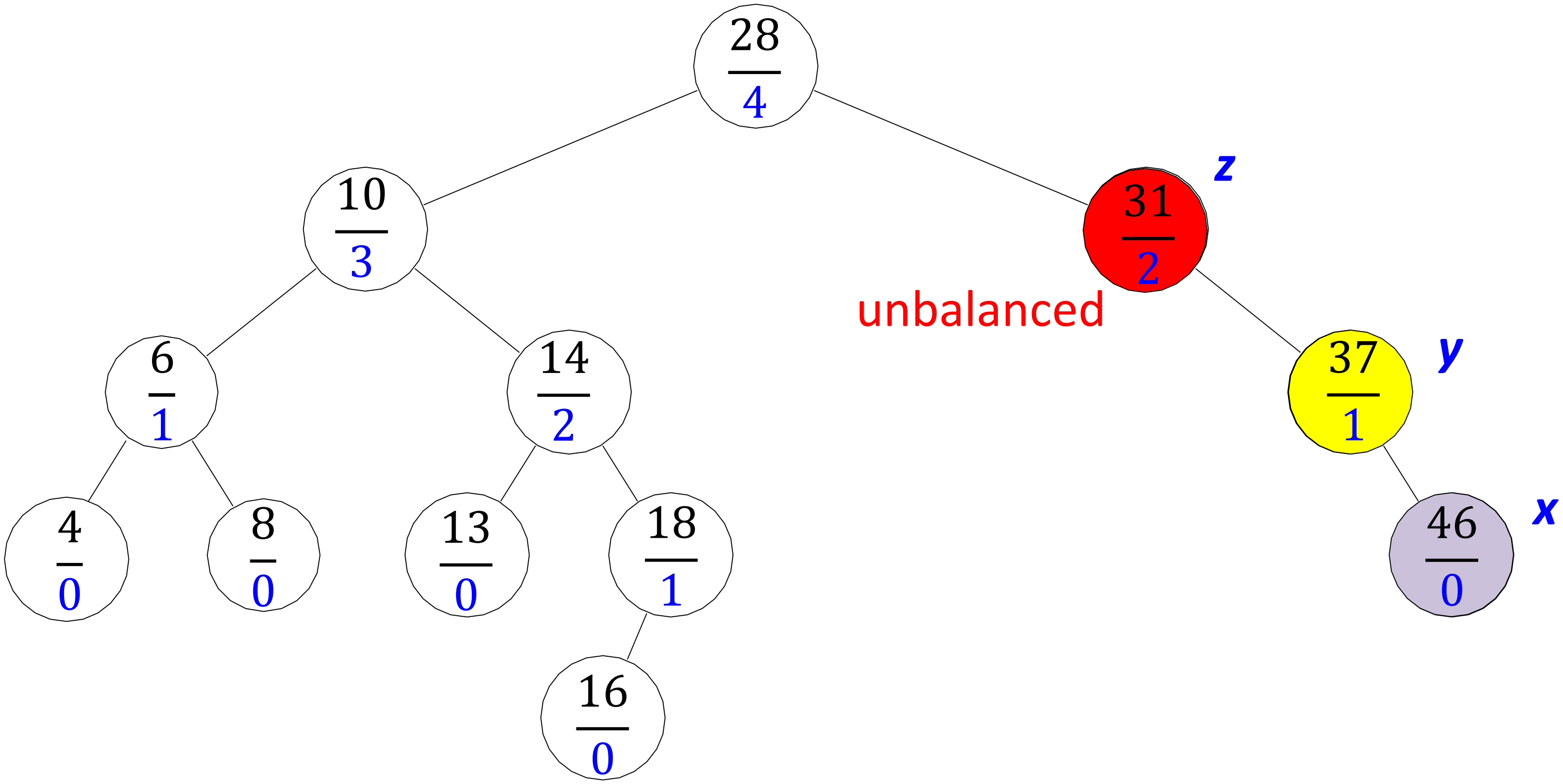




- Rebalancing is similar to that after insertion, but
  - *z* is guaranteed to have one taller child
  - *y* may have both children of the same height

