CS 240 – Data Structures and Data Management

Module 8: Range-Searching in Dictionaries for Points

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Based on lecture notes by many previous cs240 instructors

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Outline

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- Range-Searching in Dictionaries for Points
    - Range Search
    - Multi-Dimensional Data
    - Quadtrees
    - kd-Trees
    - Range Trees
    - Conclusion

Range Searches

- *search*(k) looks for one specific item
- New operation *RangeSearch* (x, x')
  - look for all items that fall within given range
  - input a range, i.e. interval $l = (x, x')$
    - may have open or closed ends
  - want to report all KVPs in the dictionary with $k \in l$
  - example

5	**10**	**11**	**17**	**18**	**33**	**45**	**51**	**55**	**77**

RangeSearch $(17,45]$ should return $\{18, 33, 45\}$

- Let s be the output-size, i.e. the number of items in the range
- Need $\Omega(s)$ time just to report the items in the range
  - s can be anything between 0 and n
- Running time depends both on s and n
  - keep s as a parameter when analyzing runtime
  - $O(\log n + s)$ time would be the best possible with comparison based search

Range Search in Existing Dictionary Realizations

- *Unsorted list/array/hash table*
  - range search requires $\Omega(n)$ time
    - must check for each item explicitly if it is in the range
- *Sorted array*
  - range search can be done in $O(\log n + s)$ time

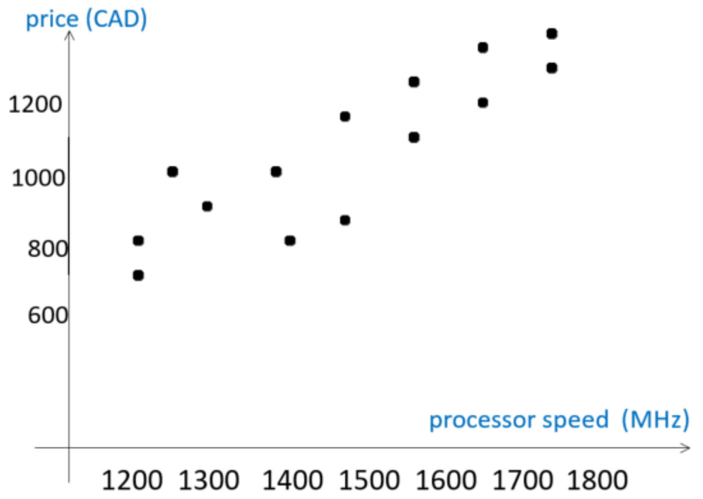
5	**10**	**11**	**17**	**18**	**33**	**45**	**51**	**55**	**77**
			i				i'		

  - *RangeSearch* $(16{,}50)$
  - use binary search to find i s.t. x is at (or would be at) $A[i]$
  - use binary search to find i' s.t. x' is at (or would be at) $A[i']$
  - report all items in $A[i+1 \ldots i'-1]$
  - Report $A[i]$ and $A[i']$ if they are in the range
- *BST*
  - can do range search in $O(height + s)$ time, will see in detail later

Outline

- Range-Searching in Dictionaries for Points
    - Range Search Query
    - Multi-Dimensional Data
    - Quadtrees
    - kd-Trees
    - Range Trees
    - Conclusion

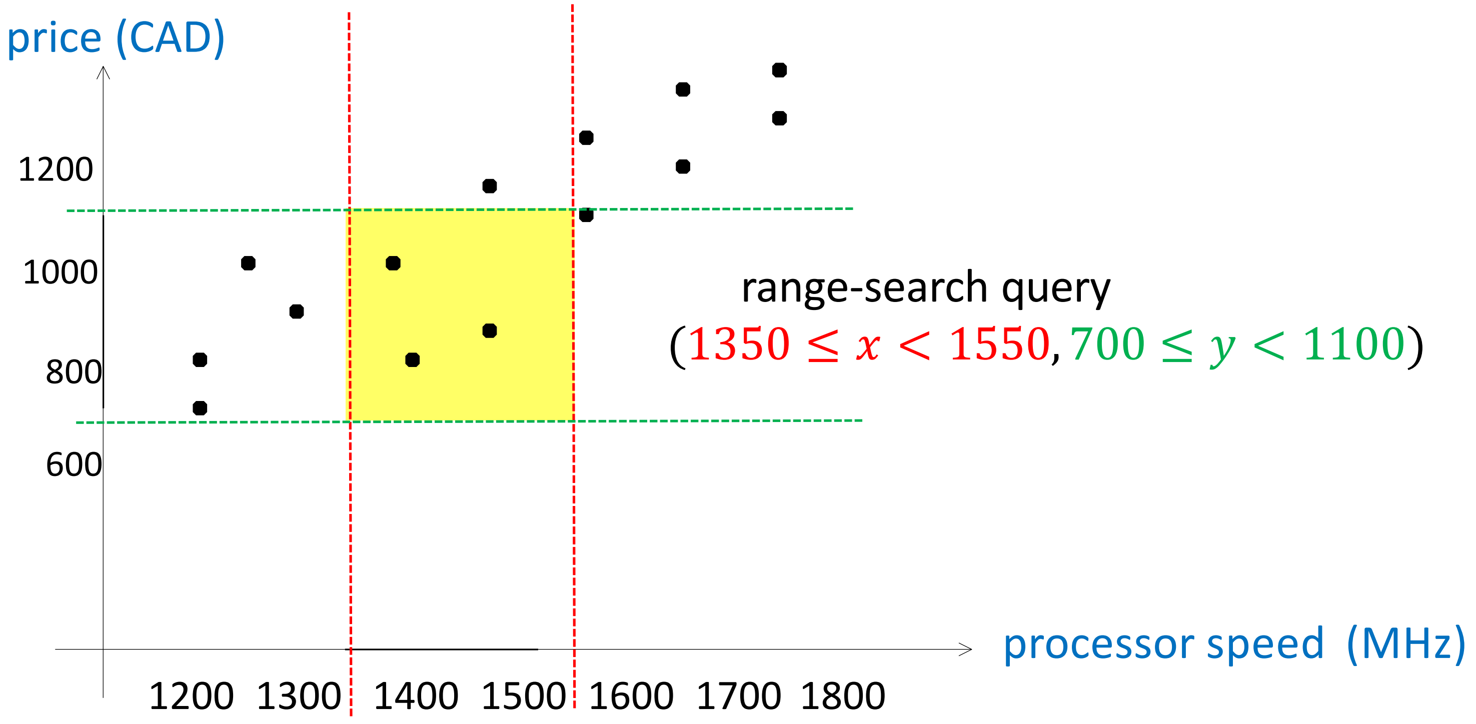
Multi-dimensional Data



- Data with multiple aspects of interest
  - laptops: price, screen size, processor speed, etc.
  - employees: name, age, salary, …
- Dictionary for multi-dimensional data
  - collection of d-dimensional items (or points)
  - each item has d aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  - operations: insert, delete, search, range search
  - Range search
    - example: laptops with
      1) 11 inches < screen size < 13 inches
      2) 8GB < RAM < 16 GB
      3) 1,500 CAD < price < 2,000 CAD
- We focus on $d = 2$, i.e. points in Euclidean plane

Multi-Dimensional Range Search

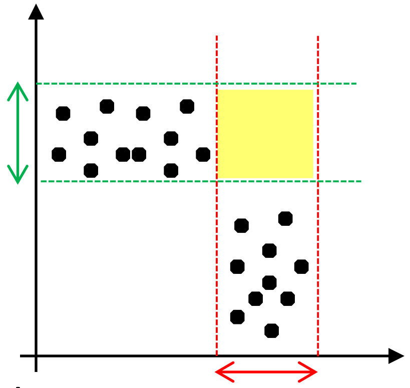
- (Orthogonal) d-dimensional range search
  - given a *query rectangle* A, find all points that lie within A



$(1350 \leq x < 1550, 700 \leq y < 1100)$

Multi-Dimensional Range Search

- Options for implementing d dimensional dictionaries
  - Reduce to one-dimensional dictionary
    - combine d-dimensional key into one dimensional key
      - i.e. $(x, y) \rightarrow x + y \cdot n^2$
    - problem: range search on one aspect is not straightforward
  - Use several dictionaries, one for each dimension
    - problem: wastes space, inefficient search
  - Example
    - $n/2$ points retrieved in each direction
    - $O(n \log n)$ time to consolidate results of both searches, with AVL trees
    - Total time is $O(n \log n)$, worse than exhaustive search
      - far from $O(s + \log n)$, especially since $s = 0$



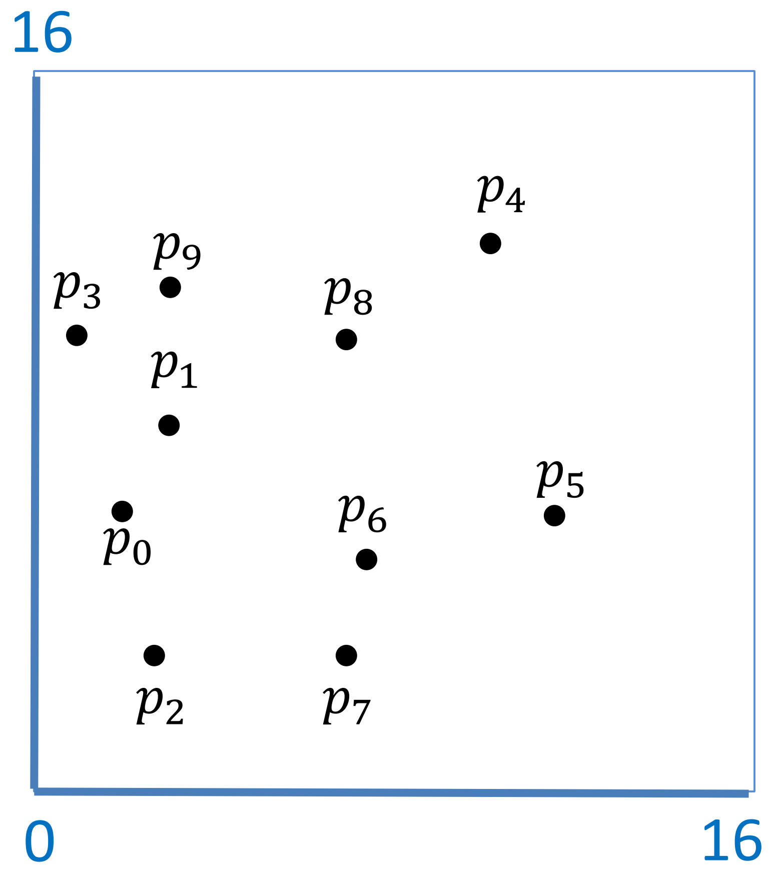
- Better idea
  - design new data structures specifically for points
  - will assume points are in *general position*: no two x-coordinates or y-coordinates are the same
    - simplifies presentation, data structures can be generalized

Multi-Dimensional Range Search

- Partition trees
  - organize space to facilitate efficient multidimensional search
    - internal nodes are associated with spatial regions
    - actual dictionary points stored only at leaves
  - quadtrees, kd-trees
- Multi-dimensional range trees
  - organize dictionary points to support efficient nD search with a variant of BST search
  - both internal and leaf nodes store points, similar to one dimensional BST

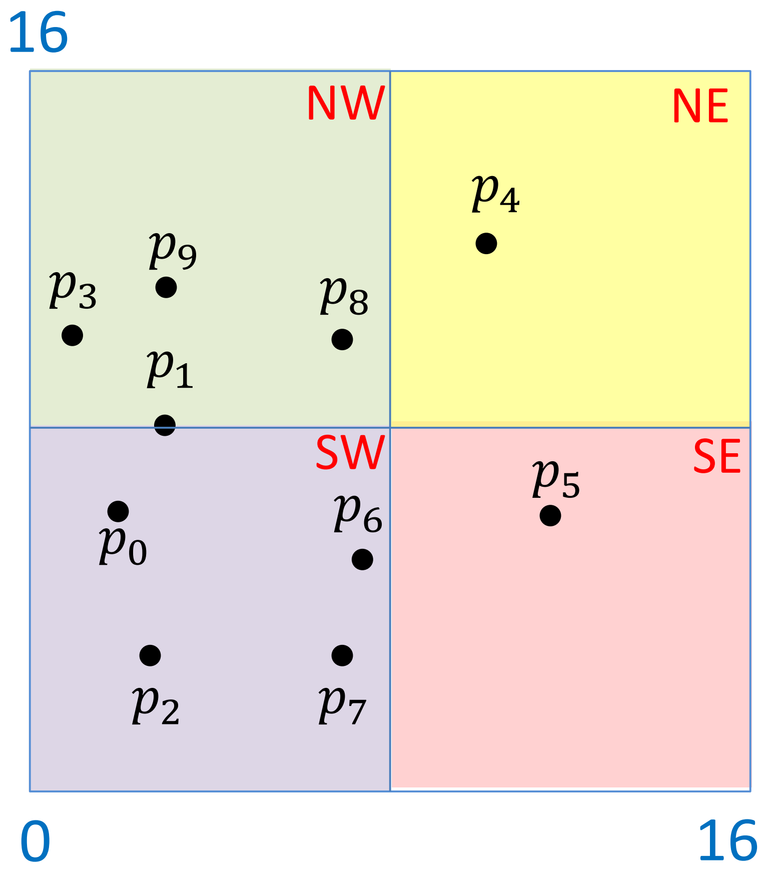
Outline

Quadtrees

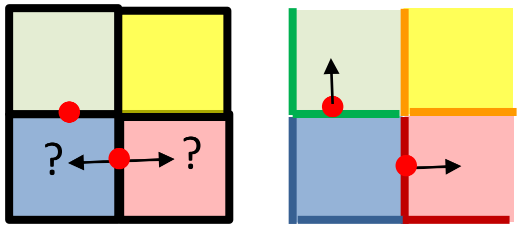


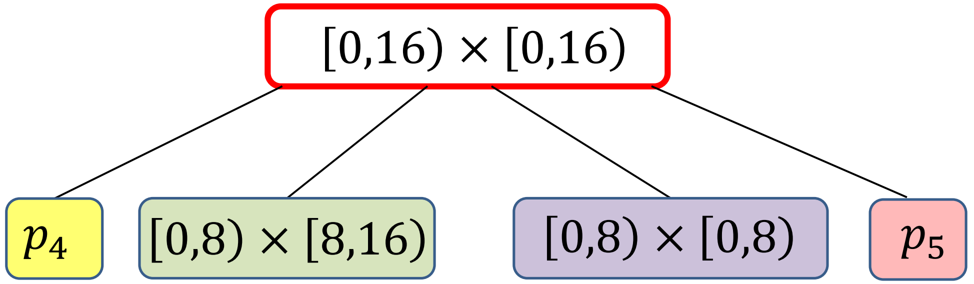
- Have a set S of n points in the plane
- Need *bounding box* R
    - width/height of R is a power of 2
    - smallest square $[0, 2^k) \times [0, 2^k)$ containing all points
    - this also simplifies insert/delete
    - find R by computing the maximum x and y values in set S
- Quadtree is a hierarchy (tree) of regions
- Higher levels responsible for larger regions
- Lower levels responsible for smaller regions
- Leaves responsible for regions small enough to store one point
- Whenever possible, search rules out regions at higher level of hierarchy, achieving efficiency

Quadtree Construction Example

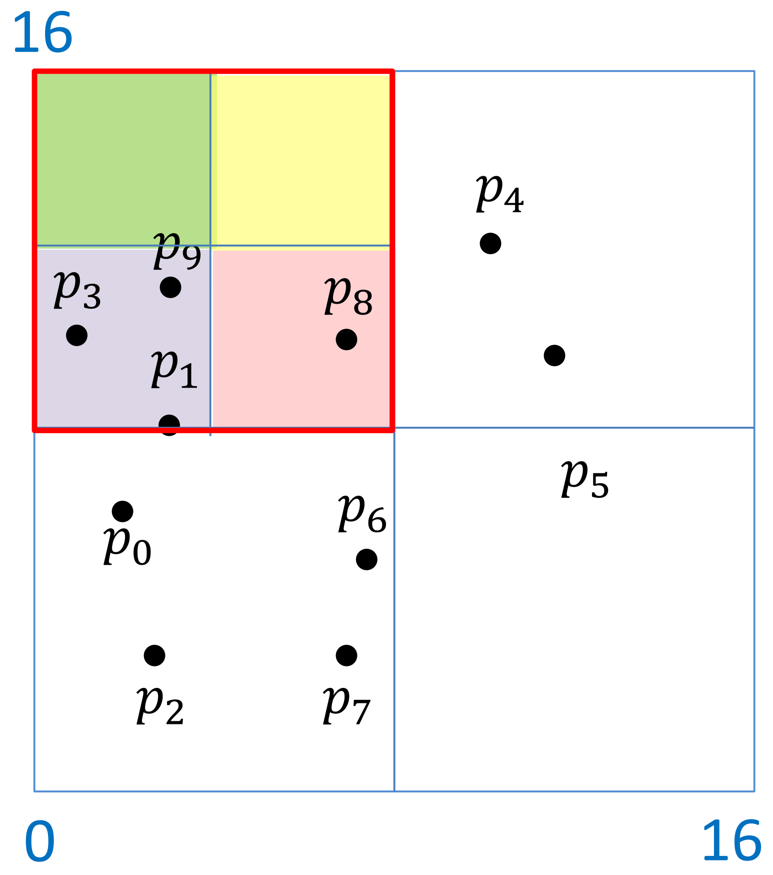


- The root corresponds to the whole square
- Split the square into 4 equal regions
- Convention: points on split lines belong to region on the right (or top)

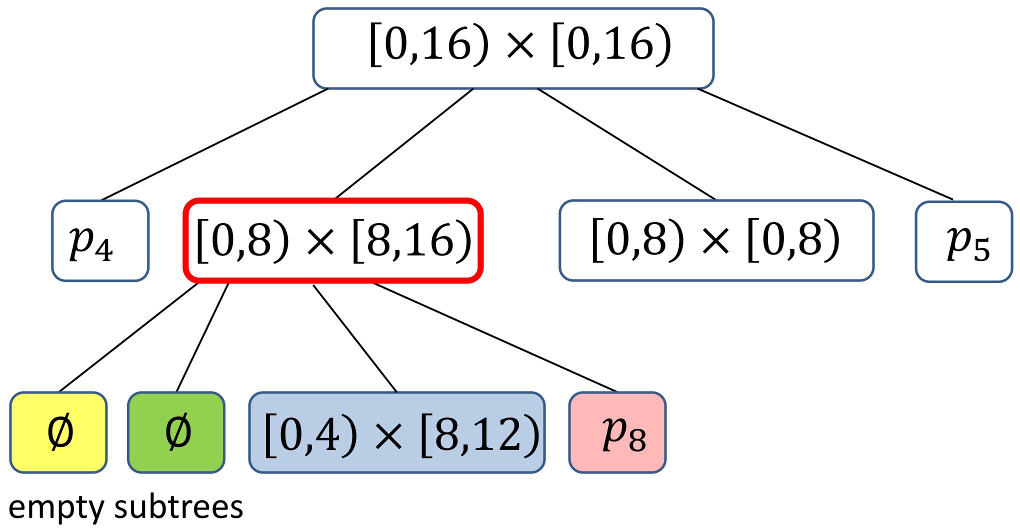




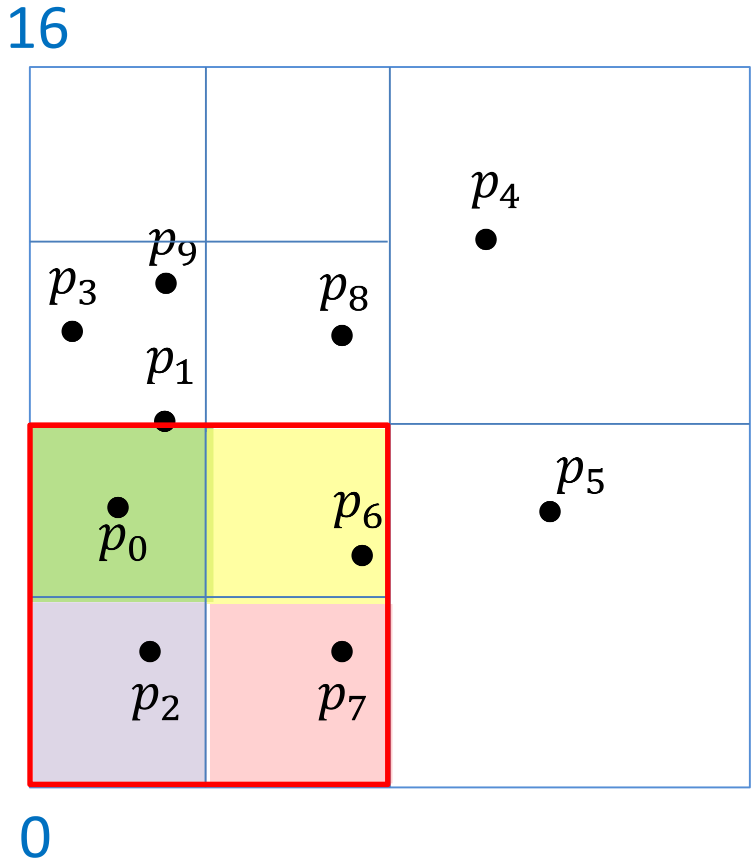
Quadtree Construction Example



- keep subdividing regions (recursively) into smaller region until each region has one point



Quadtree Construction Example



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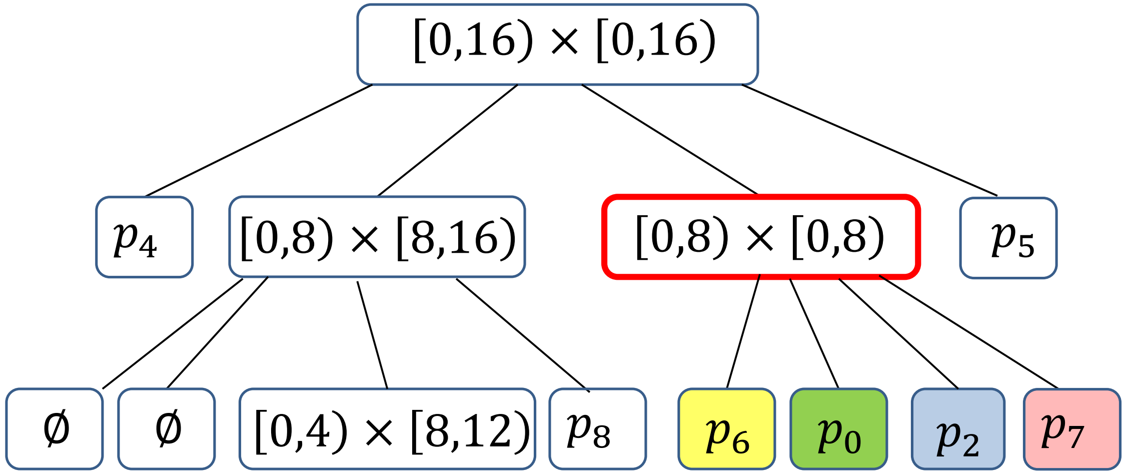
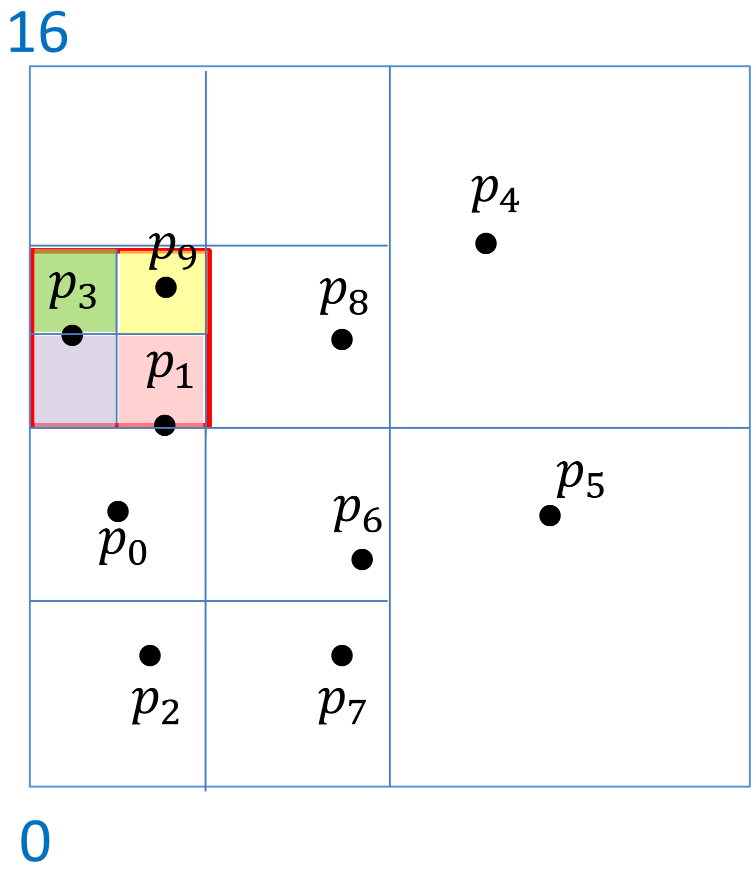


Quadtree Construction Example



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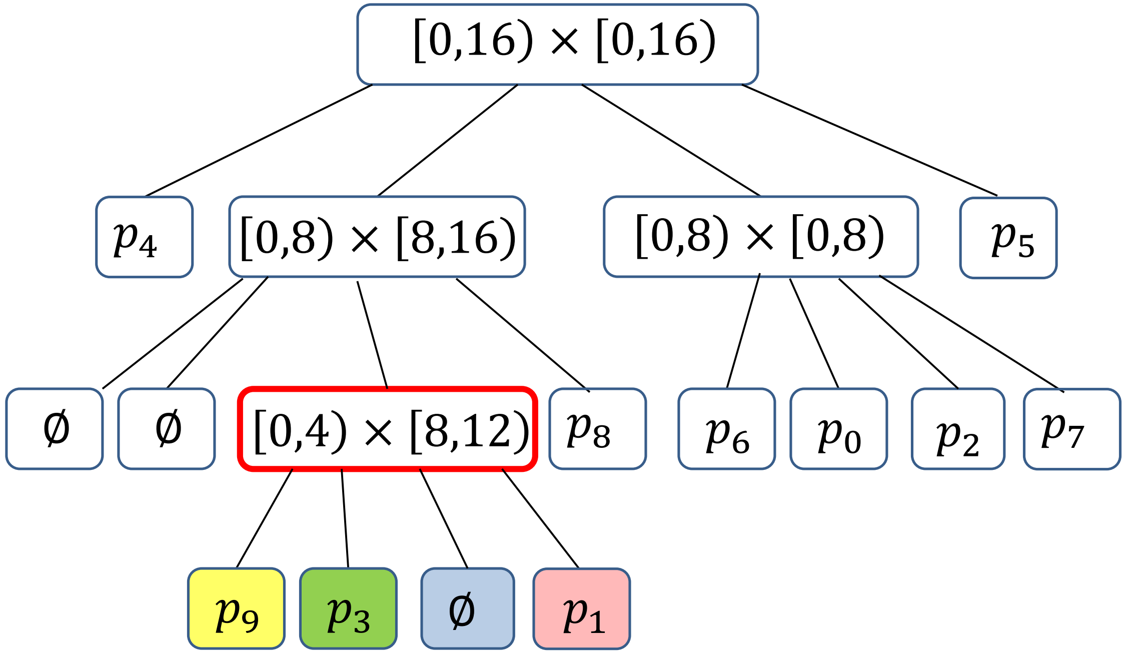
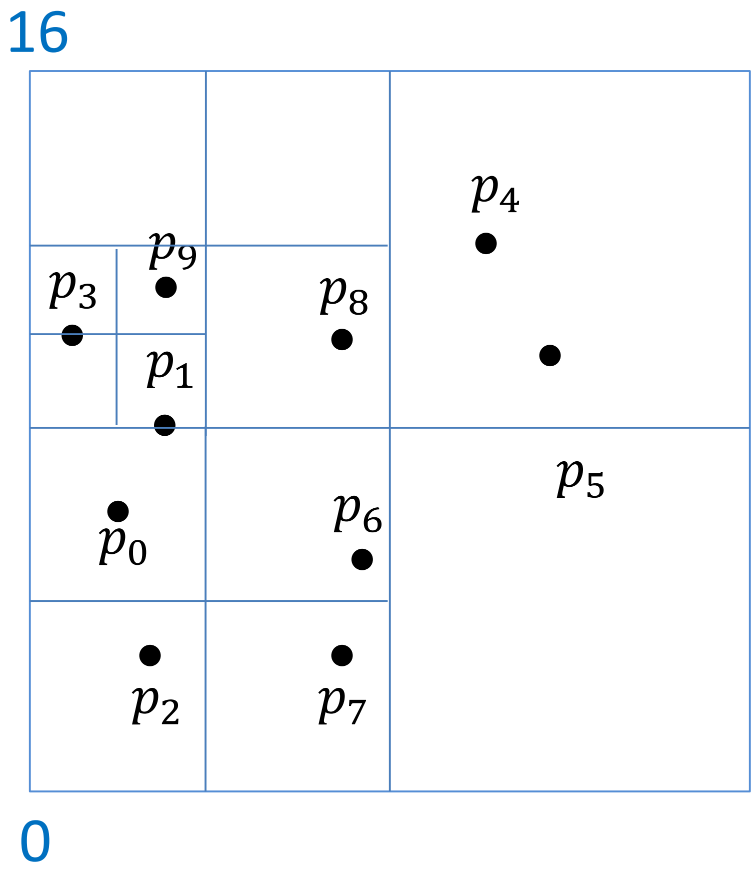
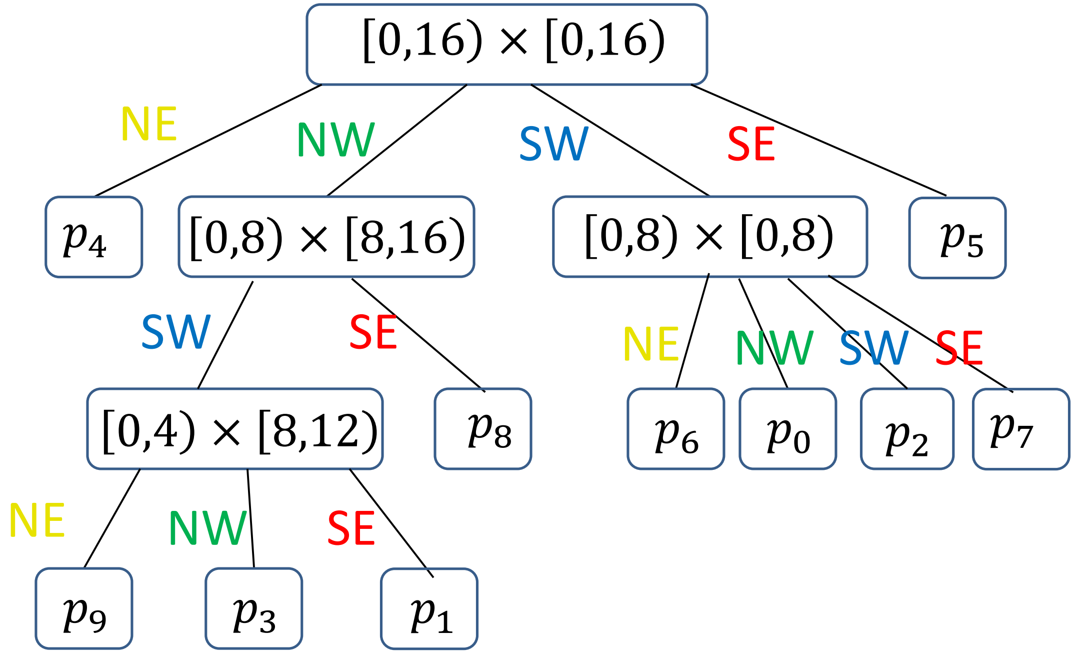


Quadtree: Omit Empty Subtrees



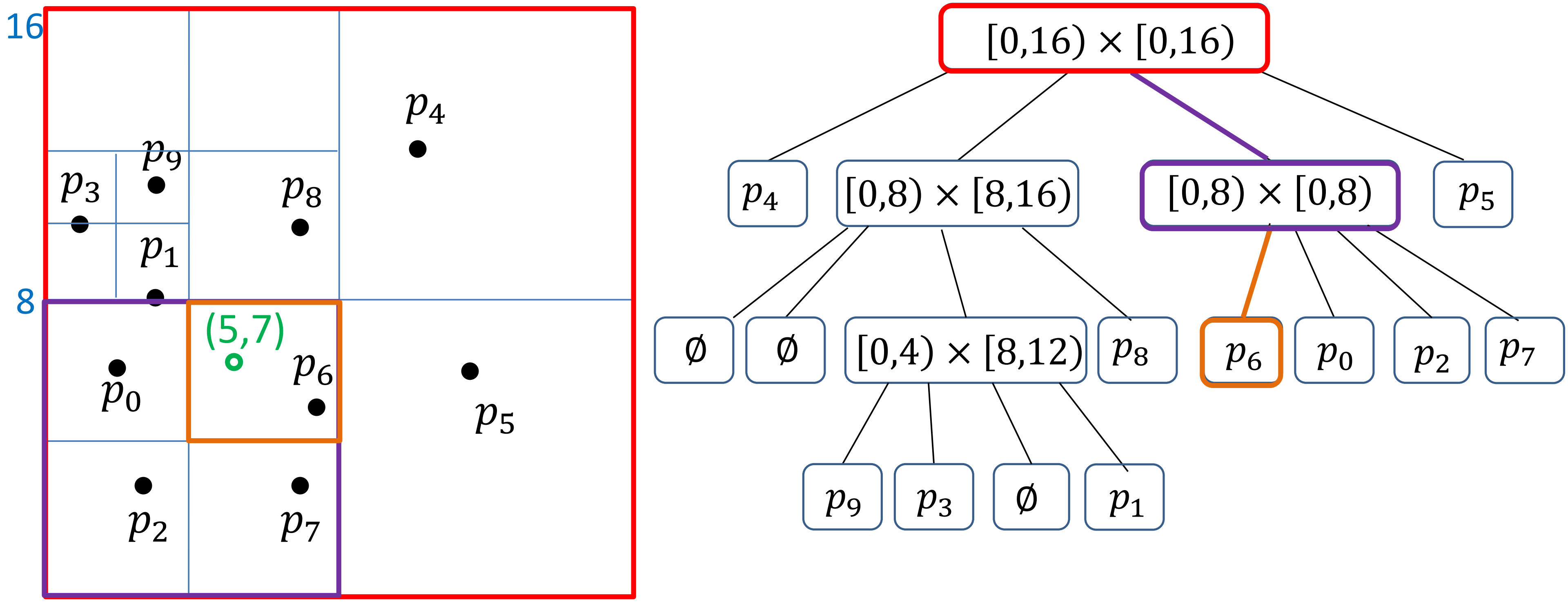
- Easier for humans
  - omit empty subtrees, label edges



Quadtree Building Summary

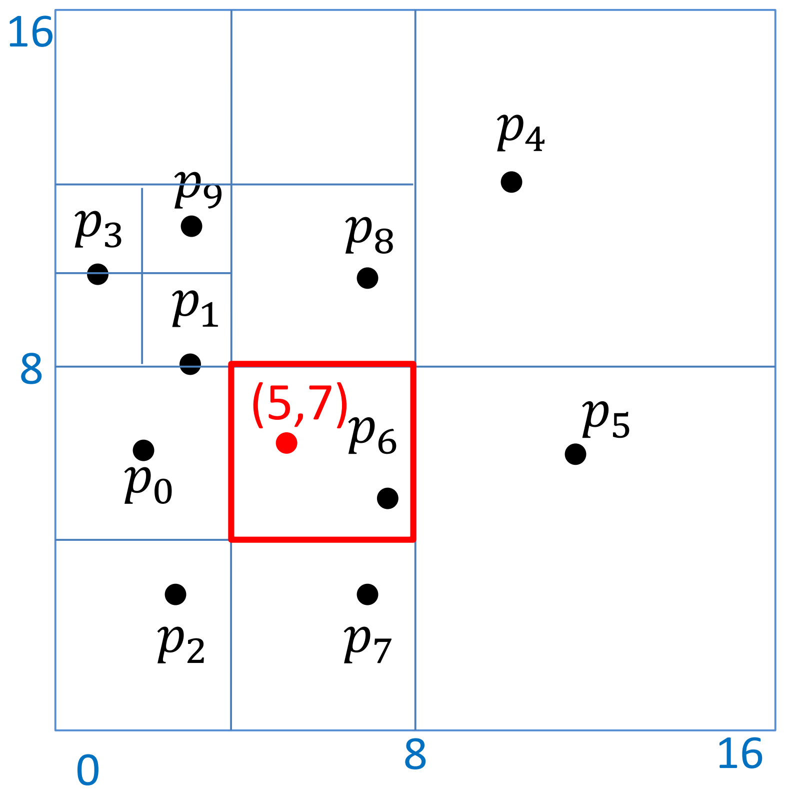
- Have n points $S = \{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$
    - all points are within a square R
- To build quadtree on S
    - root r corresponds to R
    - if R contains 0 (or 1) point
        - then root r is empty (or a leaf that stores 1 point)
        - else partition R into four equal subsquares (quadrants)
            - $R_{NE}, R_{NW}, R_{SW}, R_{SE}$
            - for each region, root has four subtrees $v_{NE}, v_{NW}, v_{SW}, v_{SE}$
    - recursively repeat this process at each nonleaf child
    - convention: points on split lines belong to region on the right (or top)

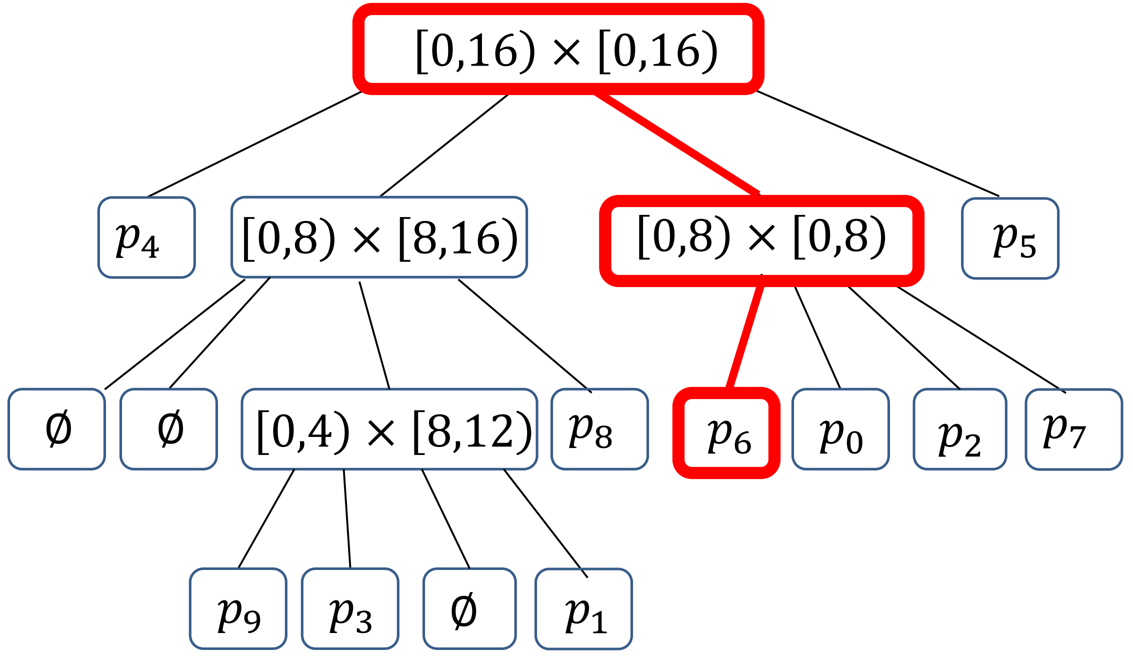
Quadtree Search



- Analogous to trie or BST
- Three possibilities for where search ends
    1. leaf storing point we search for (found)
    2. leaf storing point different from search point (not found)
    3. empty subtree (not found)
- Example: search(5,7) (not found)
- Search is efficient if quadtree has small height