AVL Deletion Example

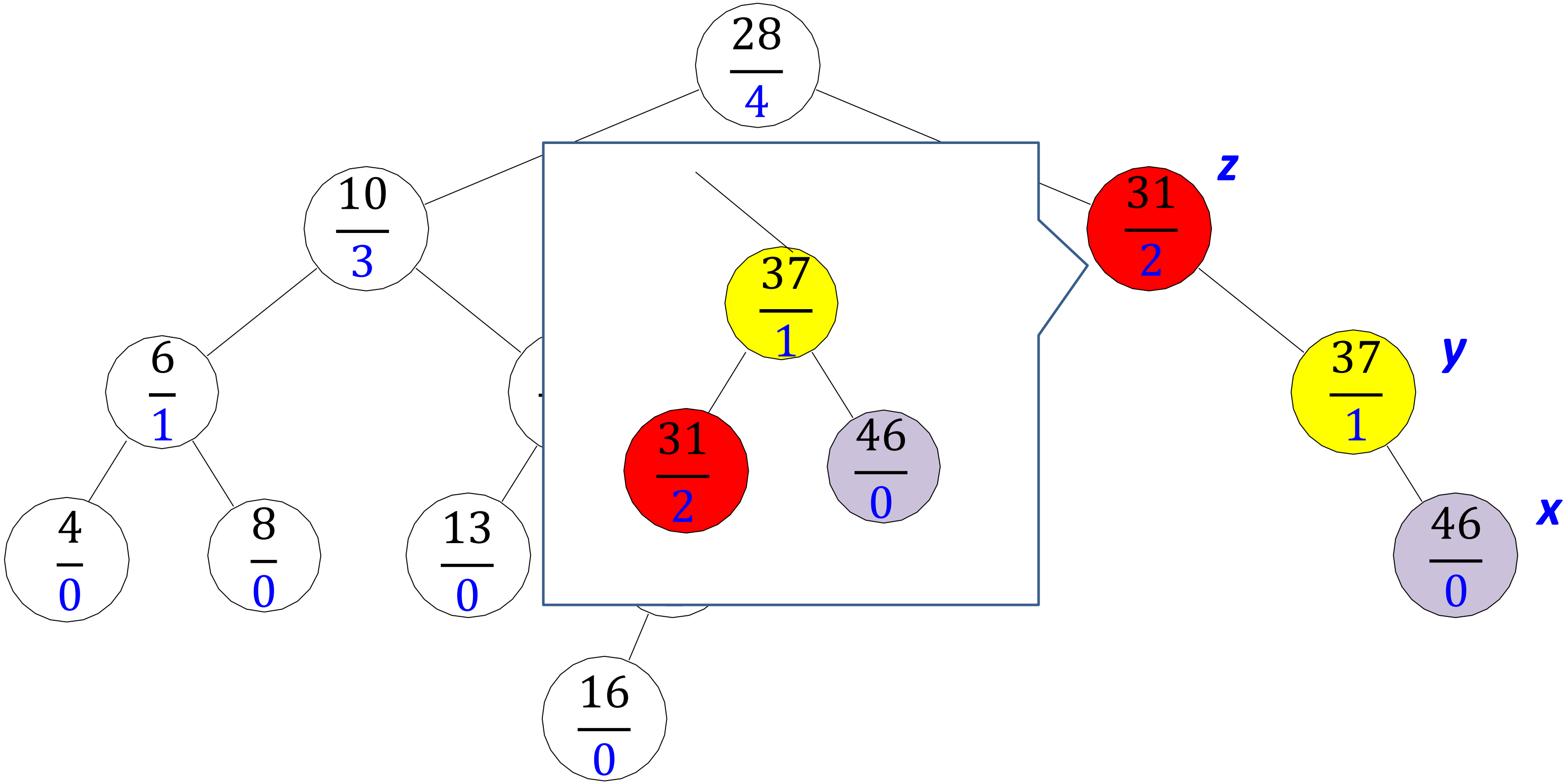
Example: *AVL::delete*(22)



- Fix with left rotation on node ***z***
- Or trinode restructuring on node ***z***