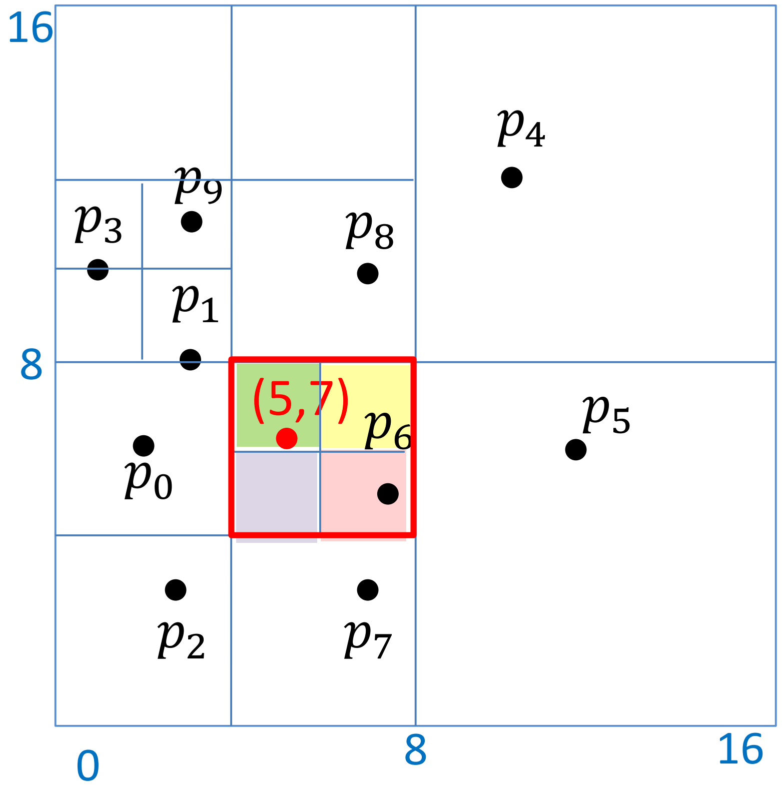
Quadtree Insert

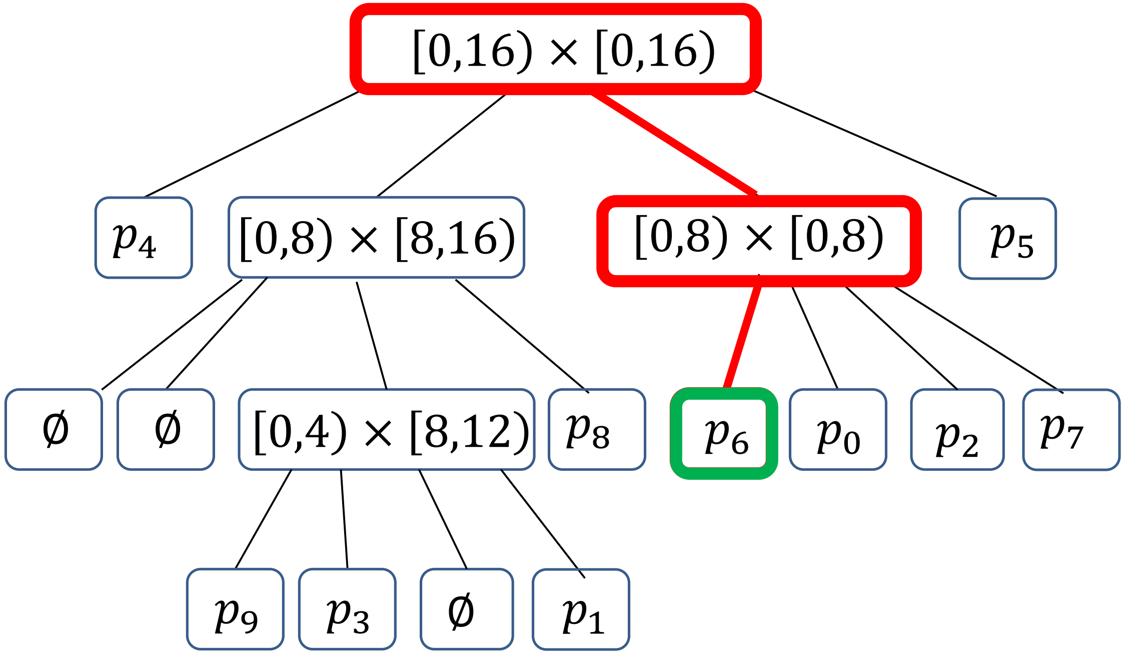




- First perform search
- Two cases
    1. search finds a leaf storing one point
        - example: insert(5,7)

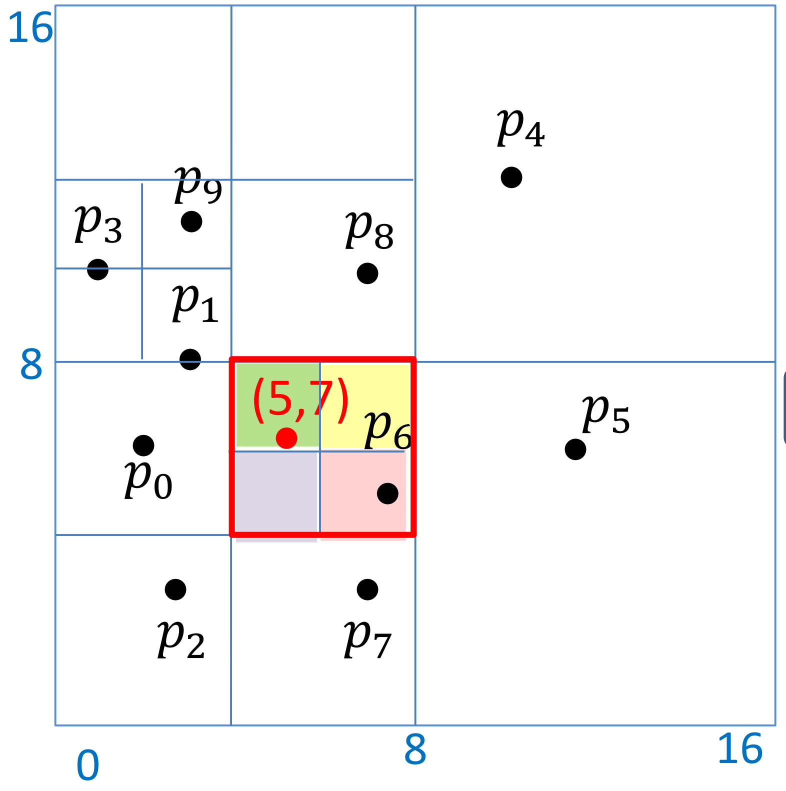
Quadtree Insert

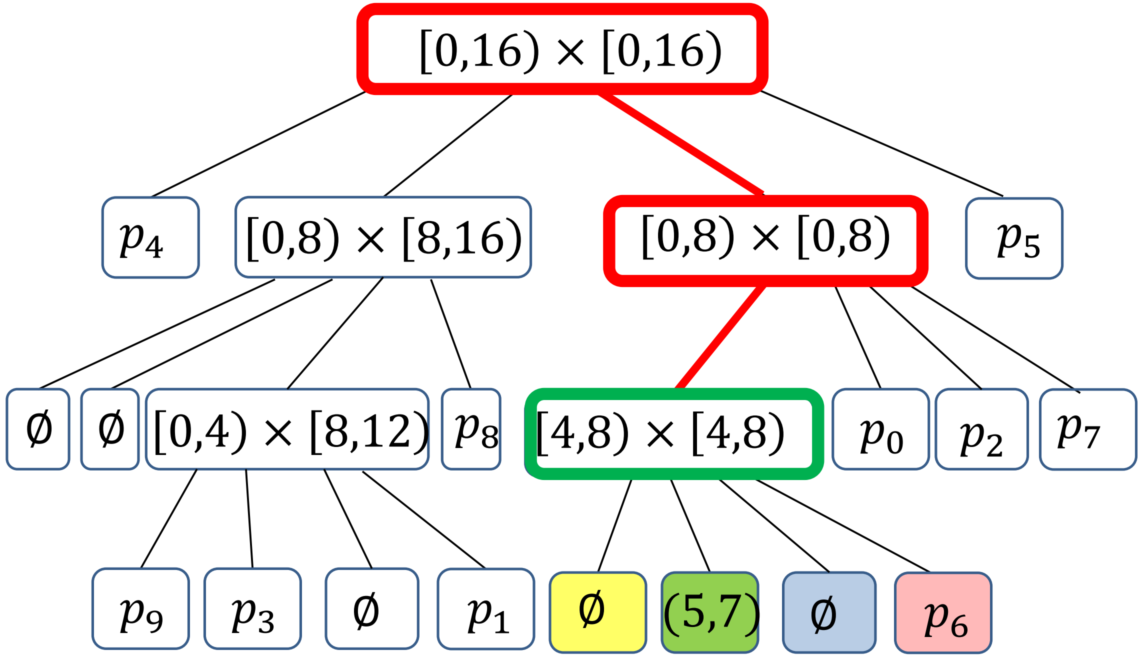




- First perform search
- Two cases
  1. search finds a leaf storing one point
     - example: insert(5,7)
     - repeatedly split the leaf **while** there are two points in one region

Quadtree Insert





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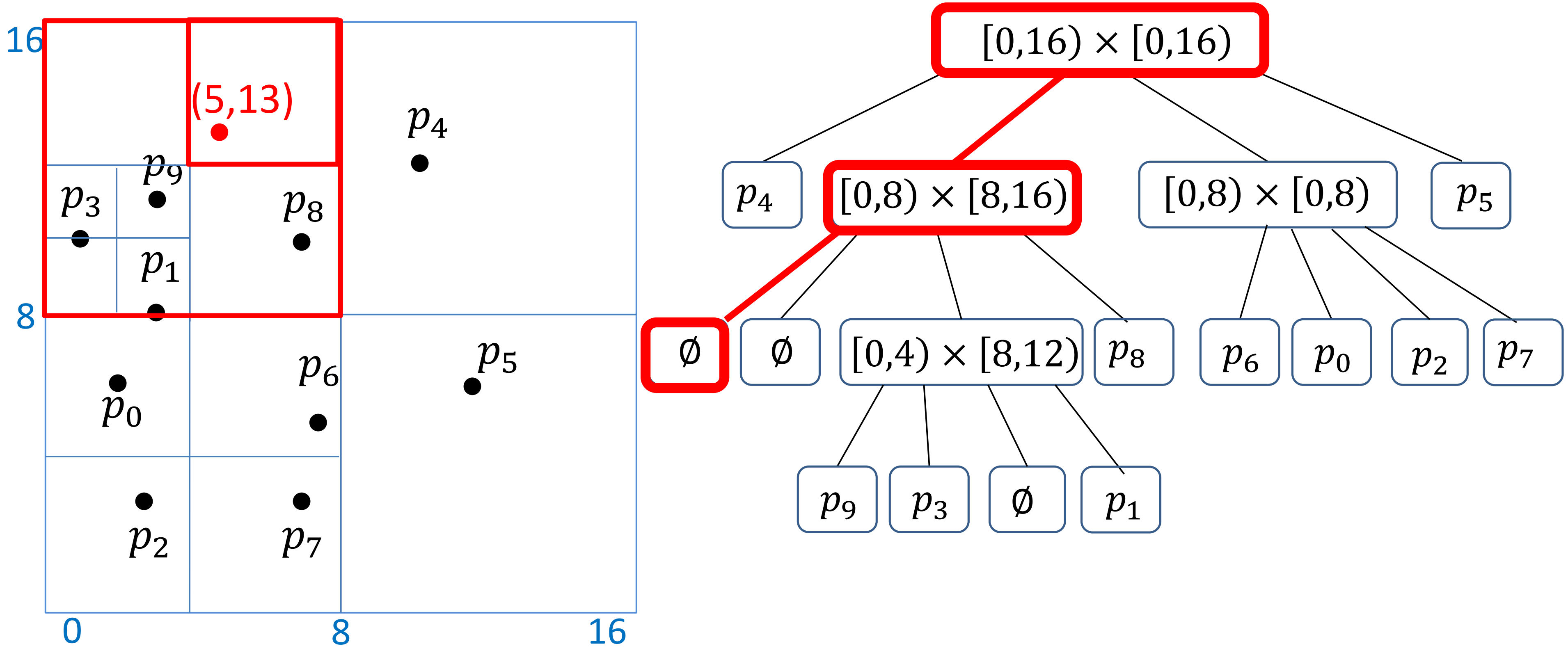
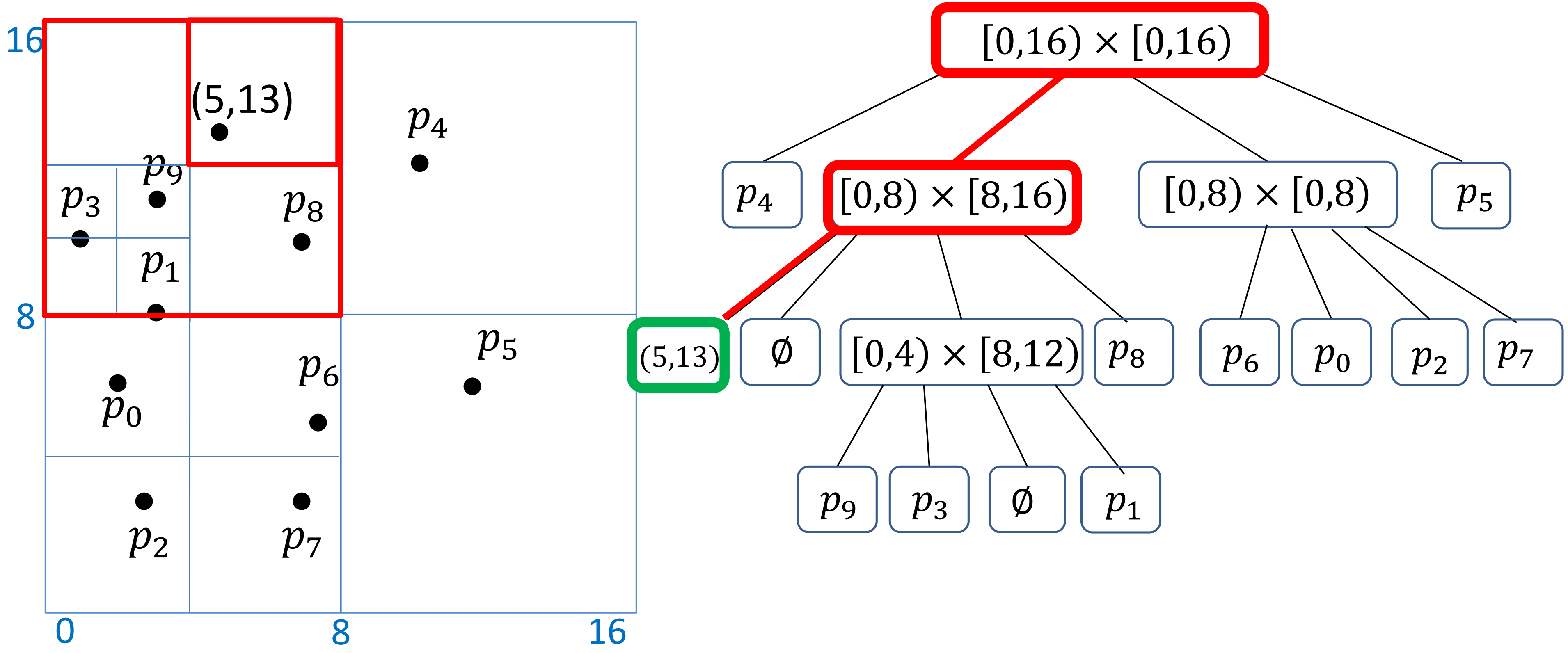


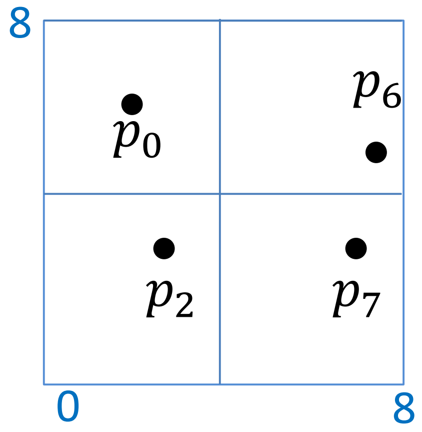
- First perform search
- Two cases
  1. search finds a leaf storing one point
  2. search finds an empty subtree
     - example: insert (5,13)

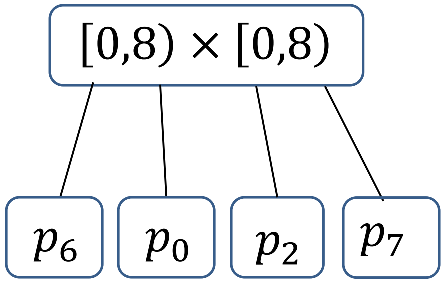
Quadtree Insert



- First perform search
- Two cases
  1. search finds a leaf storing one point
  2. search finds an empty subtree
     - example: insert(5,13)
     - expand empty subtree into a leaf storing insertion point

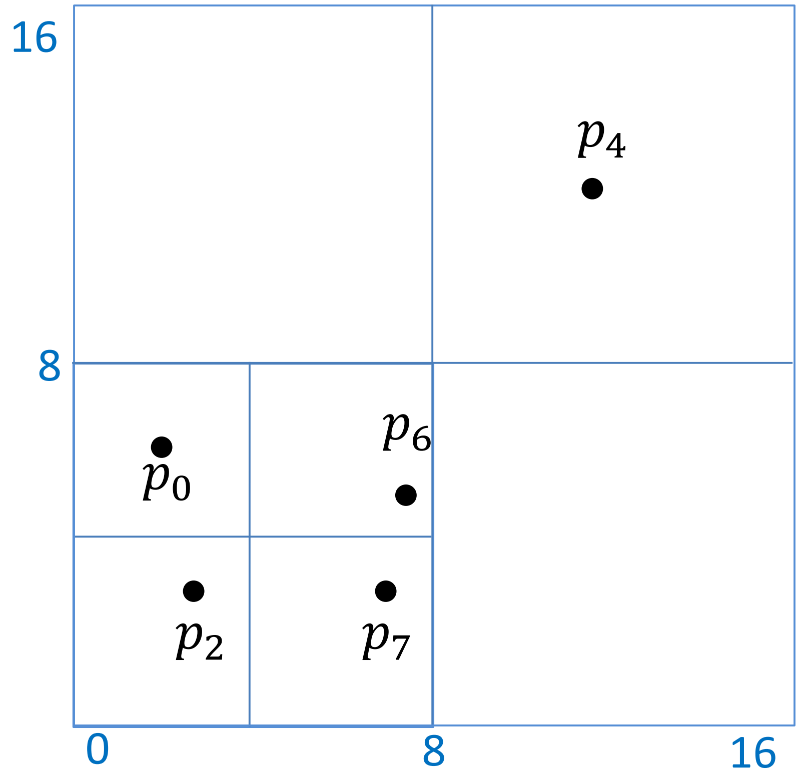
Quadtree Insert

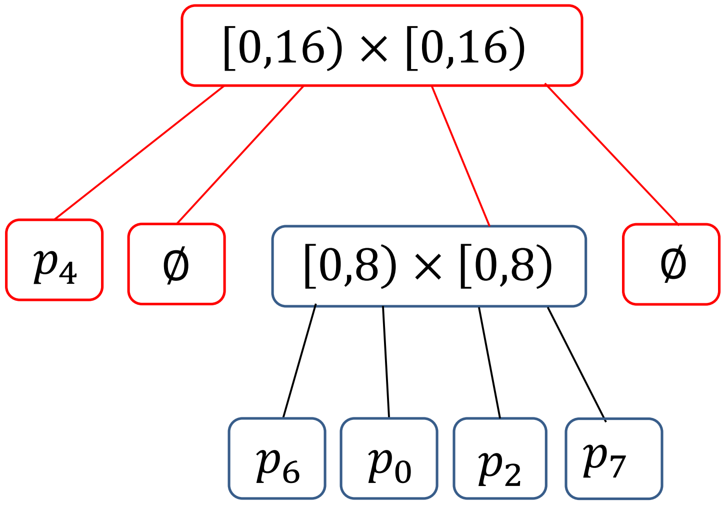




- If we insert point outside the bounding box, no need to rebuild the tree due to bounding box being $[0, 2^k) \times [0, 2^k)$

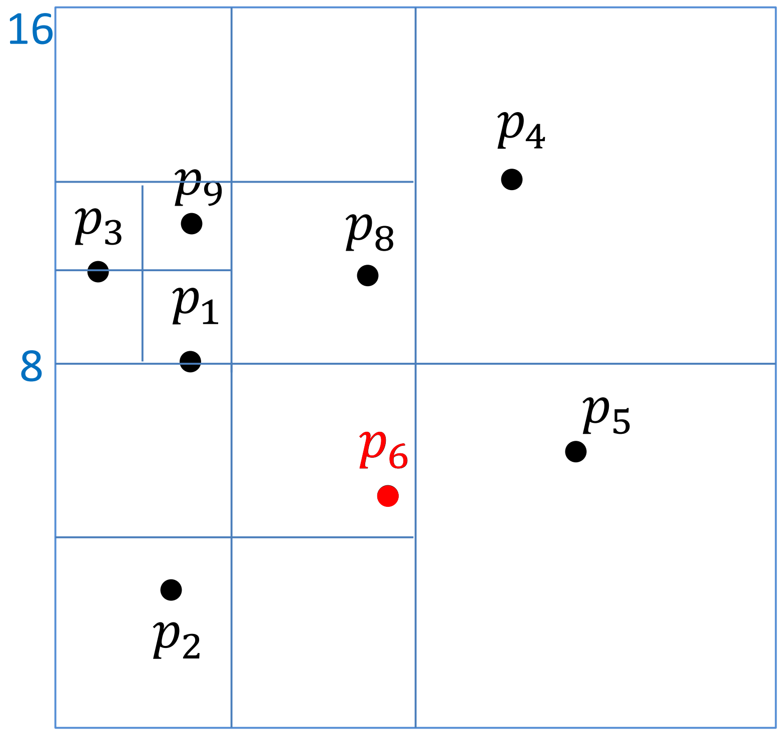
Quadtree Insert

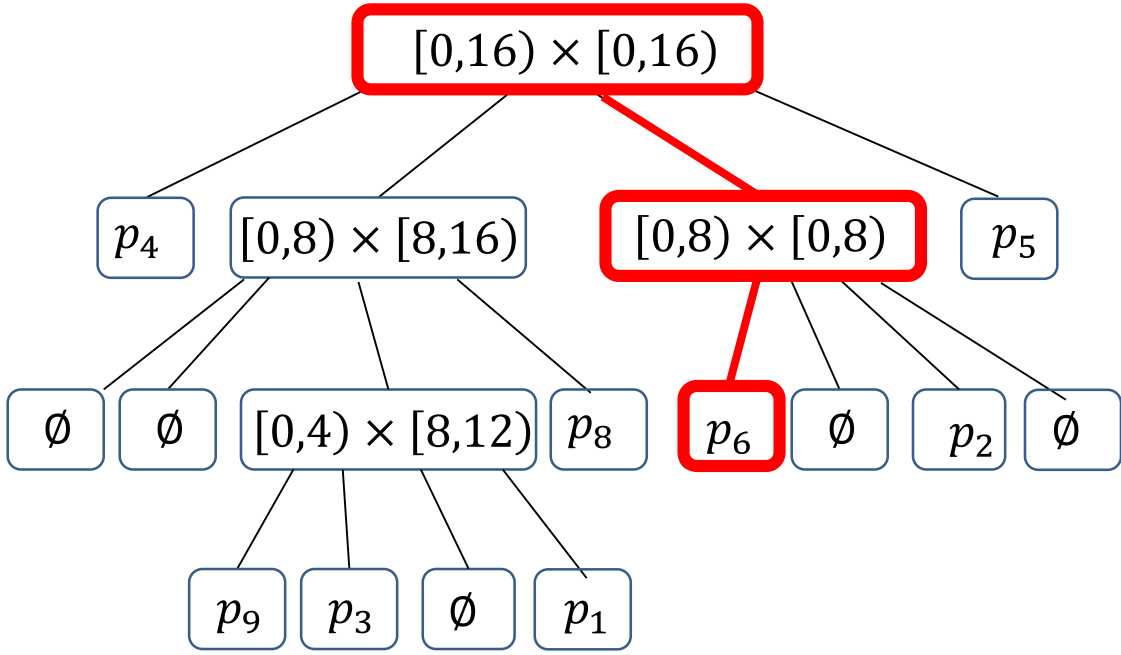




- If we insert point outside the bounding box, no need to rebuild the tree due to bounding box being $[0, 2^k) \times [0, 2^k)$

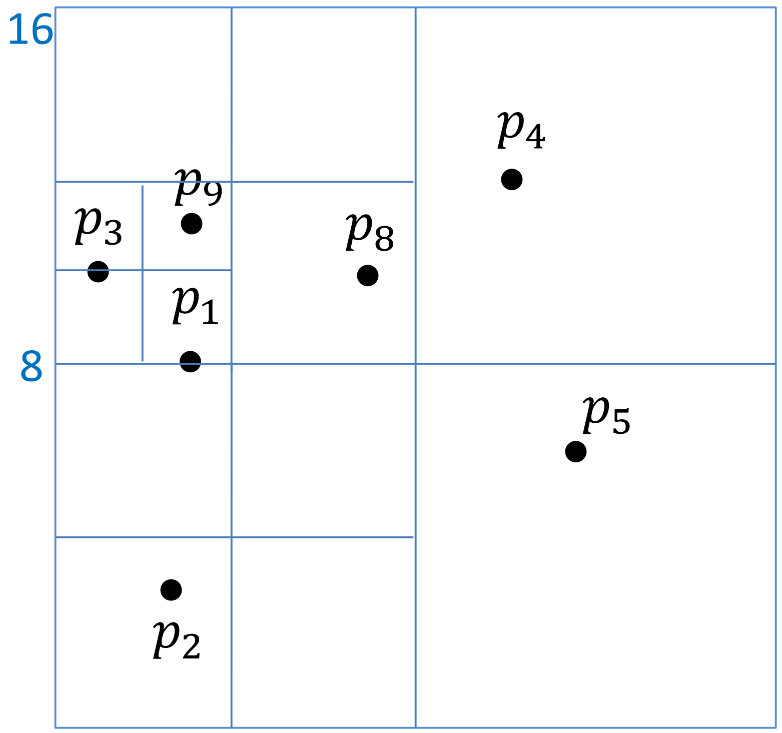
Quadtree Delete

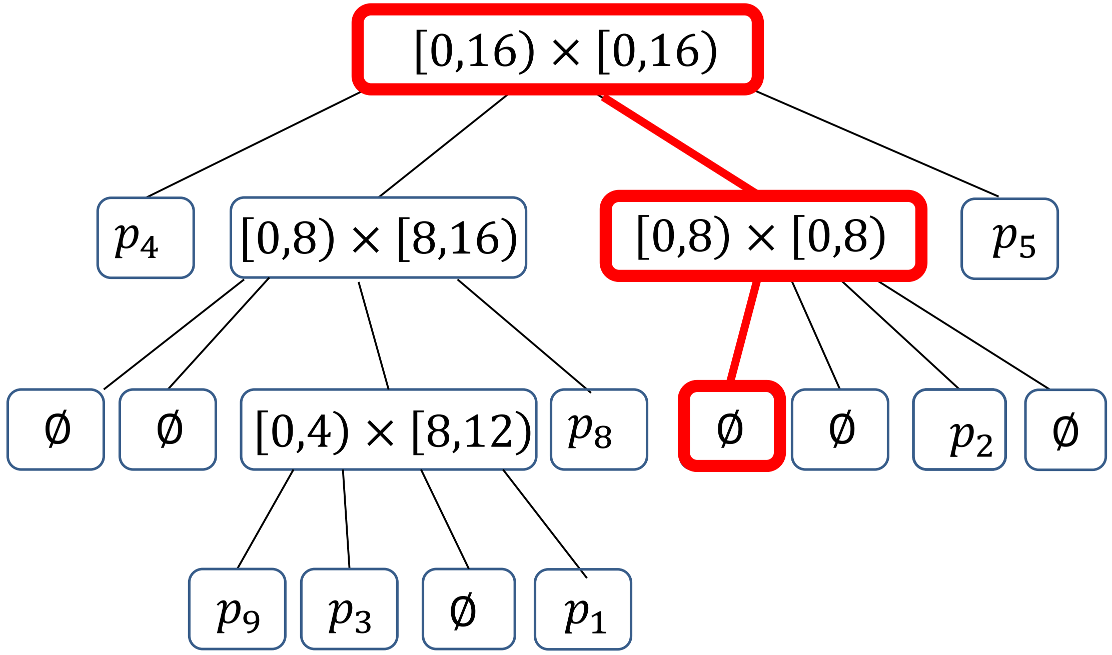




- search for leaf containing the point
    - example: delete(p_6)
- replace the leaf by empty subtree

Quadtree Delete





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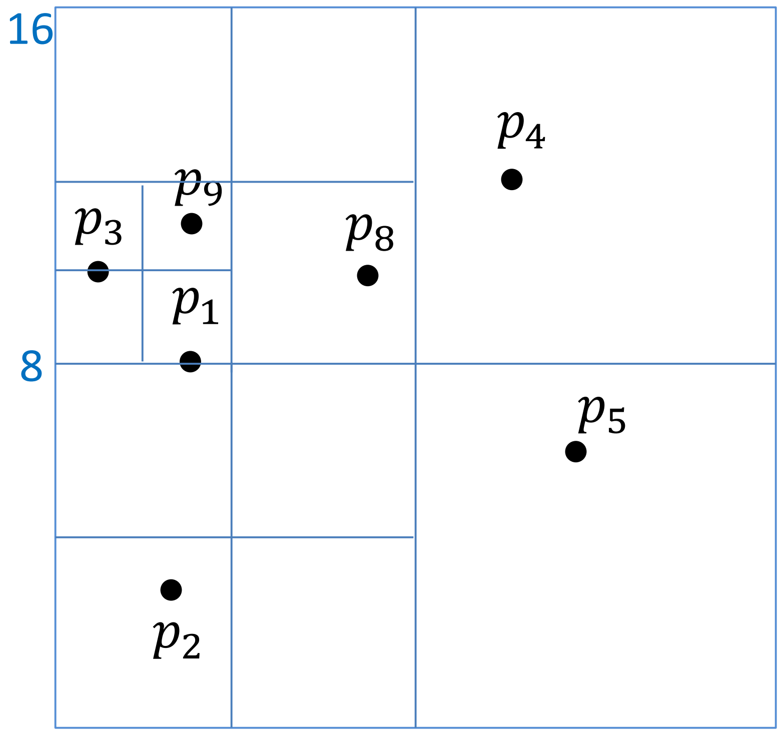
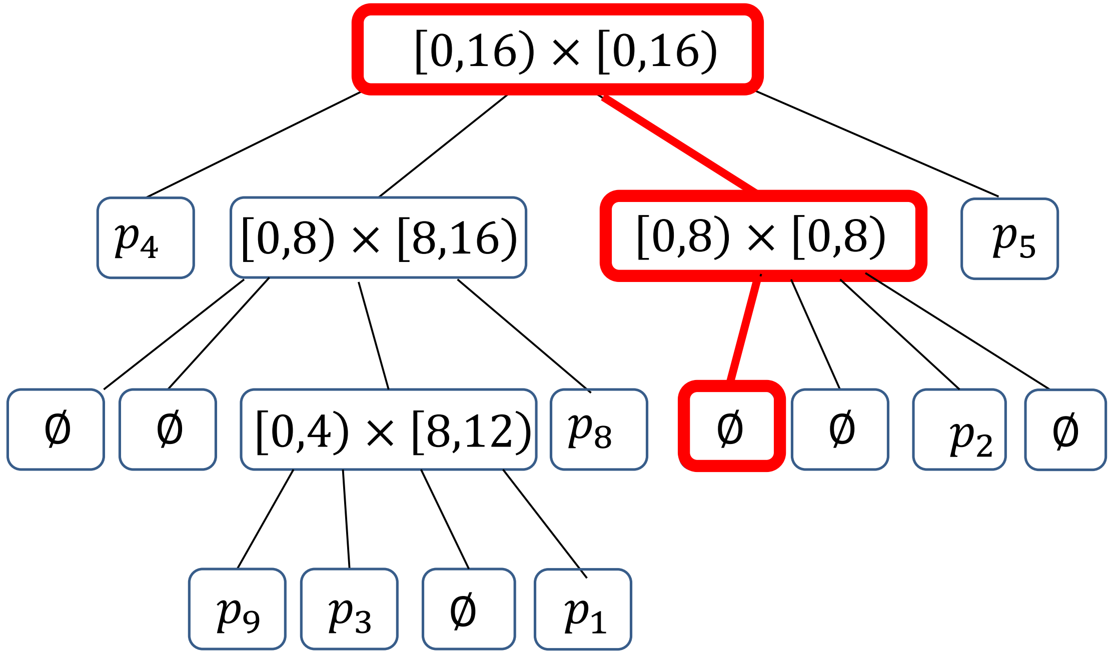
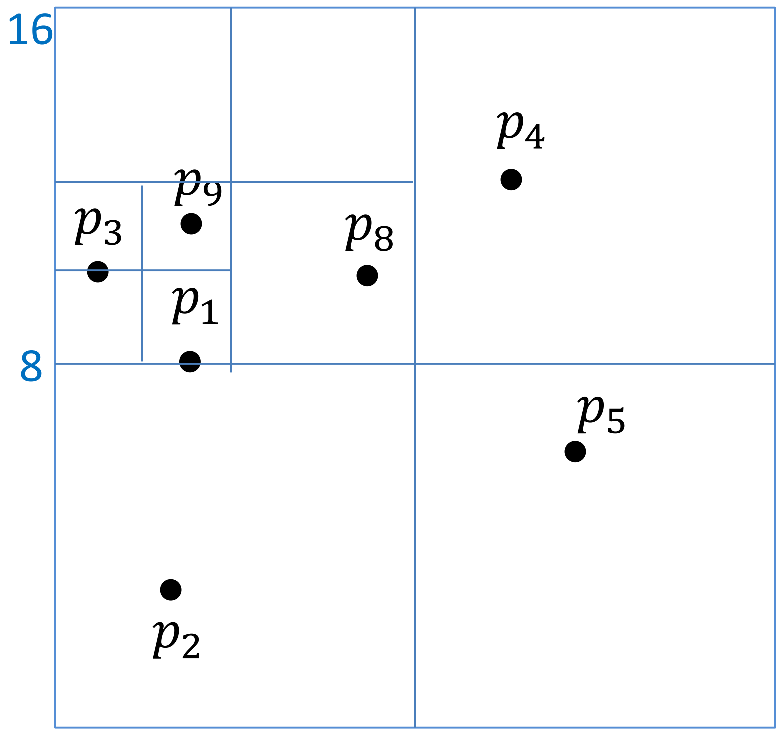
Quadtree Delete

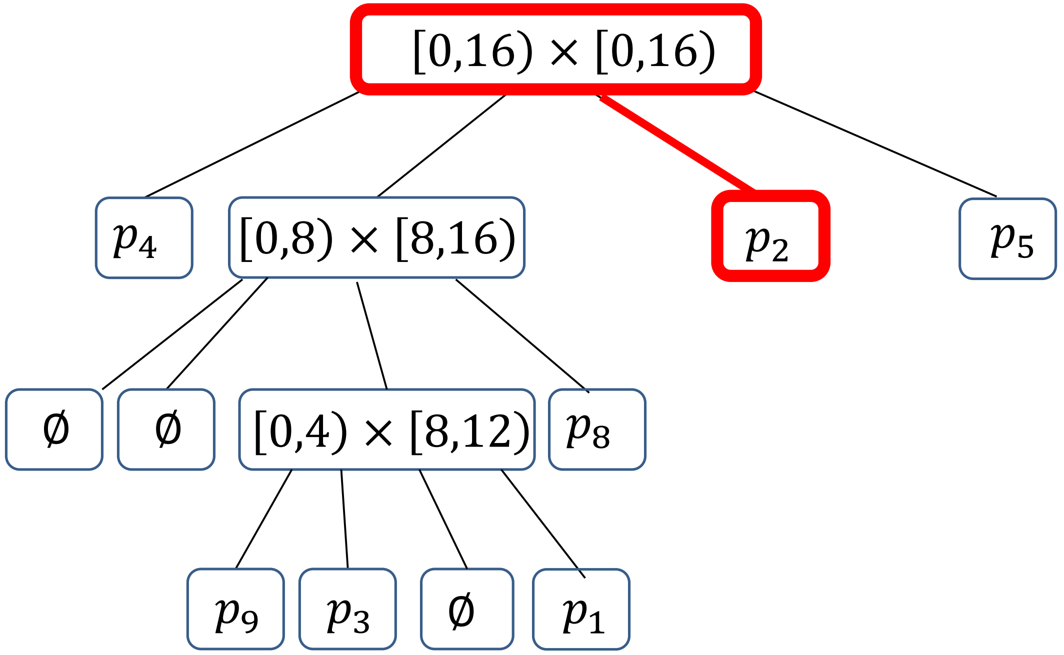




- search for leaf containing the point
    - example: delete(p_6)
- replace the leaf by empty subtree
- if parent has only one child, and the child is a *leaf*
    - delete parent, make the only leaf child to be a child of its grandparent
        - or quadtree root if no grandparent

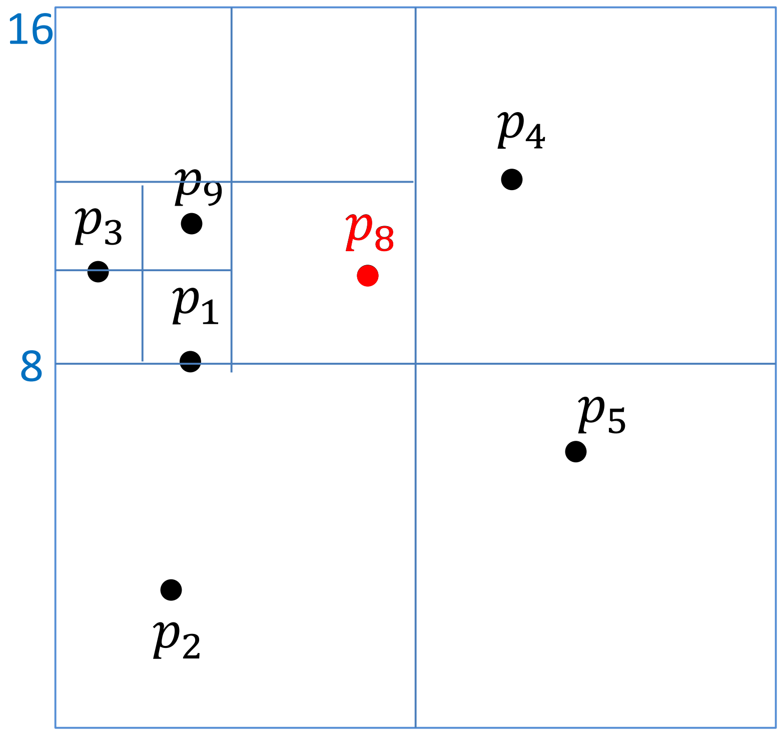
Quadtree Delete

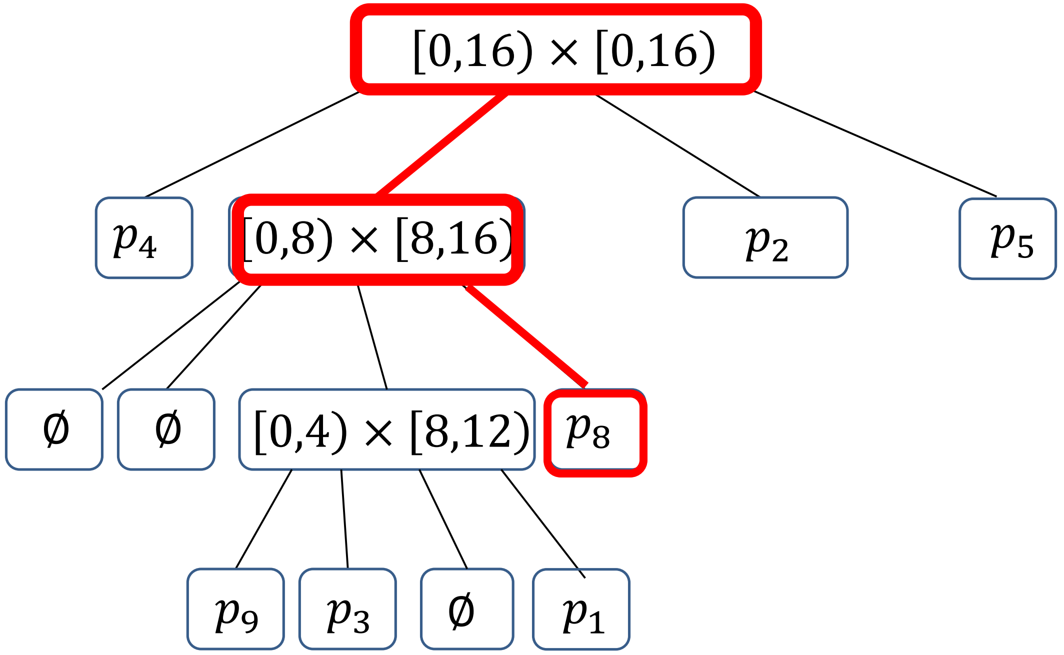




- search for leaf containing the point
    - example: delete(p_6)
- replace the leaf by empty subtree
- if parent has only one child, and the child is a *leaf*
    - delete parent, make the only leaf child to be a child of its grandparent
    - if deleted parent, may cause a series of deletes going up the tree