AVL Deletion Example

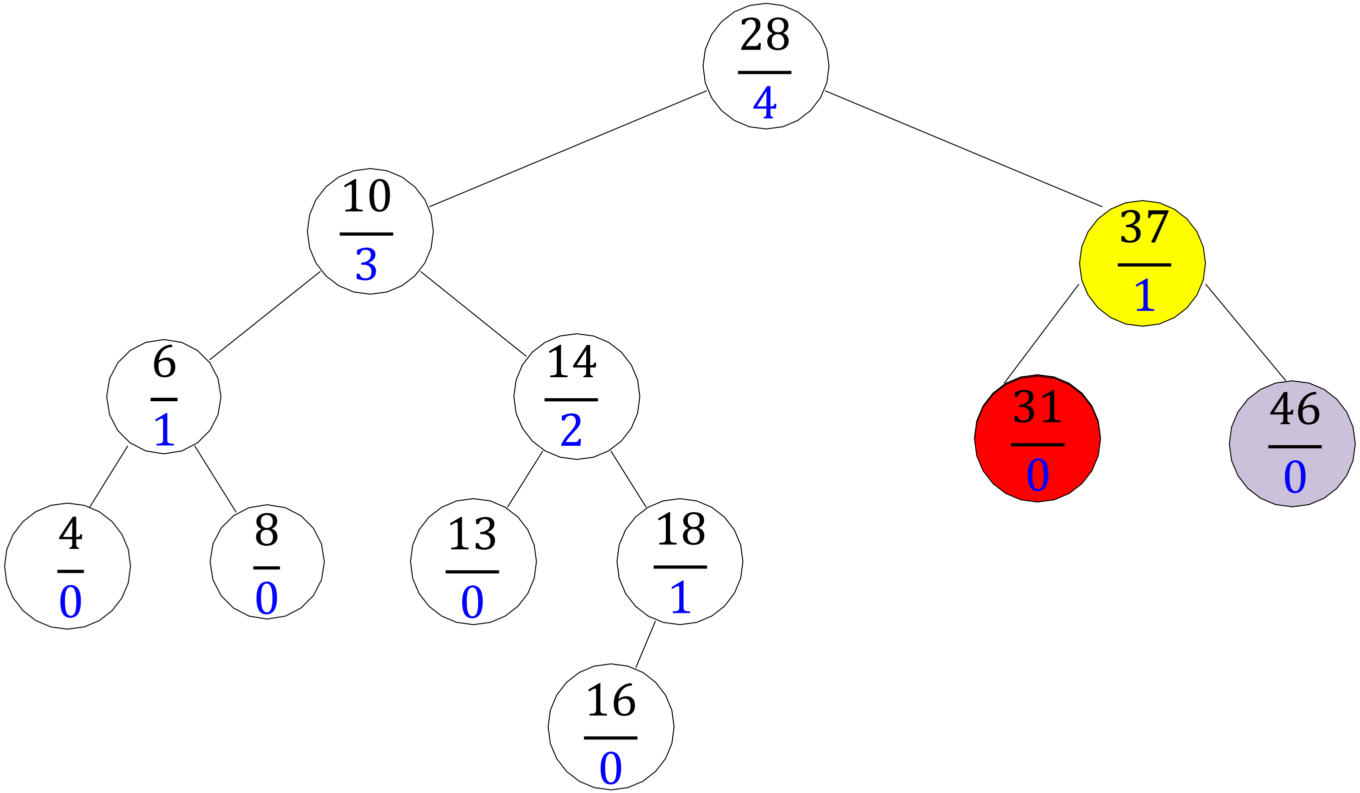
Example: *AVL::delete*(22)



- Fix with left rotation on node z
- Or trinode restructuring

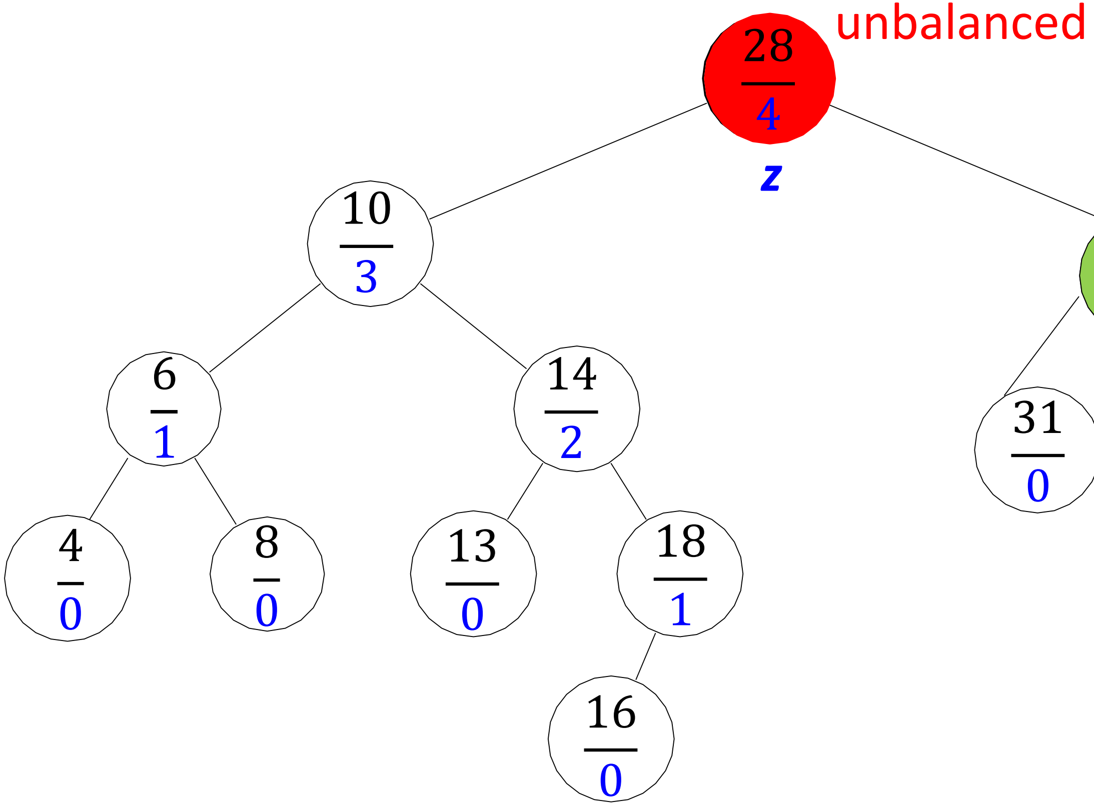
AVL Deletion Example

Example: *AVL::delete*(22)