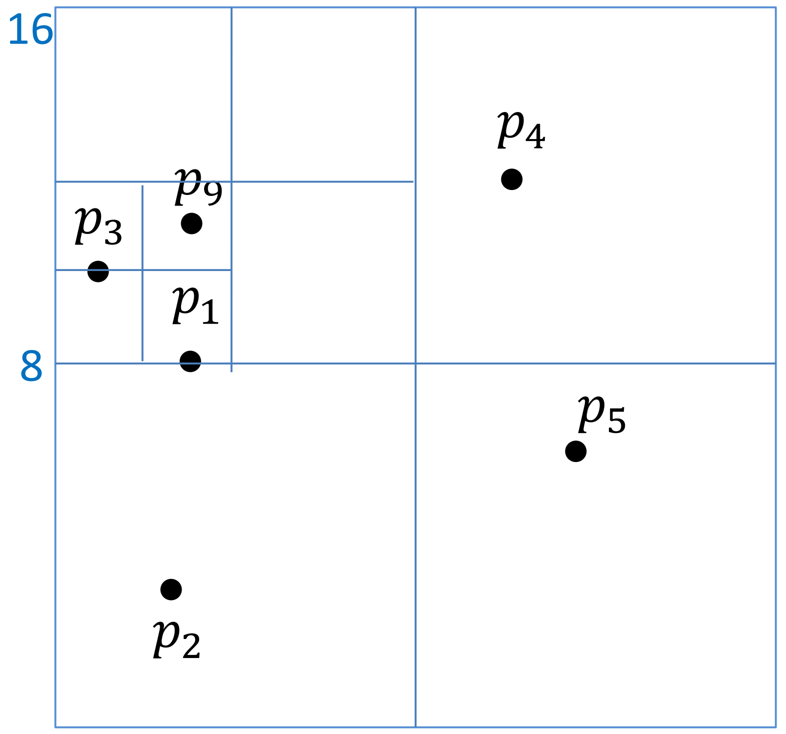
Quadtree Delete

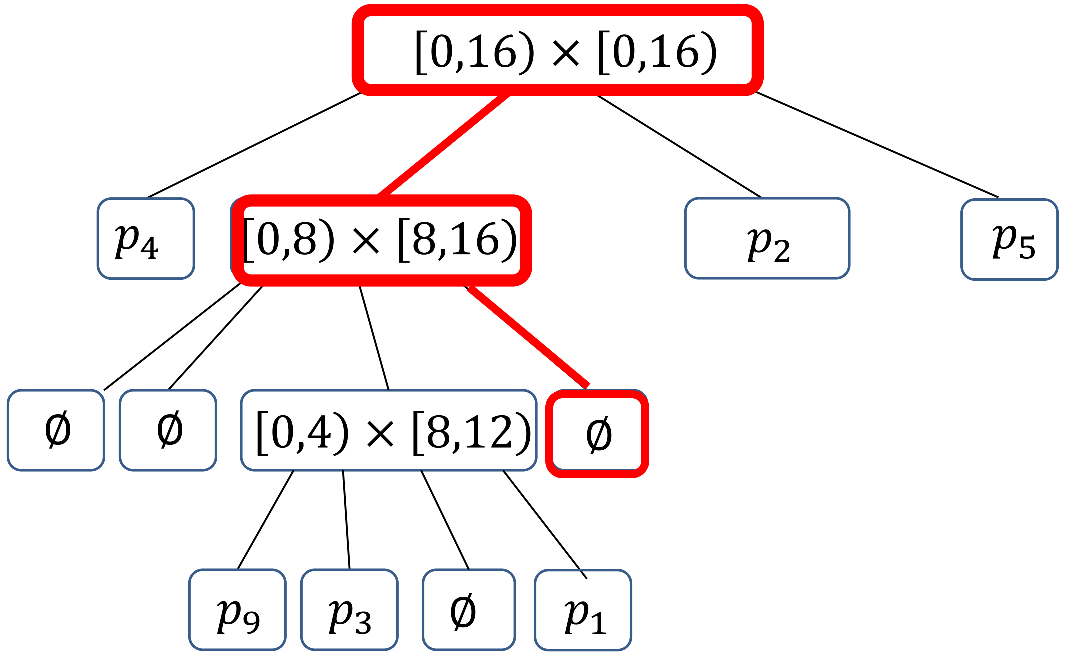




- Cannot delete if parent has a single child, but *non-leaf* child
    - example: delete(p_8)

Quadtree Delete

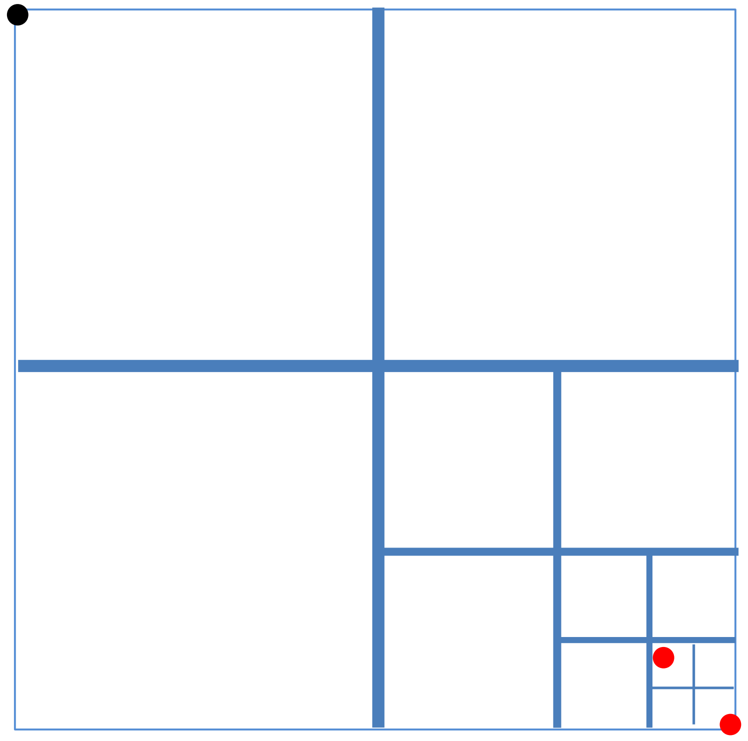




- Cannot delete if parent has a single child, but *non-leaf* child
    - non-leaf child corresponds to a region storing more than 1 point
    - example: delete(p_8)

Quadtree Analysis

$$\text{height} = 4$$



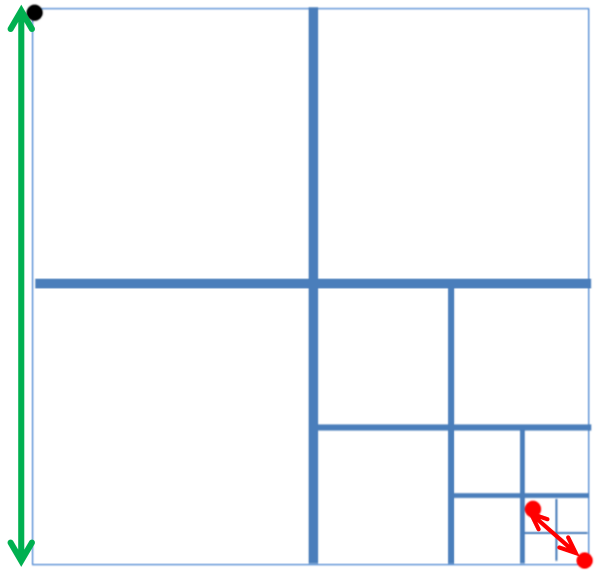
- Search, insert, delete depend on quadtree height
- What is the height of a quadtree?
    - can have very large height for bad distributions of points
    - example with just three points
    - can make height arbitrarily large by moving red points closer together

Quadtree Analysis

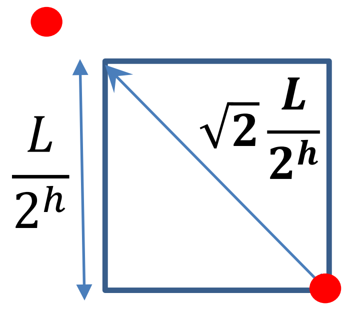
- spread factor of points S

$$\beta(S) = \frac{L}{d_{min}}$$

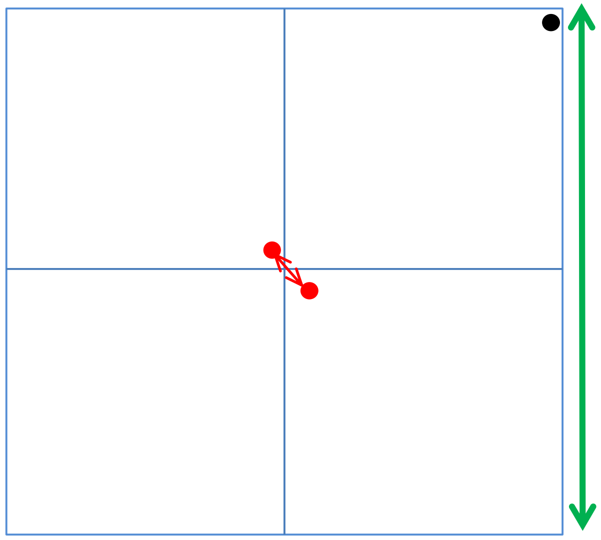
  - $L =$ side length of R
  - d_{min} is smallest distance between two points in S



- Worst case: height $h \in \Omega(\log \beta(S))$
  - until smallest region diagonal is $< d_{min}$, 2 red points are in the same region
  - if height is h, then we do h rounds of subdivisions
  - after h subdivisions, smallest regions have side length $\frac{L}{2^h}$
  - diagonal in smallest region is $\sqrt{2}\frac{L}{2^h}$
  - smallest region contains one red point $\Rightarrow \sqrt{2}\frac{L}{2^h} < d_{min}$
  - rearrange: $\sqrt{2}\frac{L}{d_{min}} < 2^h$
  - take log of both sides: $h > \log\left(\sqrt{2}\frac{L}{d_{min}}\right) = \log(\sqrt{2}\beta(S))$



Quadtree Analysis



- spread factor of points S

$$\beta(S) = \frac{L}{d_{min}}$$

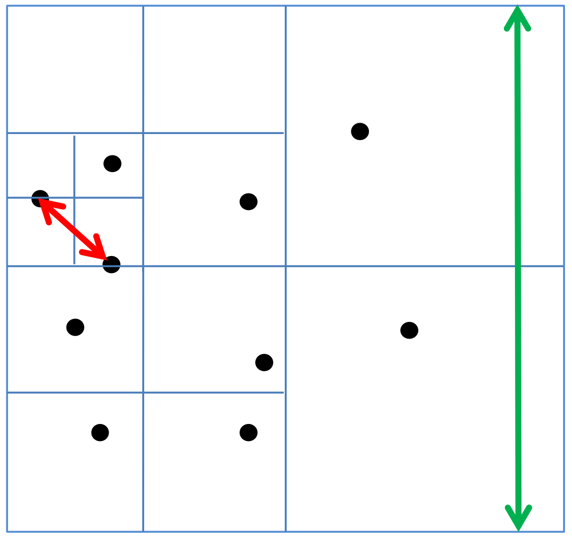
  - $L =$ side length of R
  - d_{min} is smallest distance between two points in S
- In the worst case, height $h \in \Omega(\log \beta(S))$
- However, height can be much better even if the spread is large

Quadtree Analysis

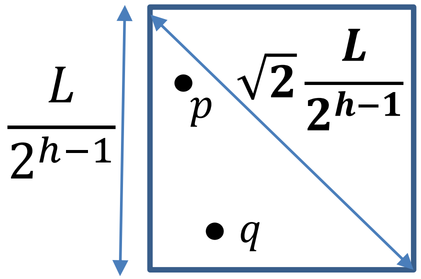
- spread factor of points S

$$\beta(S) = \frac{L}{d_{min}}$$

  - $L =$ side length of R
  - d_{min} is smallest distance between two points in S



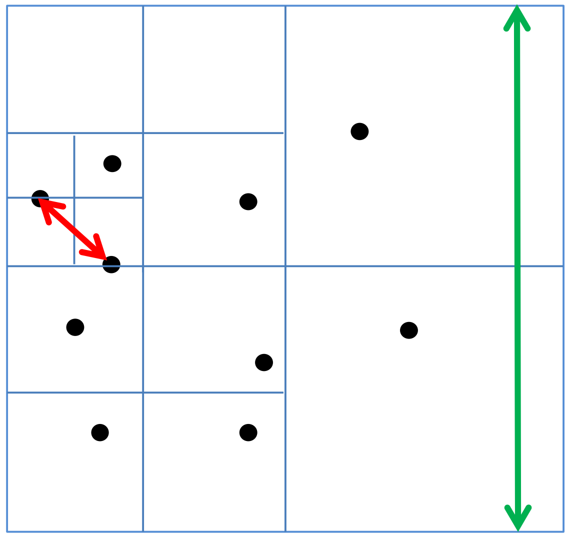
- In the worst case, height $h \in \Omega(\log \beta(S))$
- In **any case**, height $h \in O(\log \beta(S))$
  - let v be an internal node at depth $h-1$
    - there are at lest 2 points p, q inside its region
      - $d_{min} \le d(p,q)$
    - the corresponding region has side length $\frac{L}{2^{h-1}}$
    - maximum distance between 2 points in such region is $\sqrt{2}\frac{L}{2^{h-1}}$



$$d_{min} \le d(p,q) \le \sqrt{2}\frac{L}{2^{h-1}}$$

$$2^{h-1} \le \sqrt{2}\frac{L}{d_{min}} = \sqrt{2}\,\beta(S) \quad \Rightarrow h \le 1 + \log(\sqrt{2}\beta(S))$$

Quadtree Analysis



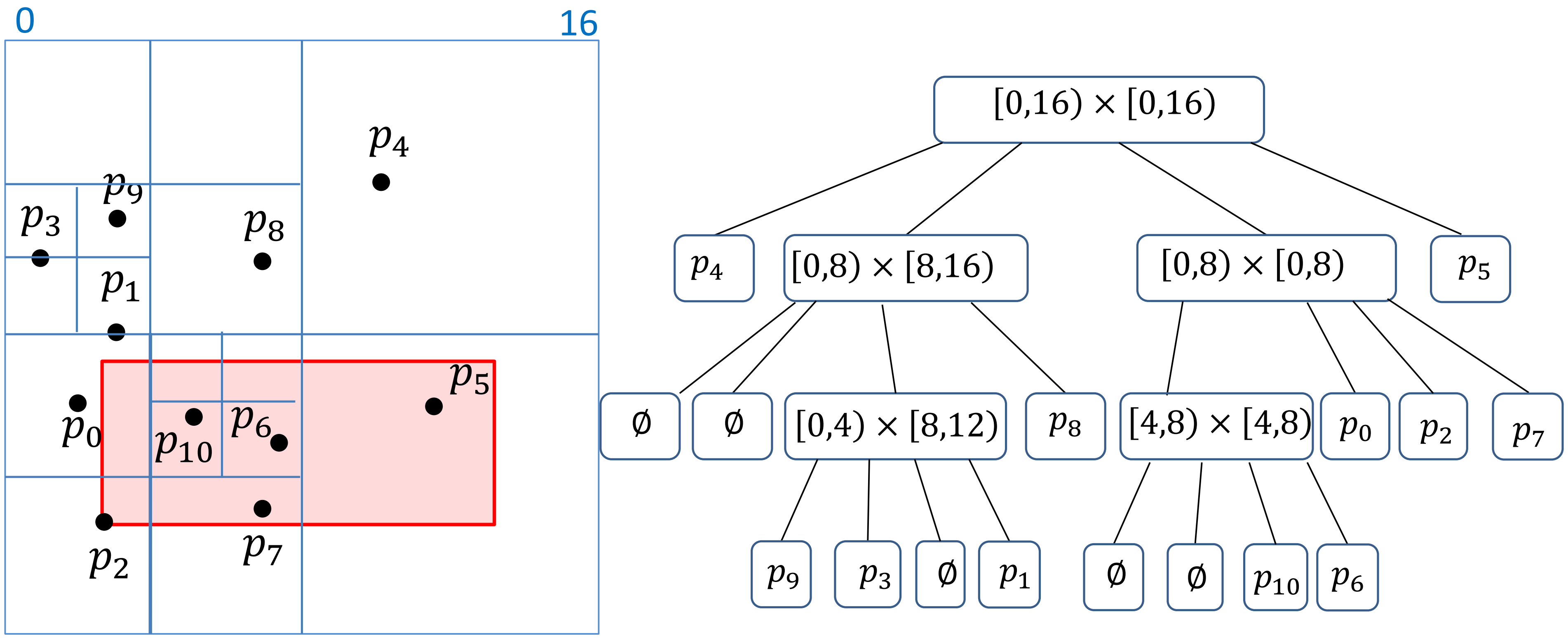
- spread factor of points S

$$\beta(S) = \frac{L}{d_{min}}$$

  - $L =$ side length of R
  - d_{min} is smallest distance between two points in S

- In the worst case, height $h \in \Omega(\log \beta(S))$
- In any case, height $h \in O(\log \beta(S))$
  - to guarantee good performance, $\log \beta(S)$ should be much smaller than n
- Complexity to build initial tree: $\Theta(nh)$ worst-case
  - expensive if large height (as compared to the number of points)

Quadtree Range Search Example



- Query rectangle A $= [3 \le x < 13, 3 \le y < 7]$
- Let R be region associated with current node, have 3 cases
  1. $R \cap A = \emptyset$: red (inside) node, do not search its children
  2. $R \subseteq A$: green (inside) node, no need to search children, report all points in R
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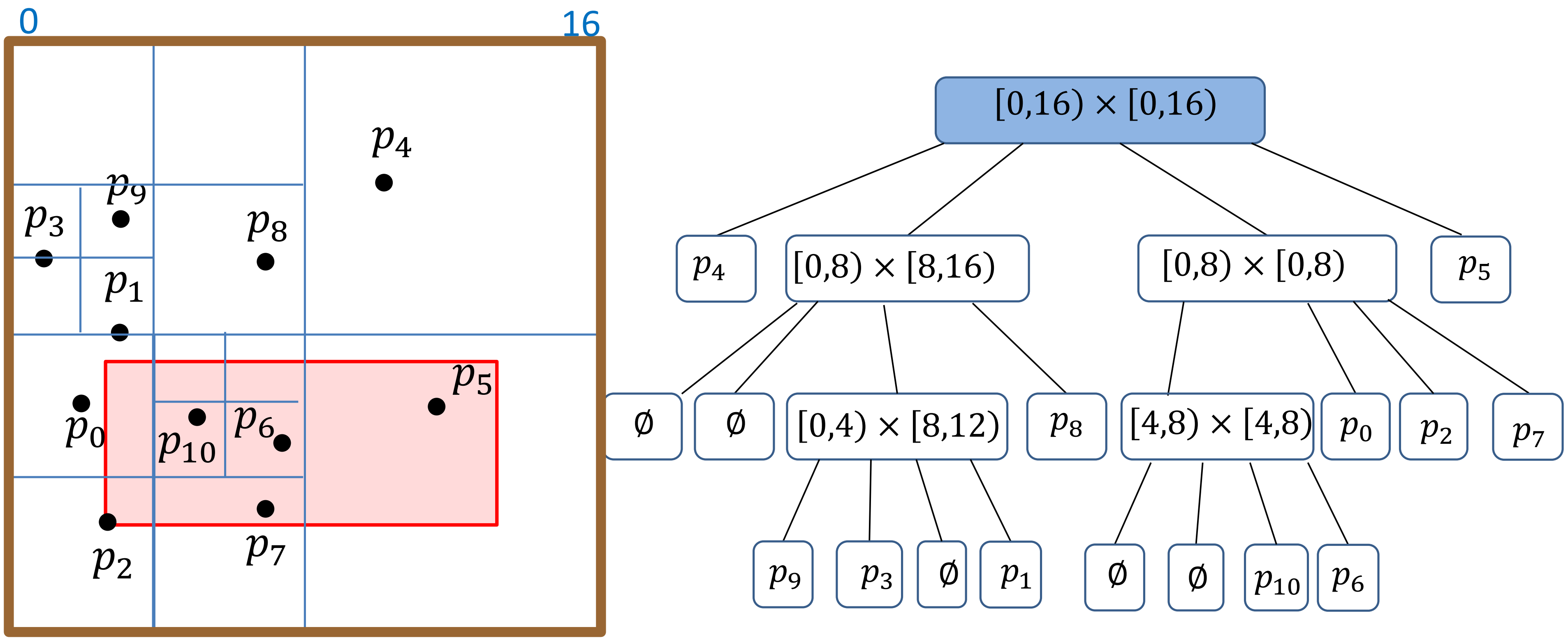

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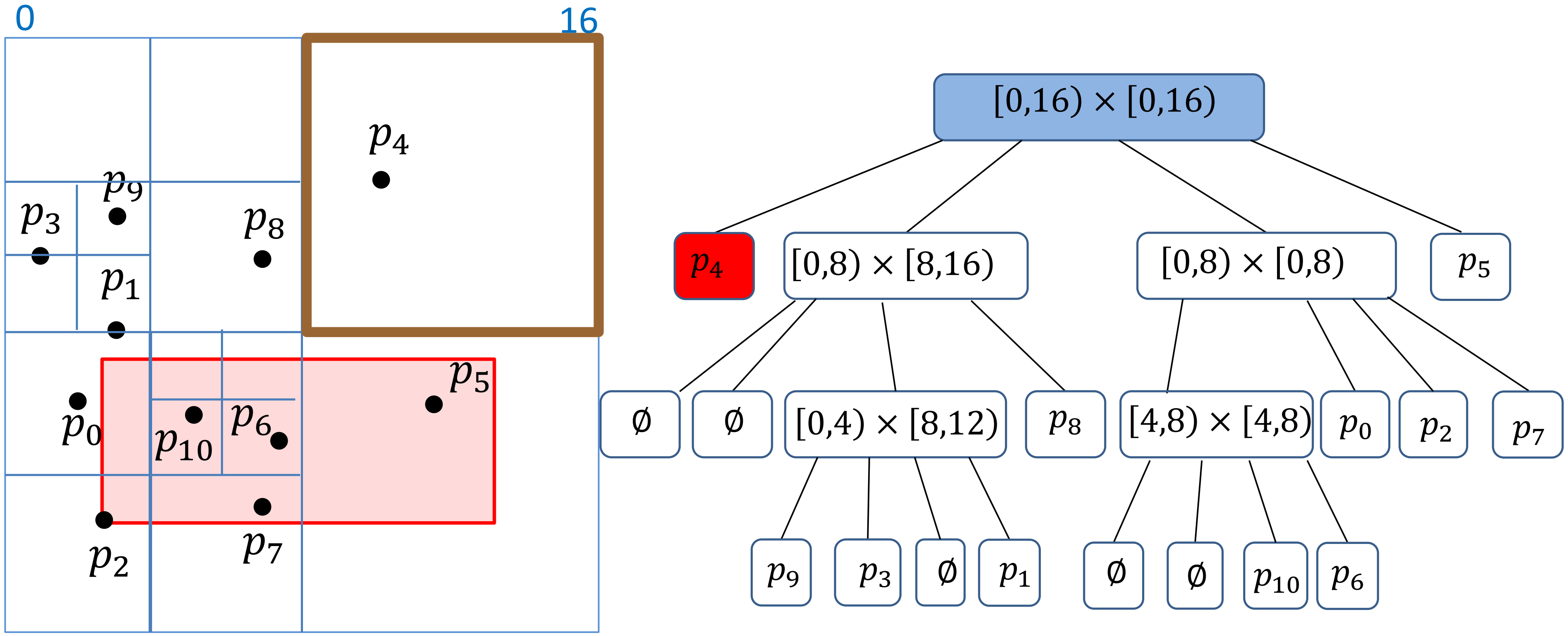

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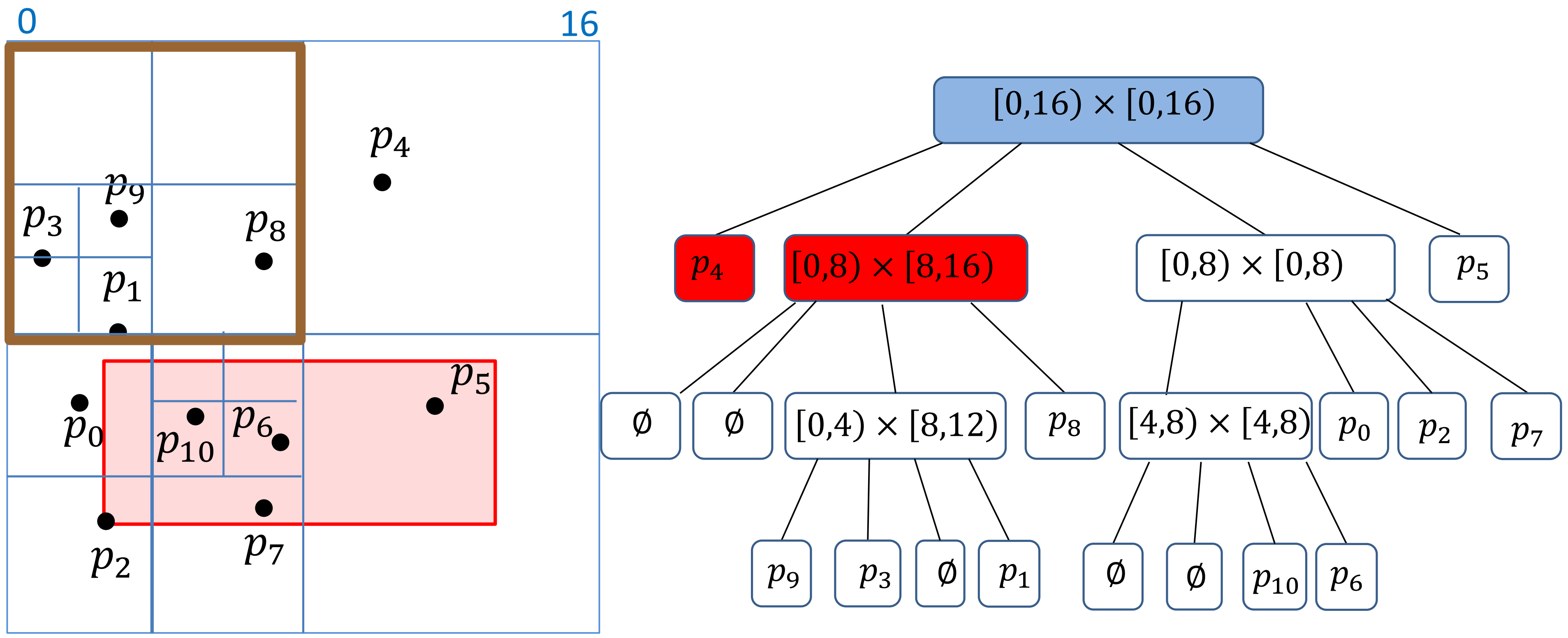

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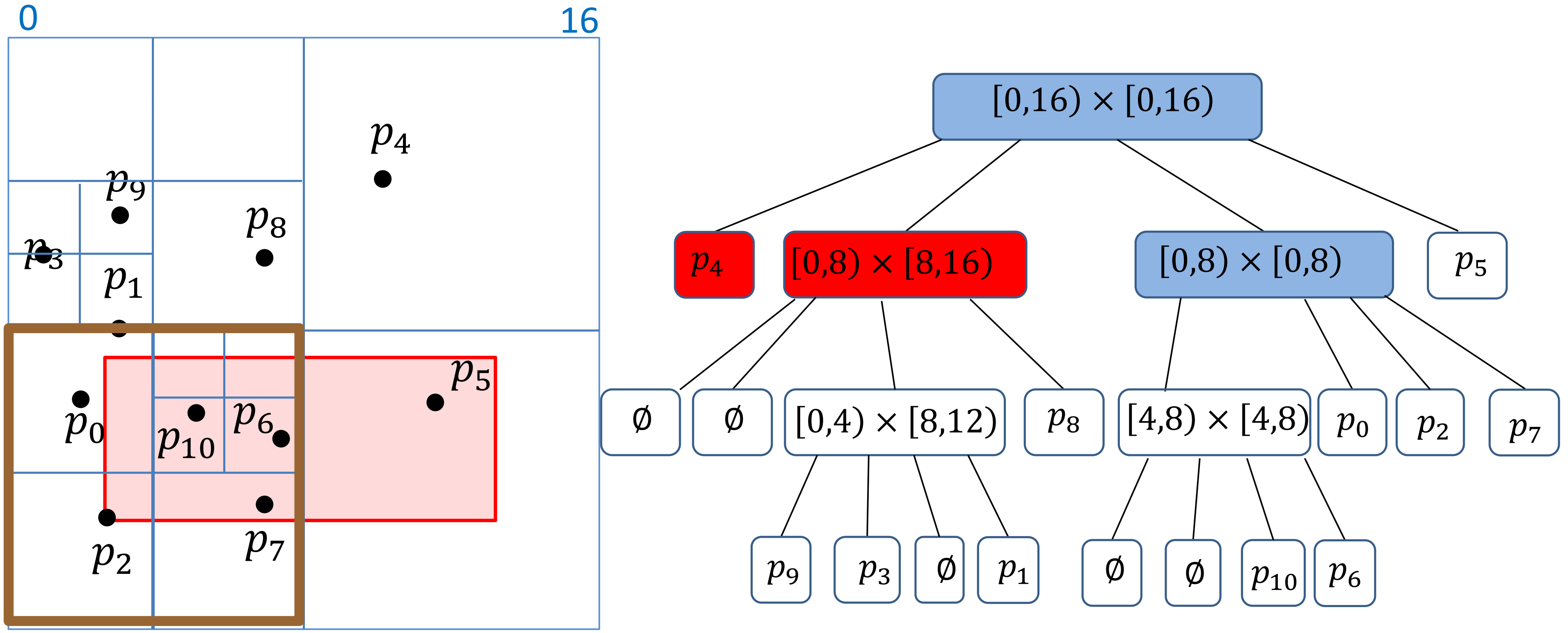

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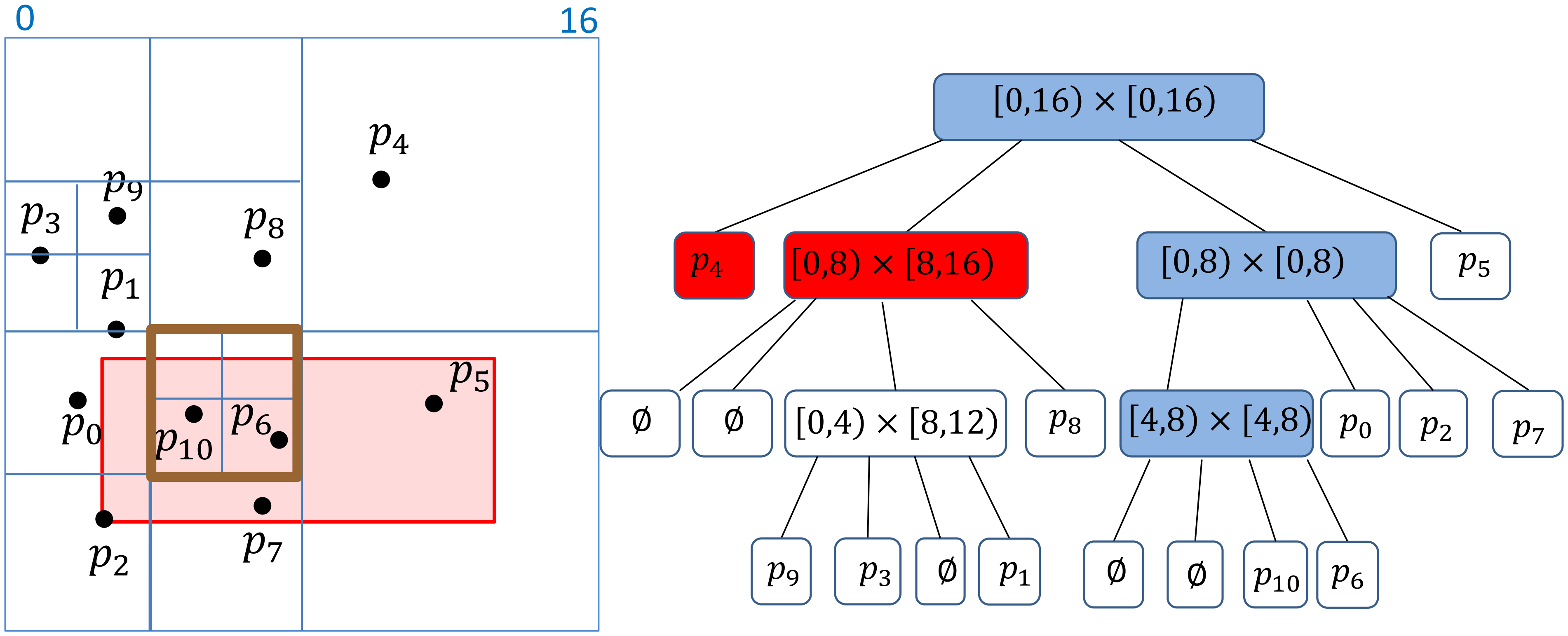

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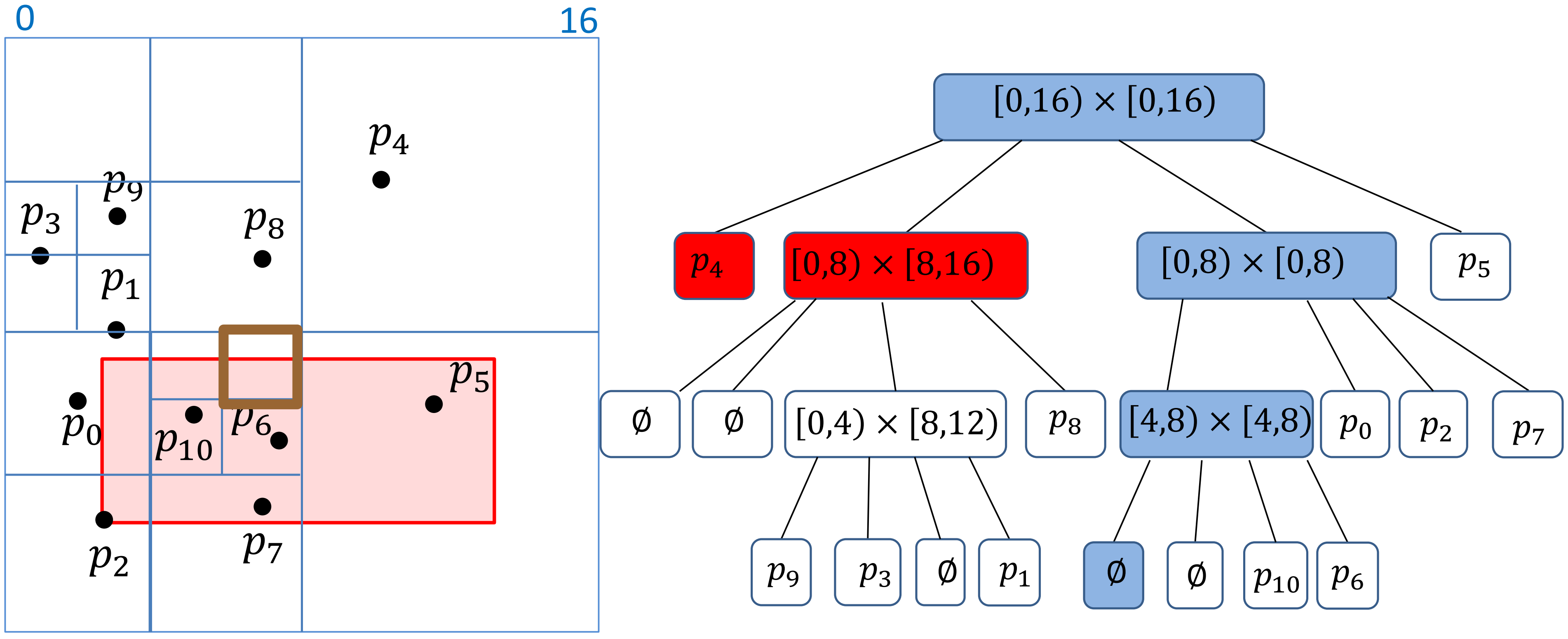

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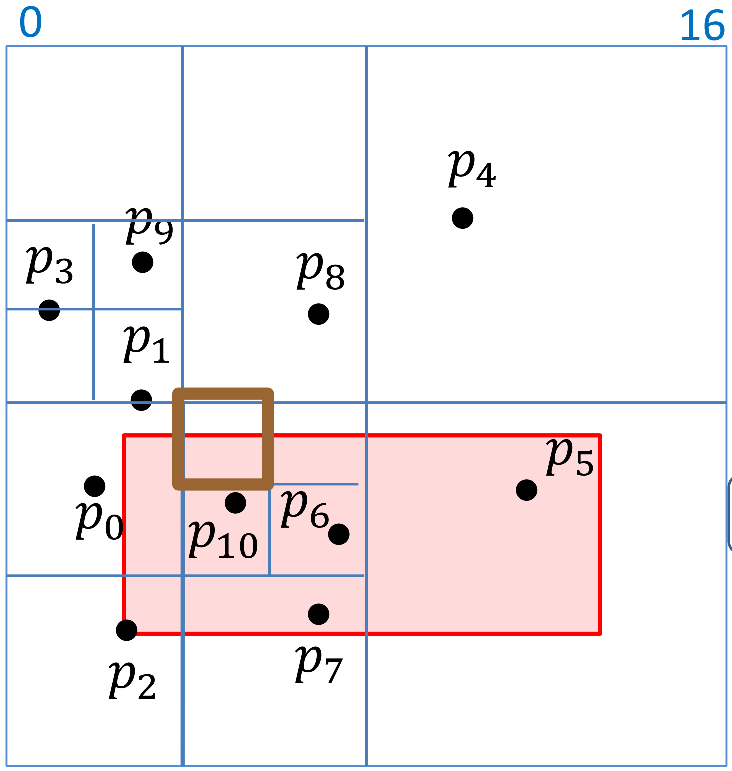

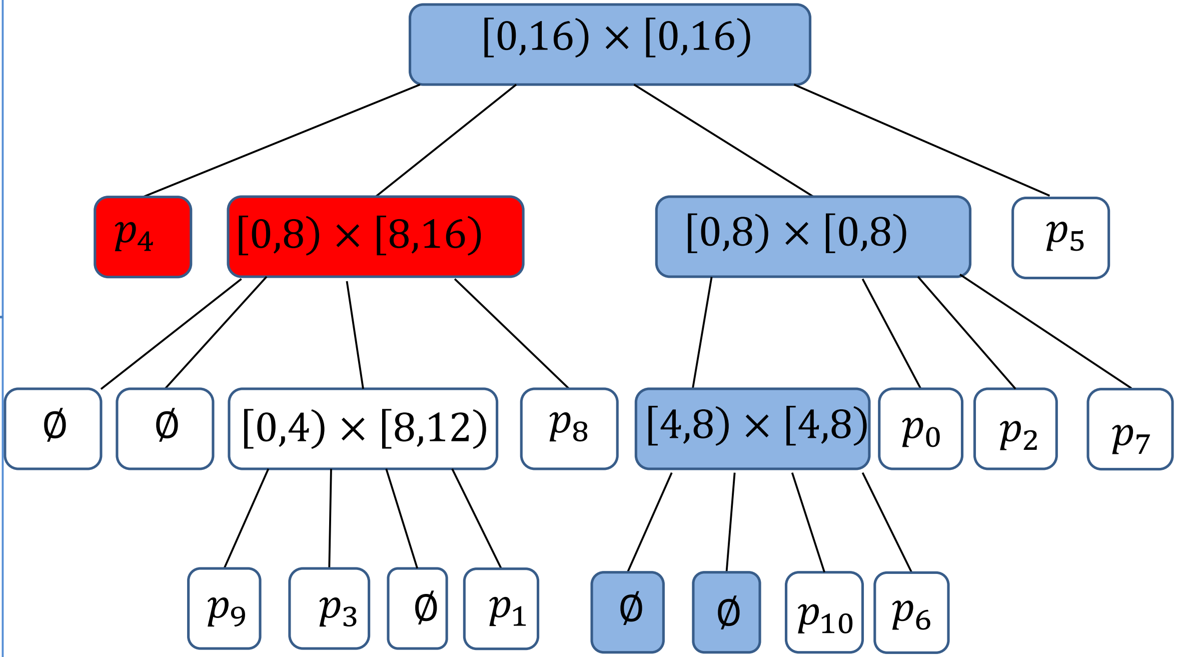

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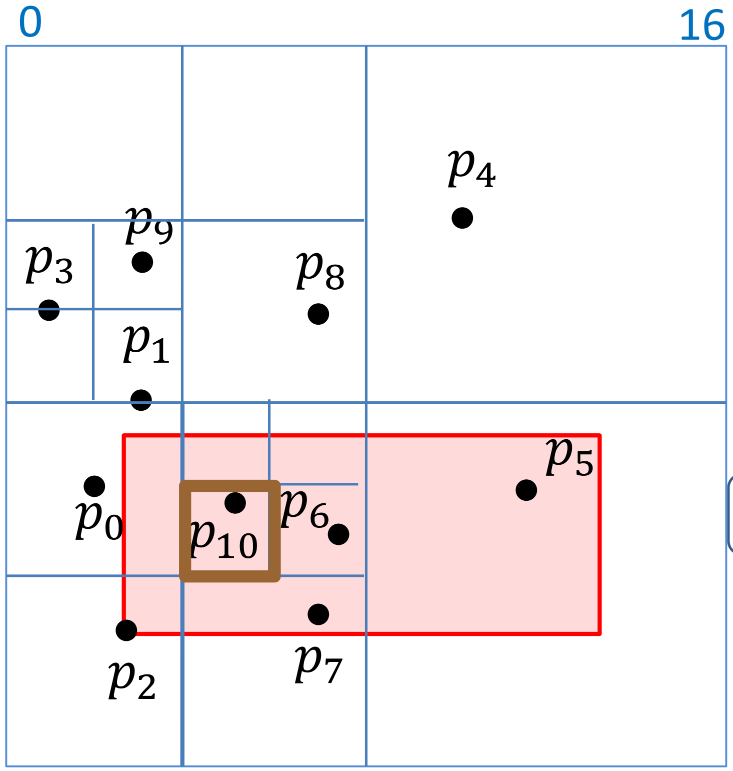