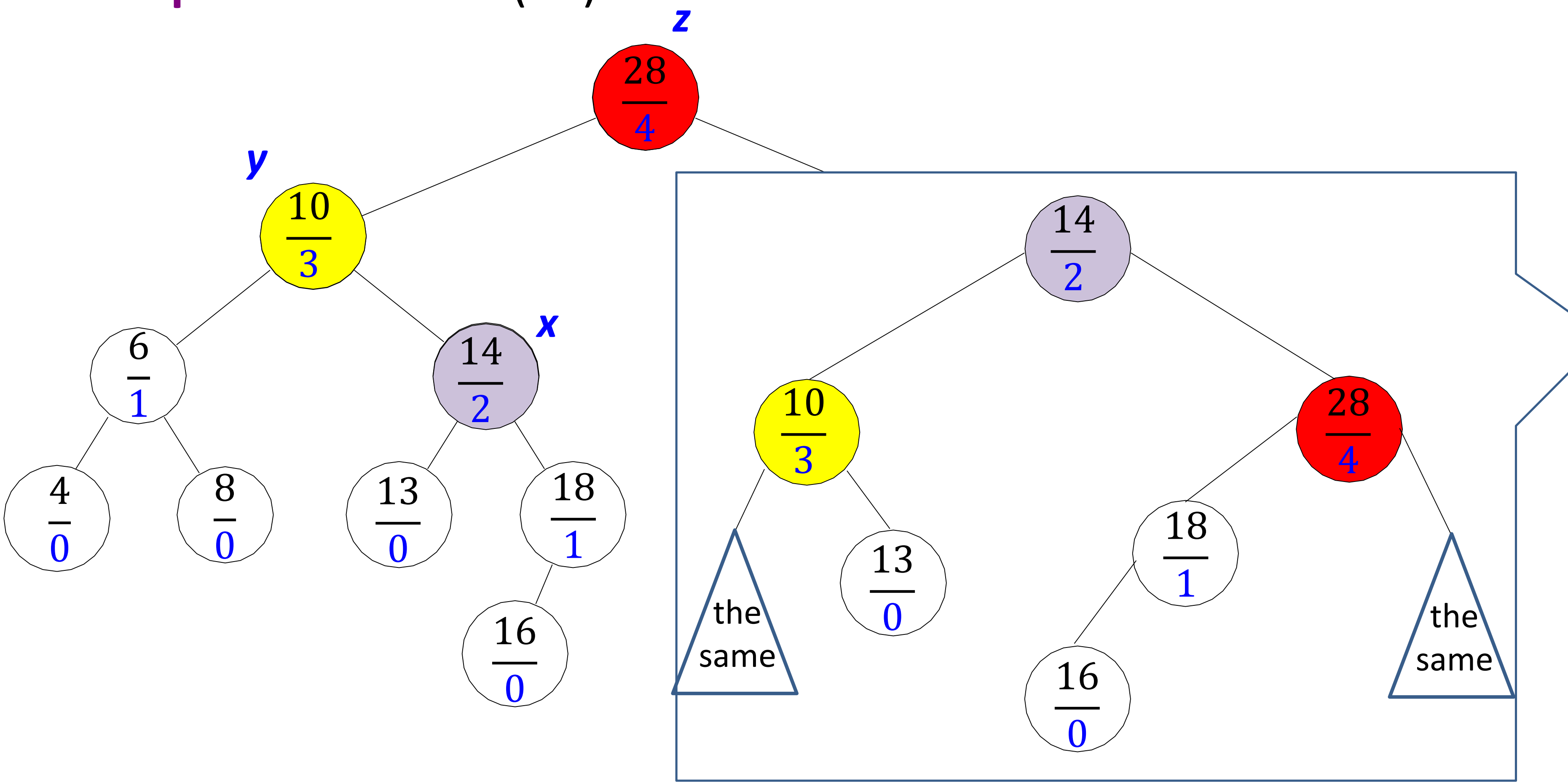
AVL Deletion Example

Example: *AVL::delete*(22)