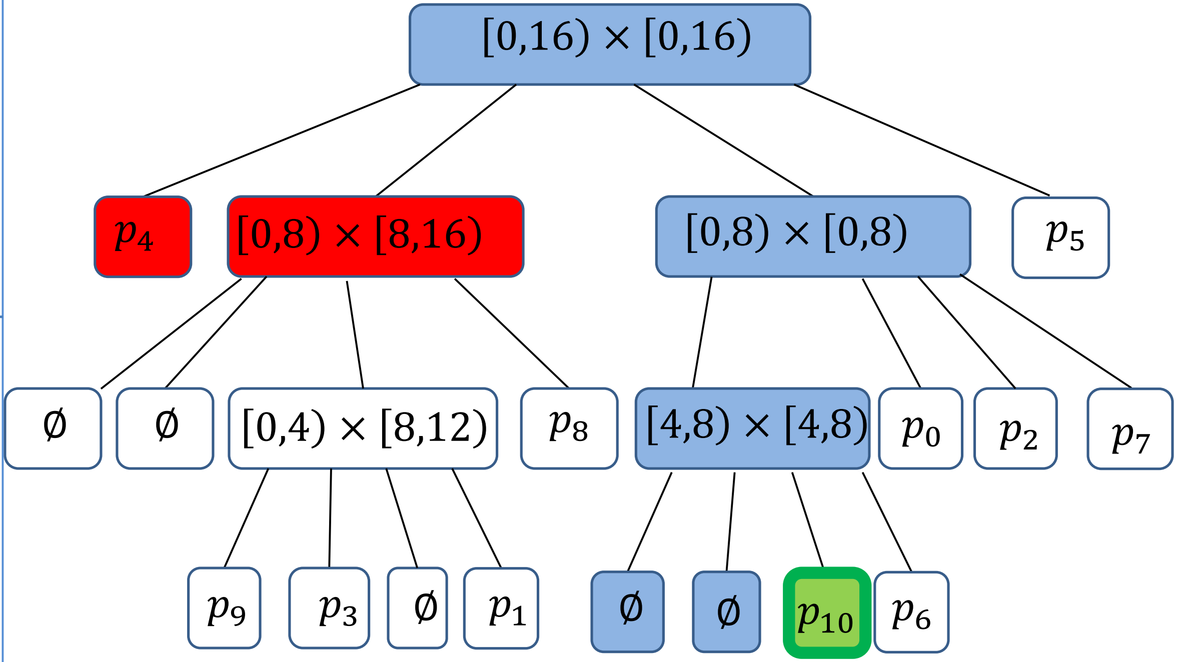

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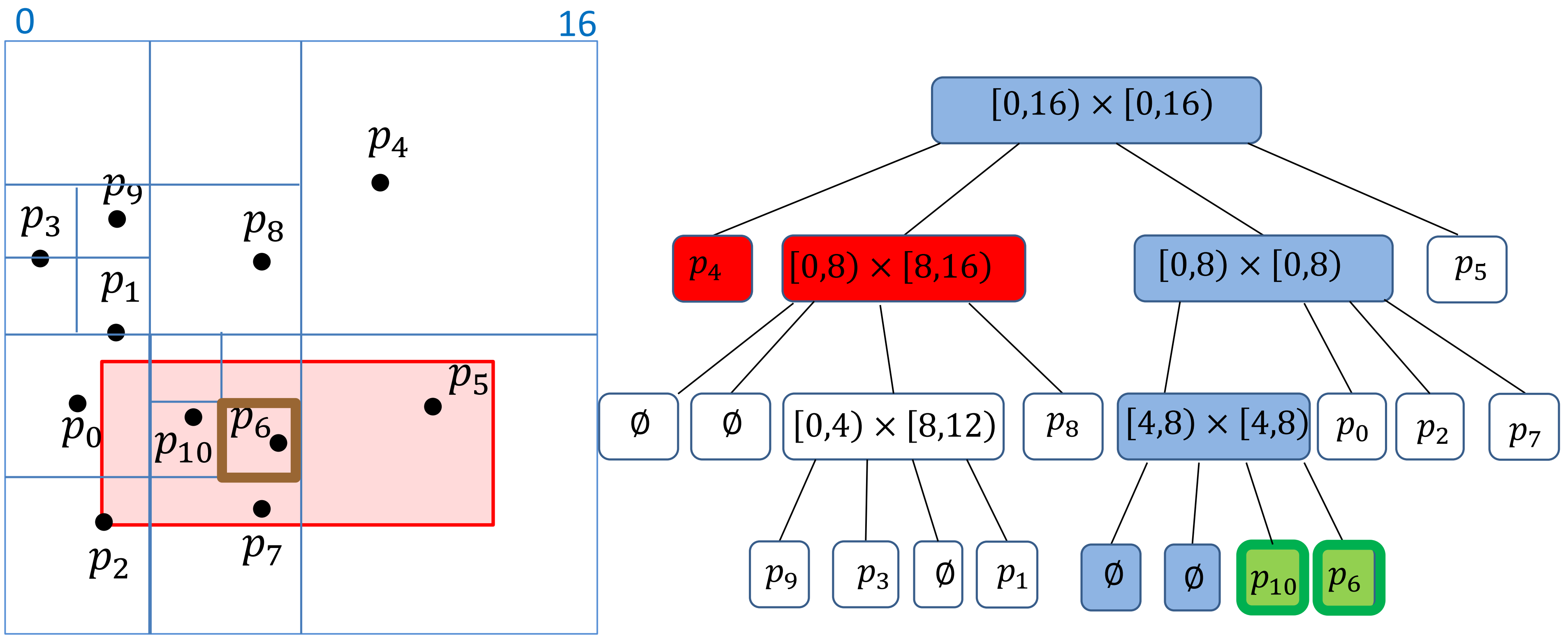

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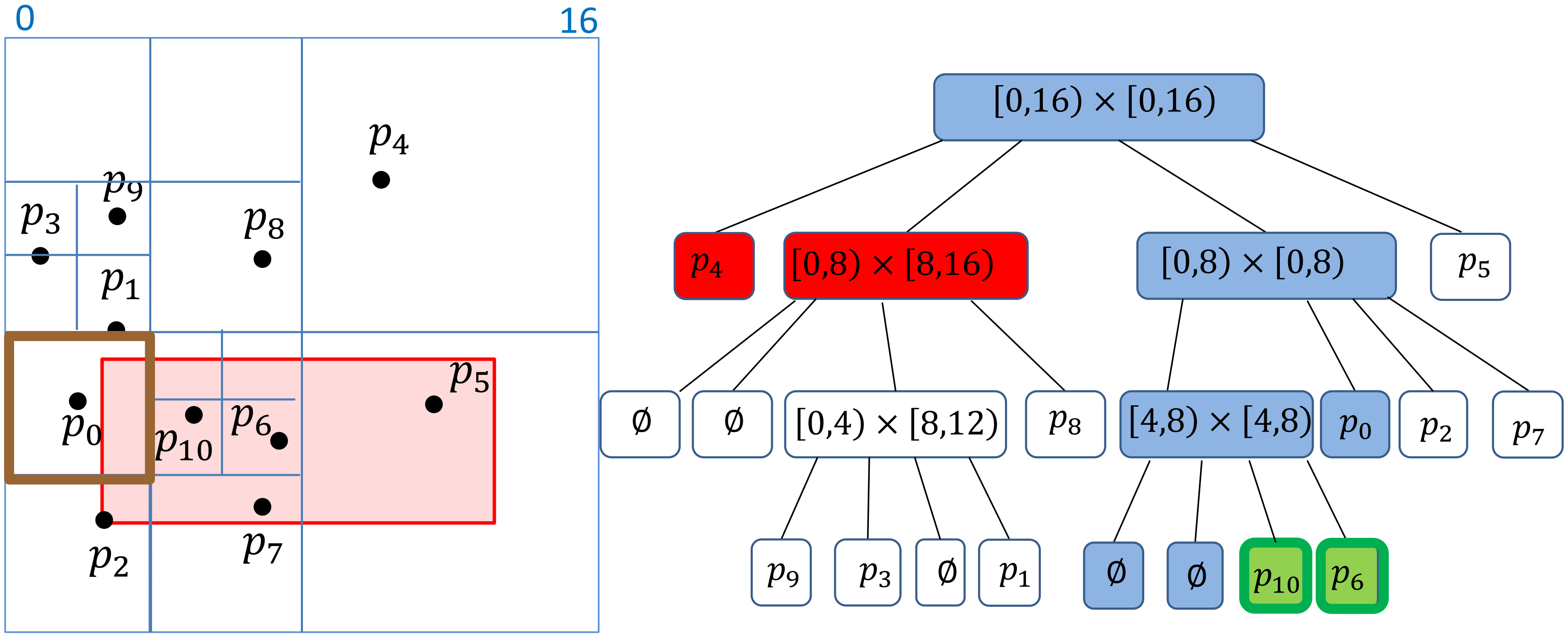

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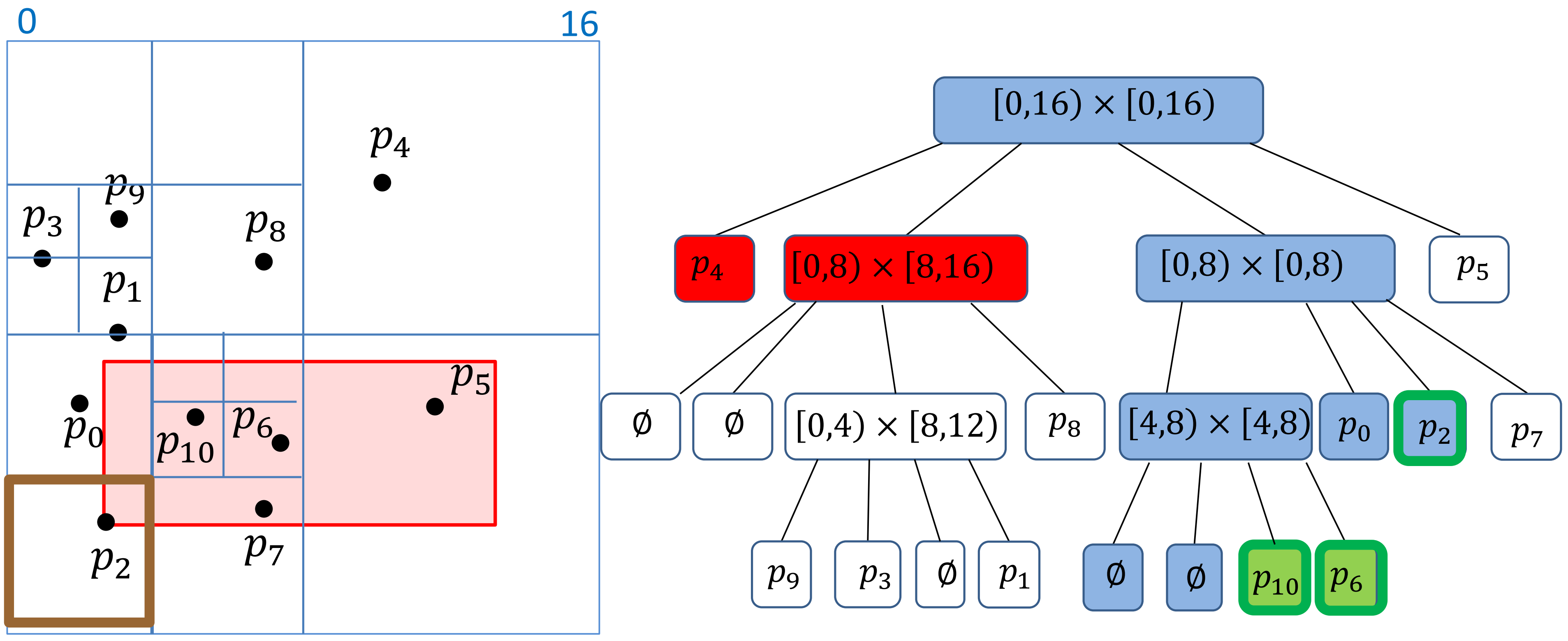

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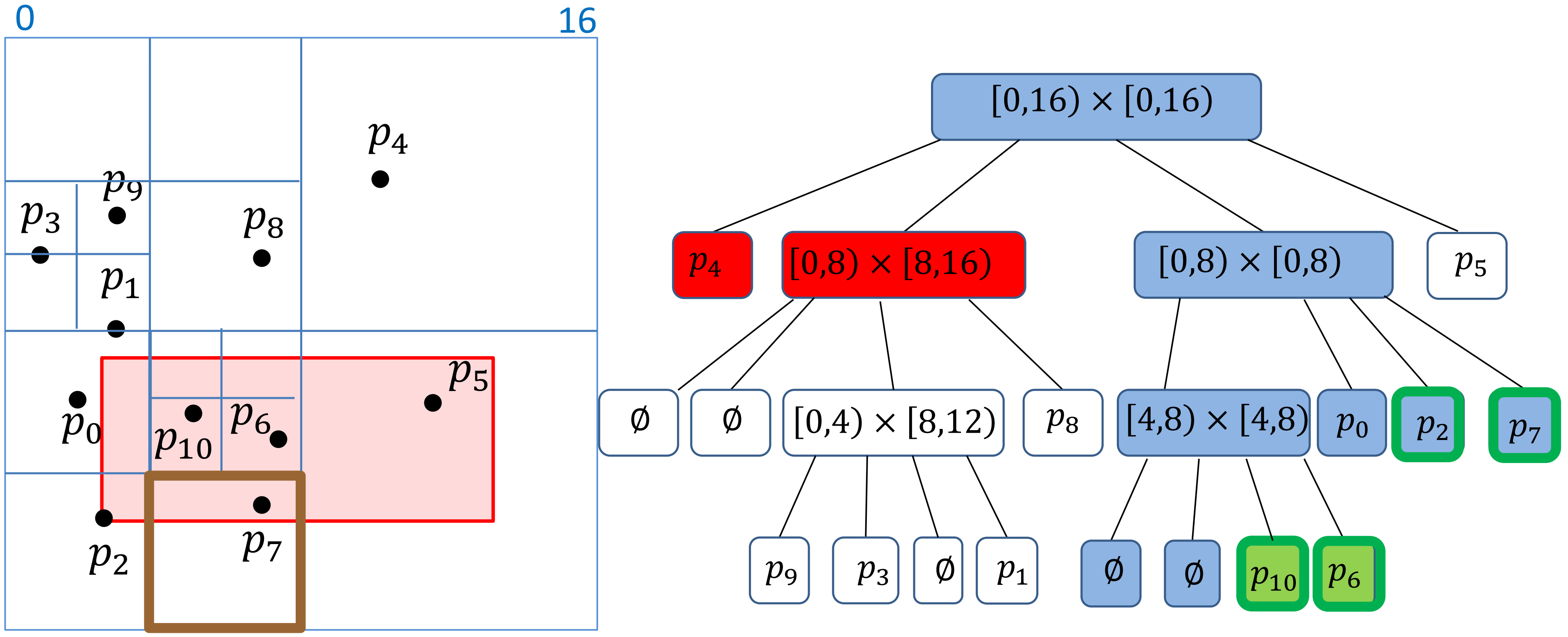

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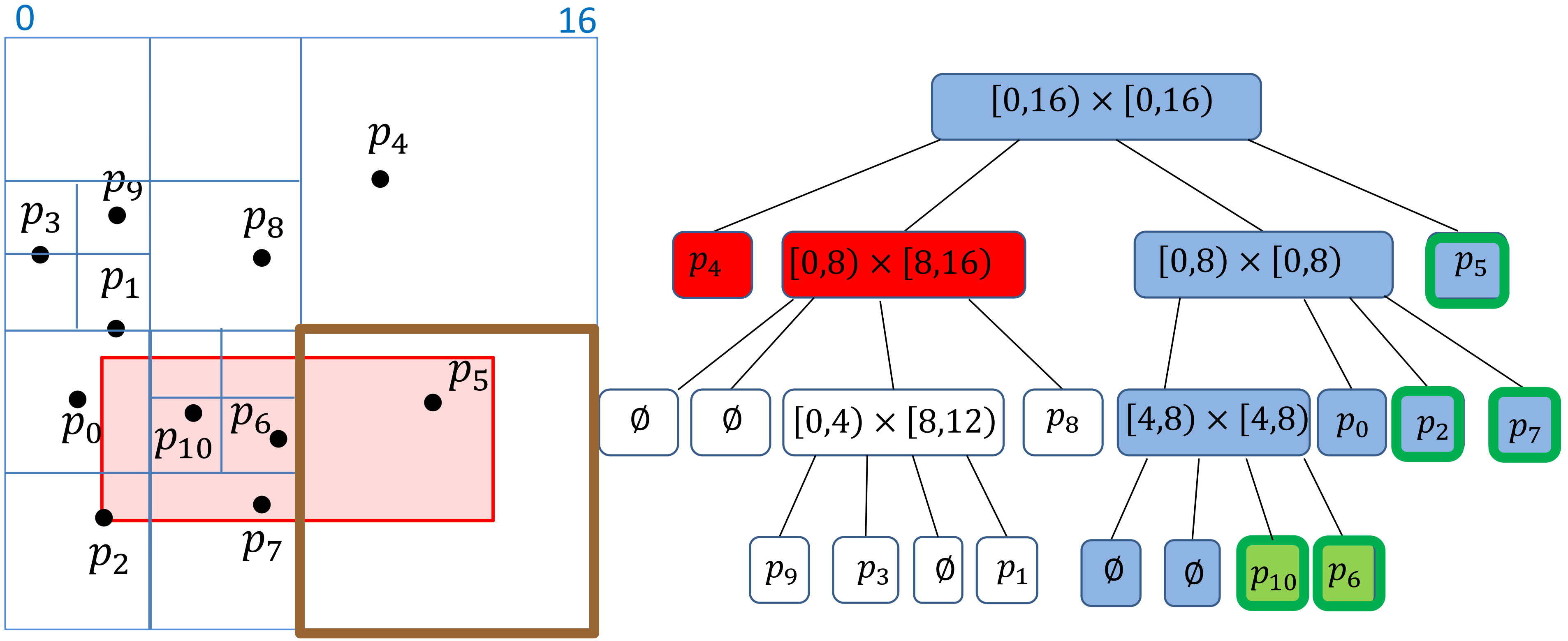