AVL Deletion Example

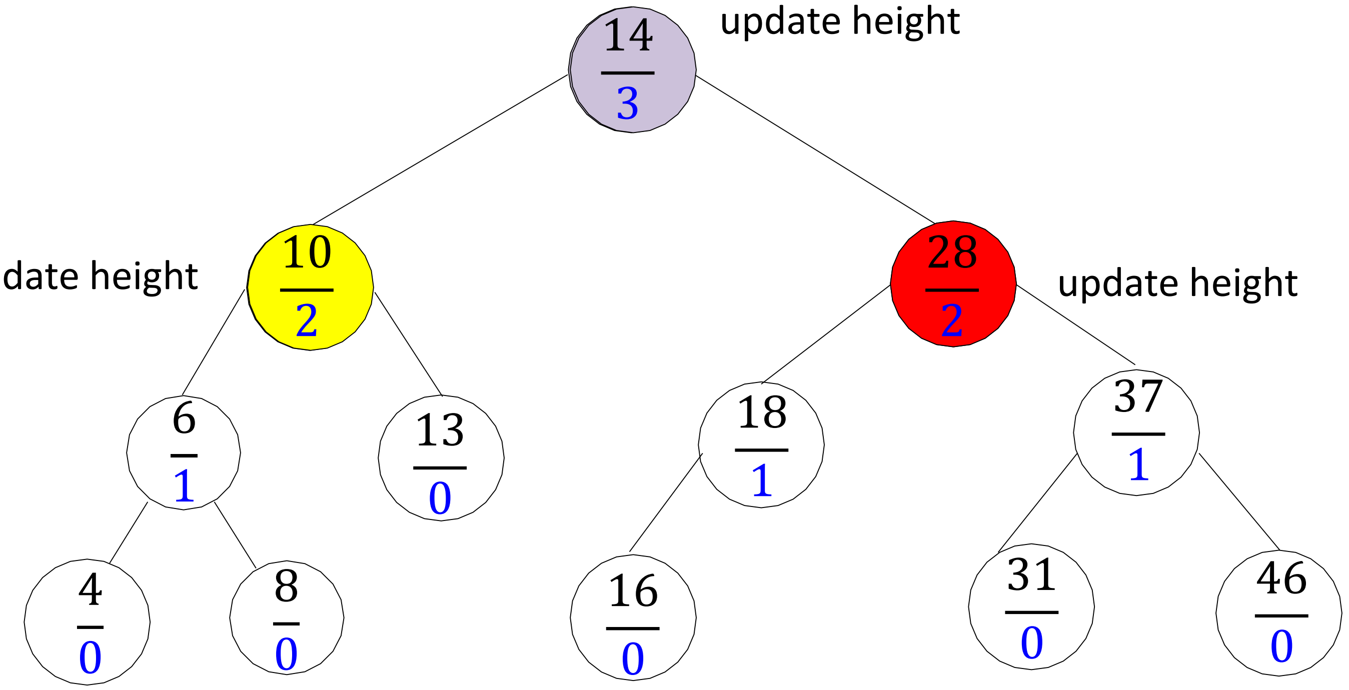
Example: *AVL::delete*(22)



- Fix with double right rotation (left rotate y, then rotate right z)
- Or trinode restructuring on node z

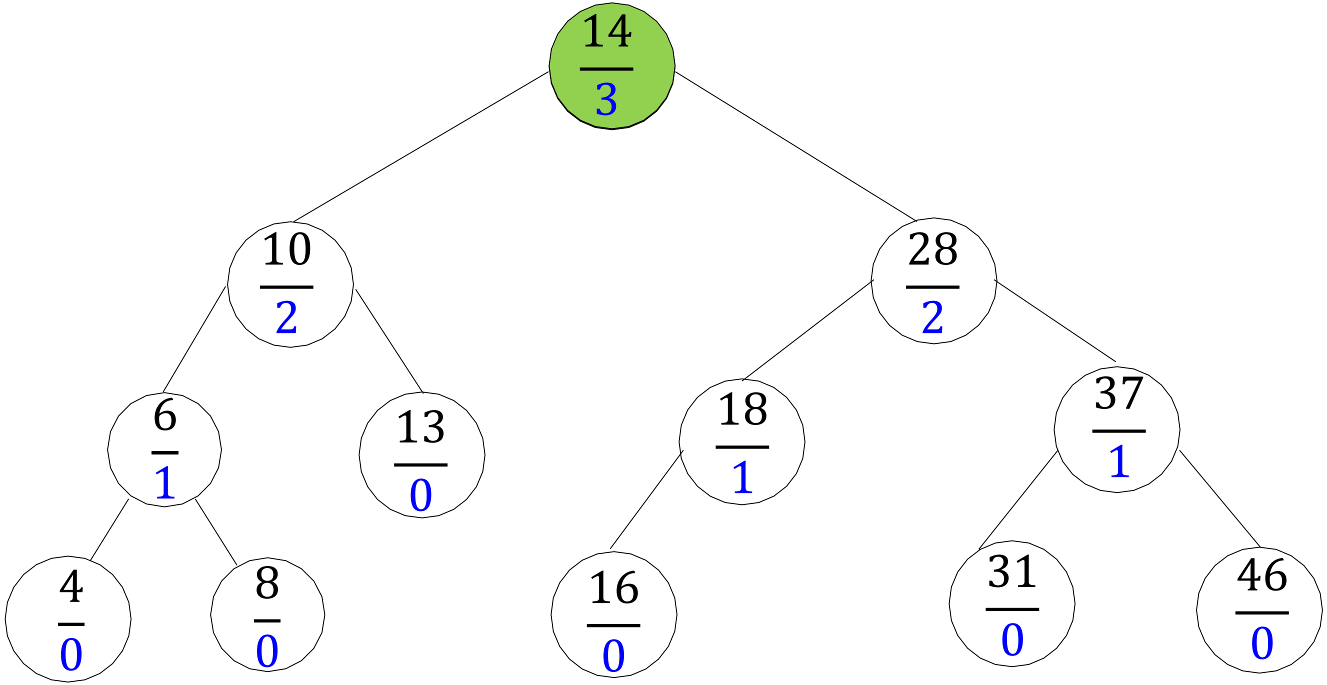
AVL Deletion Example

Example: *AVL::delete*(22)



AVL Deletion Example

Example: *AVL::delete*(22)



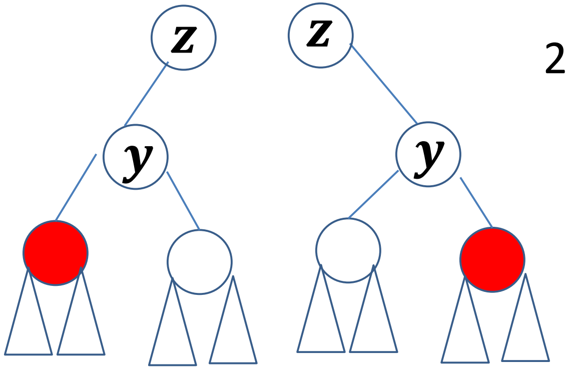
- Rebalanced

AVL Deletion

- *AVL::delete*(T, k)
  - first, delete k from T with the usual BST deletion
    - delete returns parent z of the deleted node
    - heights of nodes on path from z to root may have decreased
  - then move up the tree from z, updating heights
  - if height difference is ± 2 at node z , then z is *unbalanced*
    - re-structure tree to restore height-balance property
    - just like rebalancing for insertion, with two differences

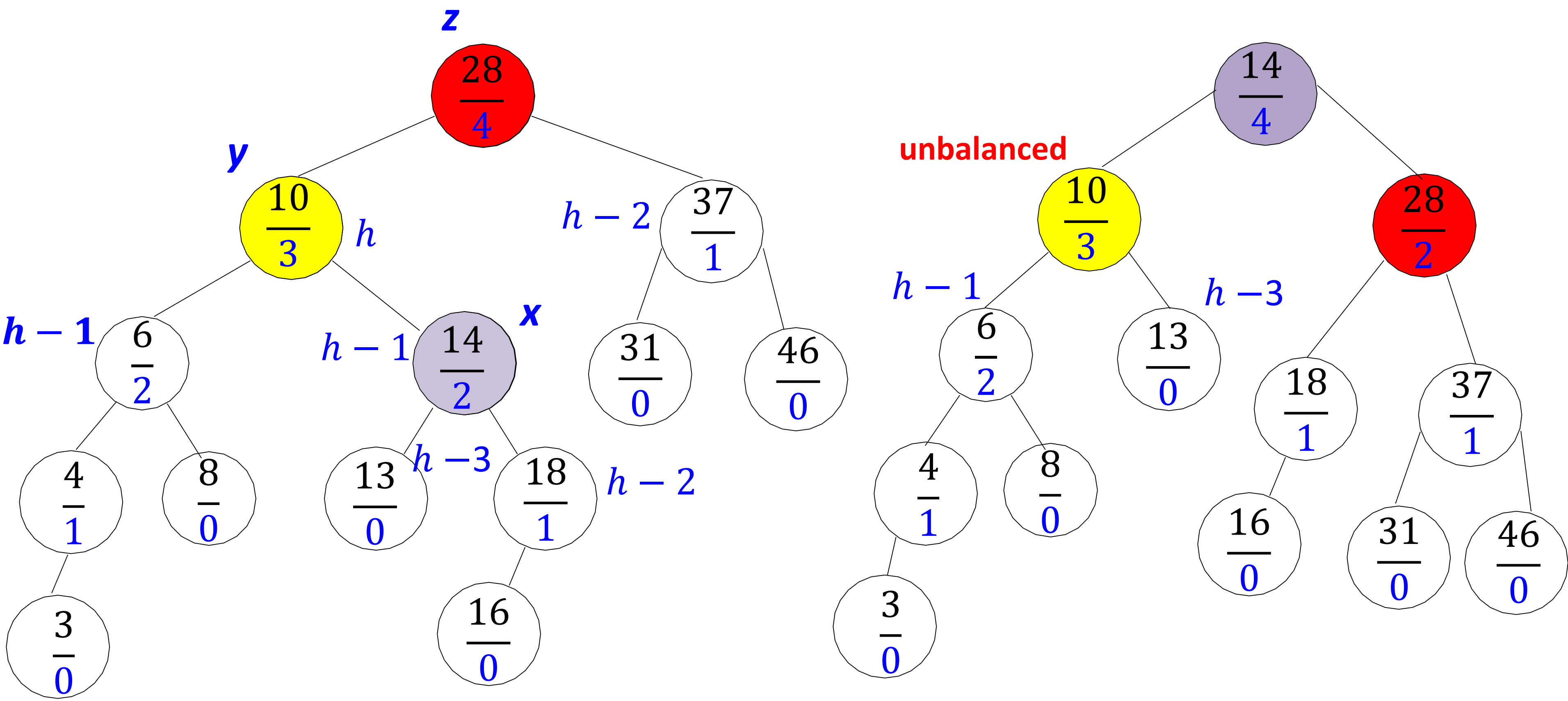
1. restructuring after deletion does not guarantee to restore tree height to what it was before deletion
   - continue the path up the tree, fixing any imbalances
2. tallerChild(y)
   - left and right children of y may have the same height
   - in case of a tie
     - return left child of y if y is itself the left child
     - return right child of y if y is itself the right child

tallerChild



AVL Deletion Example

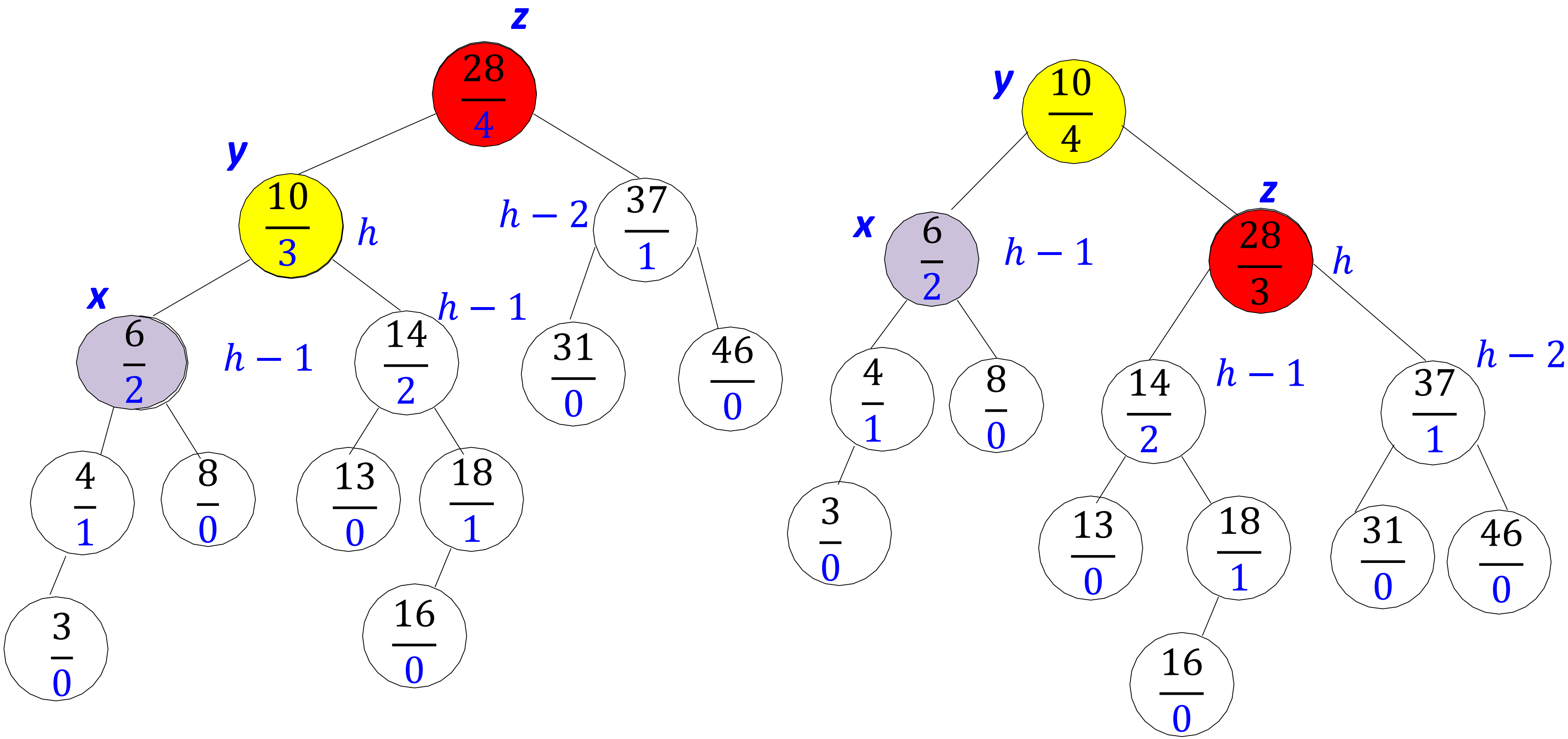
Example: incorrect if do not following the “same side” rule



- Double rotate or trinode restructuring
- For insertion, proved that the “other” child of y, not the tallest, has height $h-2$
  - cannot argue the same for deletion

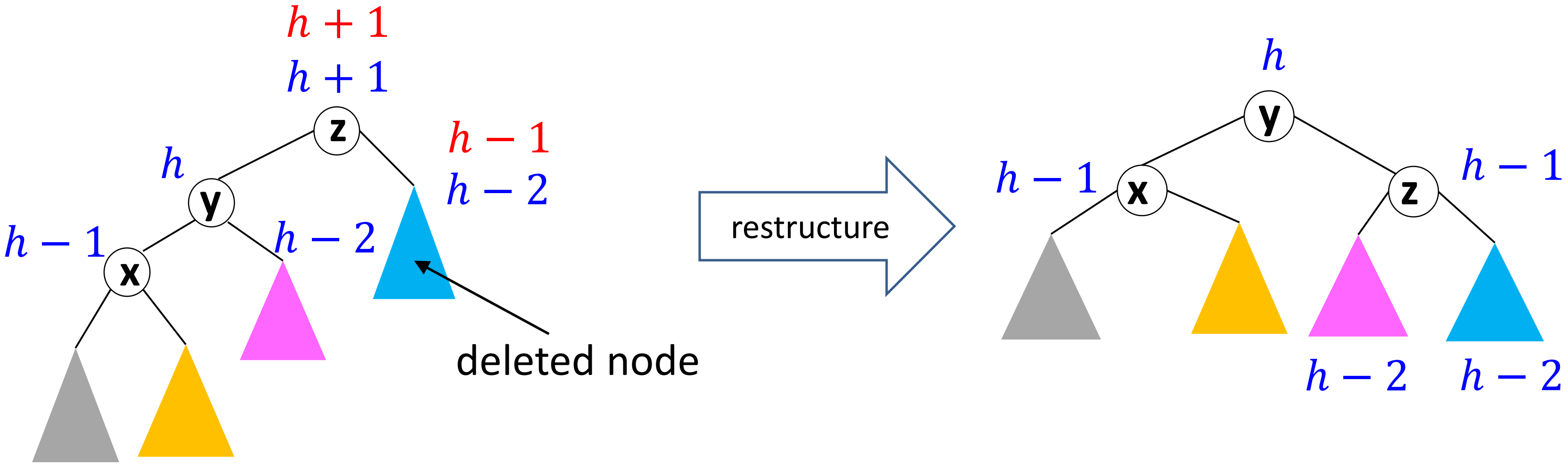
AVL Deletion Example

Example: same example, now following the “same side” rule



- Rotate or trinode restructuring
- Rebalanced, now children of x do not separate

Reduced Height after Deletion



- If ‘not the tallest’ child of $\boldsymbol{y}$ has height $h-2$, height decreases after rebalancing
    - might cause imbalance higher up the tree

AVL Delete Pseudocode

```
AVL::delete$(k, v)$
    $z \leftarrow$ BST::delete$(k, v)$
    // Assume z is the parent of the BST node that was removed
    while $(z$ is not $\mathrm{NIL})$
        if $(|z.left.height - z.right.height| > 1)$ then
            let $y$ be tallest child of $z$
            let $x$ be tallest child of $y$
            // break ties to prefer ‘the same side’
            $z \leftarrow$ restructure$(x, y, z)$
            // must continue checking the path upwards
        setHeightFromSubtrees$(z)$
        $z \leftarrow$ parent of $z$
```

AVL Tree Operations Runtime

- **AVL::search**
    - just like in BSTs, costs $\Theta(height)$
- **AVL::insert**
    - *BST::insert*
    - then check and update along path to new leaf
        - *restructure* restores the height of the tree to what it was
        - so *restructure* will be called *at most once*
    - total cost $\Theta(height)$
- **AVL::delete**
    - *BST::delete*, then check and update along path to deleted node
        - *restructure* may be called $\Theta(height)$ times
    - total cost $\Theta(height)$
- Total cost for all operations is $\Theta(height) = \Theta(\log n)$
    - but in practice, the constant is quite large