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Quadtree Range Search

```
Qtree::RangeSearch($r \leftarrow root, A$)
$r$ : quadtree root, $A$: query rectangle
    let $R$ be the region associated with $r$
    if $R \subseteq A$ then      //inside node
        report all points below $r$
        return
    if $R \cap A = \emptyset$ then //outside node
        return
    // boundary node, recurse if not a leaf
    if $r$ is a leaf then // leaf, do not recurse
        $p \leftarrow$ point stored at $r$
        if $p$ is in $A$ return $p$
        else return
    for each child $v$ of $r$ do
        QTree-RangeSearch($v, A$)
```

- Code assumes each quadtree node stores the associated square
- Alternatively, these could be re-computed during search
    - space-time tradeoff

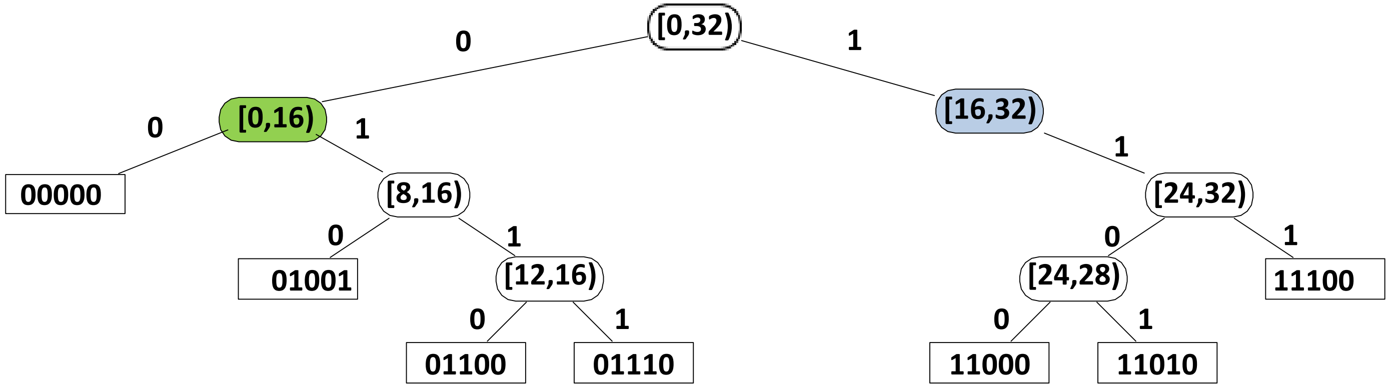
RangeSearch Analysis

- Running time is number of visited nodes + output size
- No good bound on number of visited nodes
    - may have to visit nearly all nodes in the worst case
    - $\Theta(nh)$ worst-case
        - this is worse than exhaustive search
        - even if the range search returns empty result
        - but in practice usually much faster

Quadtrees in other dimensions

points	0	9	12	14	24	26	28
base 2	00000	01001	01100	01110	11000	11010	11100

- Quad-tree of 1-dimensional points



- Same as a trie
    - with splitting stopped once key is unique

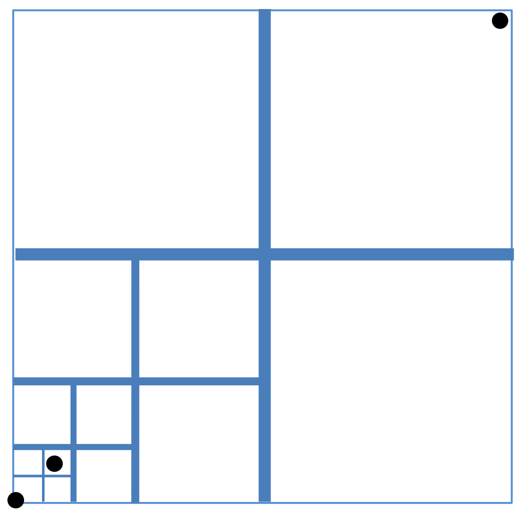
Quadtree summary

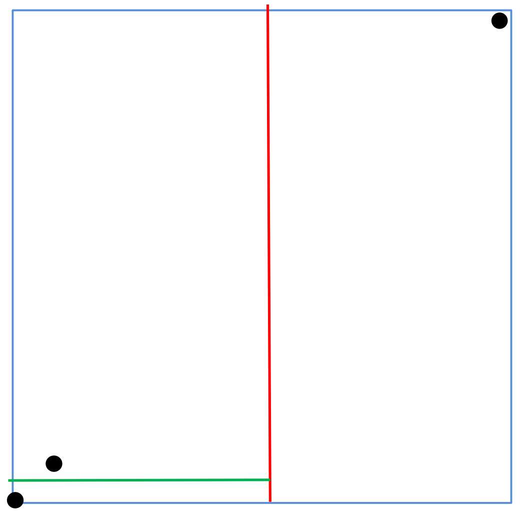
- Quadtrees easily generalize to higher dimensions
    - octrees, *etc.*
    - but rarely used beyond dimension 3
- Easy to compute and handle
- No complicated arithmetic, only divisions by 2
    - bit-shift if the width/height of R is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation
    - stop splitting earlier and allow up to k points in a leaf for some fixed k

Outline

- Range-Searching in Dictionaries for Points
    - Range Search Query
    - Multi-Dimensional Data
    - Quadtrees
    - kd-Trees
    - Range Trees
    - Conclusion

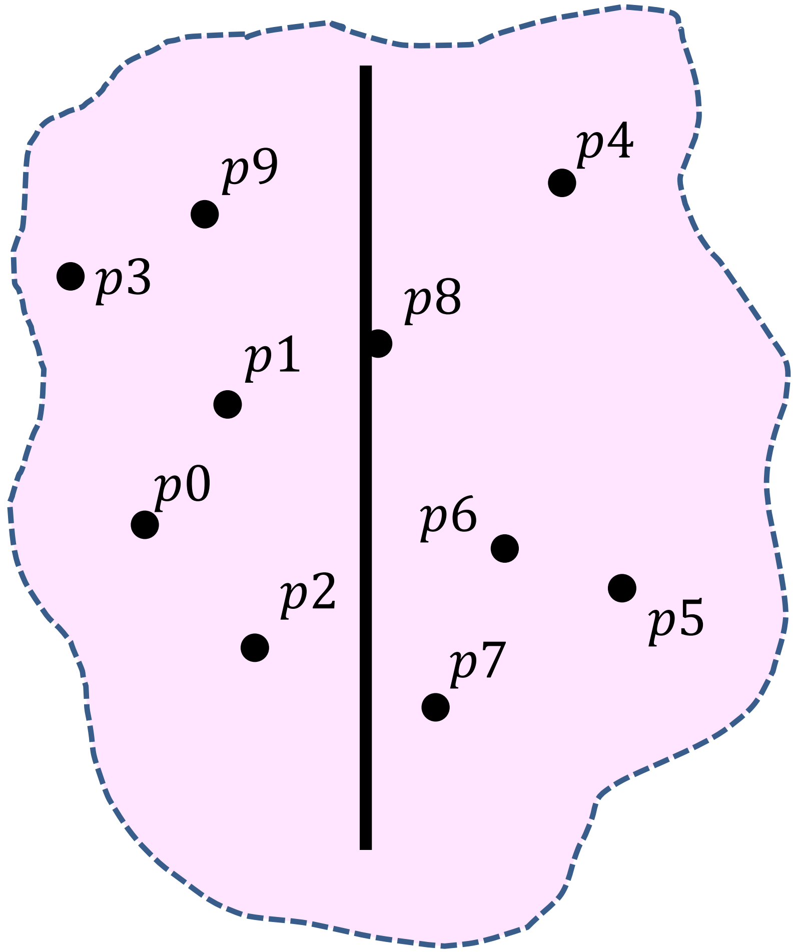
kd-tree motivation





- Quadtree can be very unbalanced
- kd-tree idea
  - split into regions with equal number of points
  - easier to split into two regions with equal number of points (rather than four regions)
  - can split either vertically or horizontally
  - alternating vertical and horizontal splits gives range search efficiency

kd-tree example

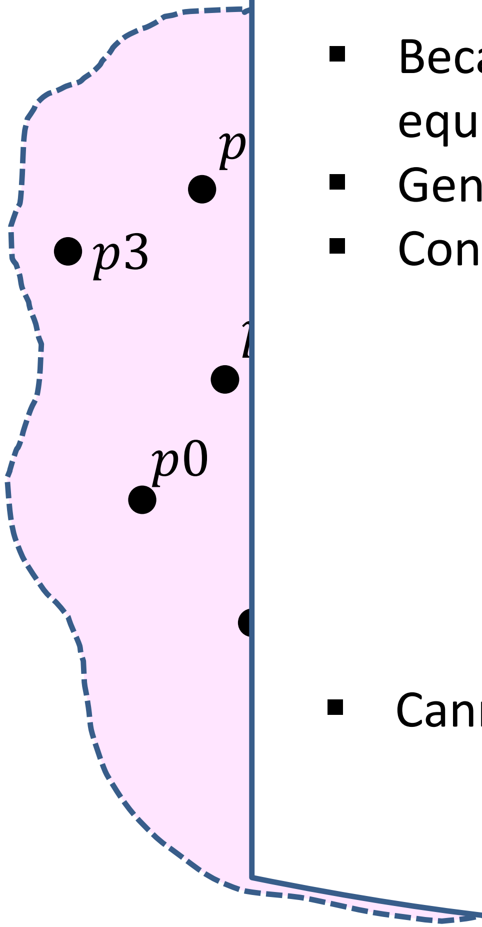


$\mathcal{R}^2$ is split into two half regions

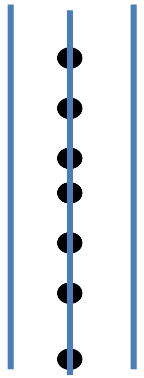
- No need for bounding box
- Root corresponds to the whole $\mathcal{R}^2$
- First find the best vertical split
- $\left\lfloor \frac{n}{2} \right\rfloor$ on one side and $\left\lceil \frac{n}{2} \right\rceil$ and points on the other

$x < p8.x$

kd-tree example



- Because points are in general position, always can split in two equal (or almost equal subsets)
- General position means no two x or y coordinates are the same
- Consider the points below **not** in general position



- Cannot divide them in two equal subsets by a vertical line

$\mathcal{R}^2$ is split into two half regions

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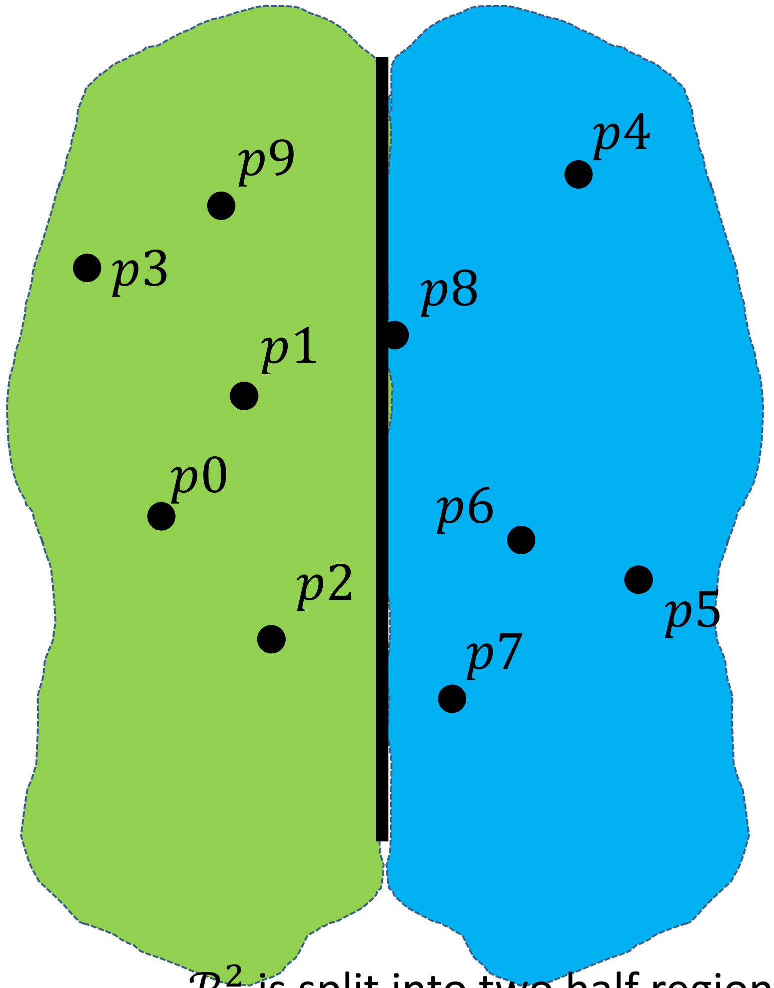

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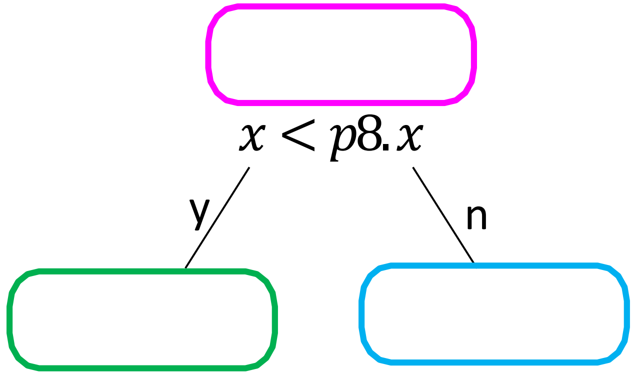

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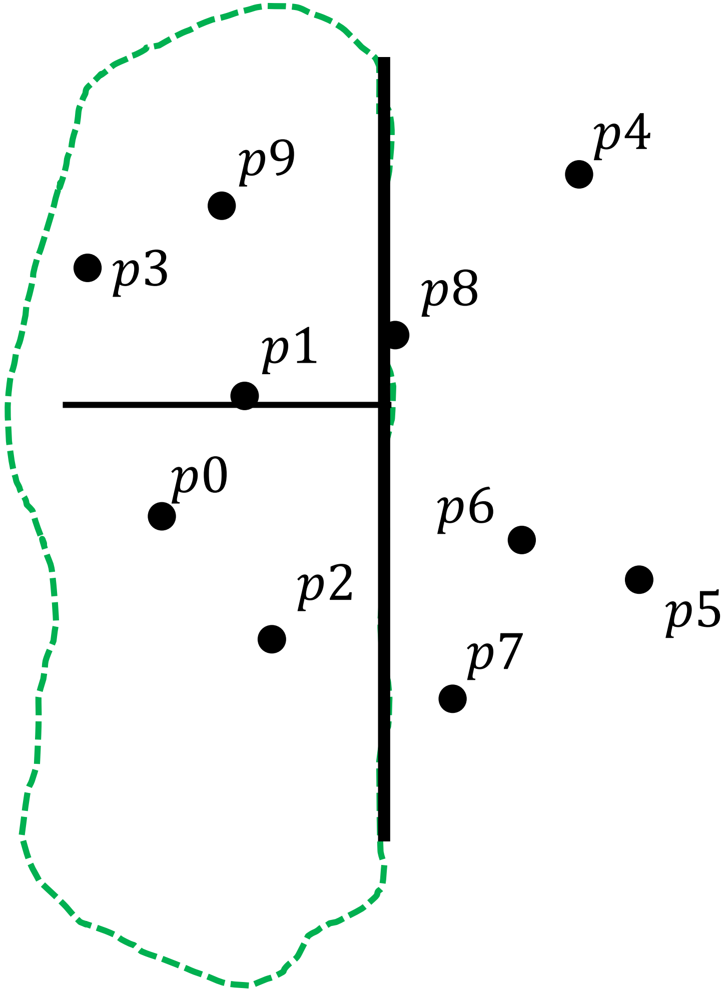
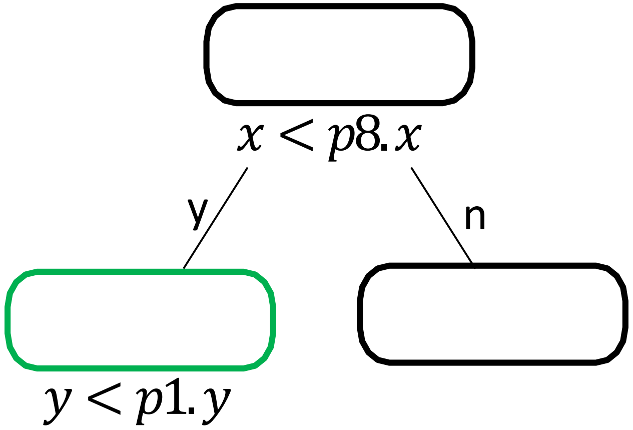


- Recurse on the resulting regions
    - if they have more than one point
- Alternate split direction



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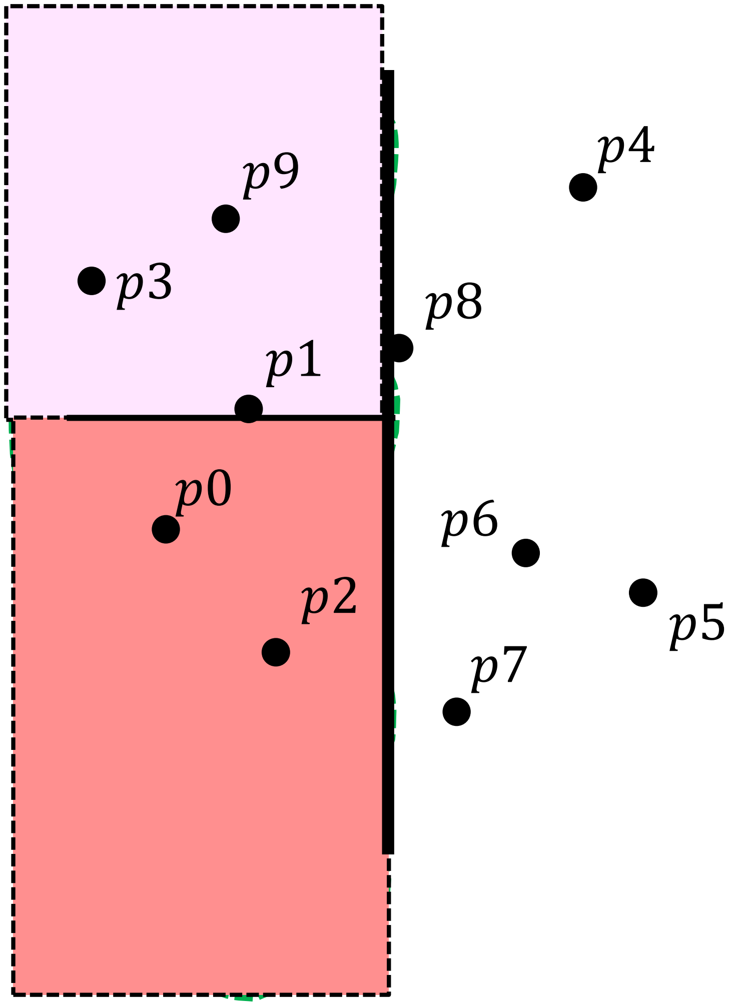

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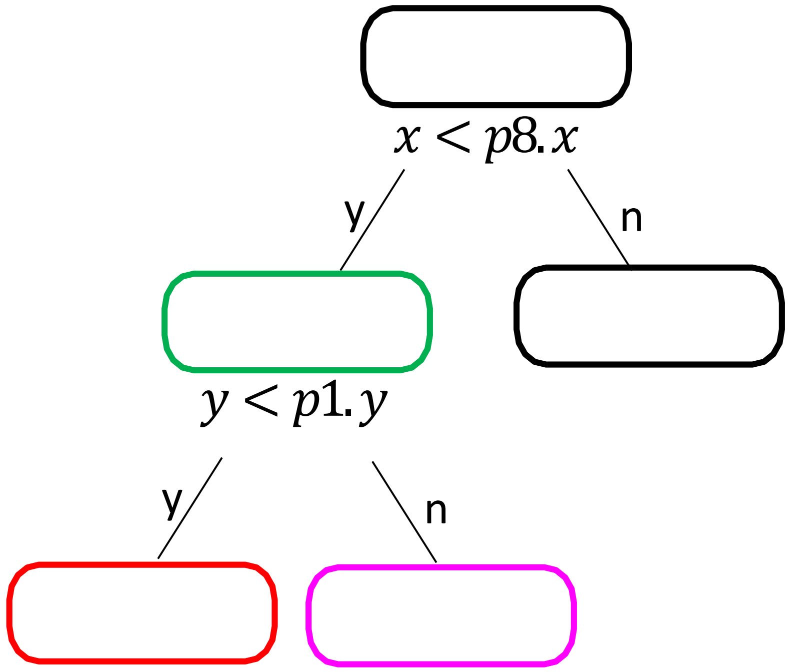

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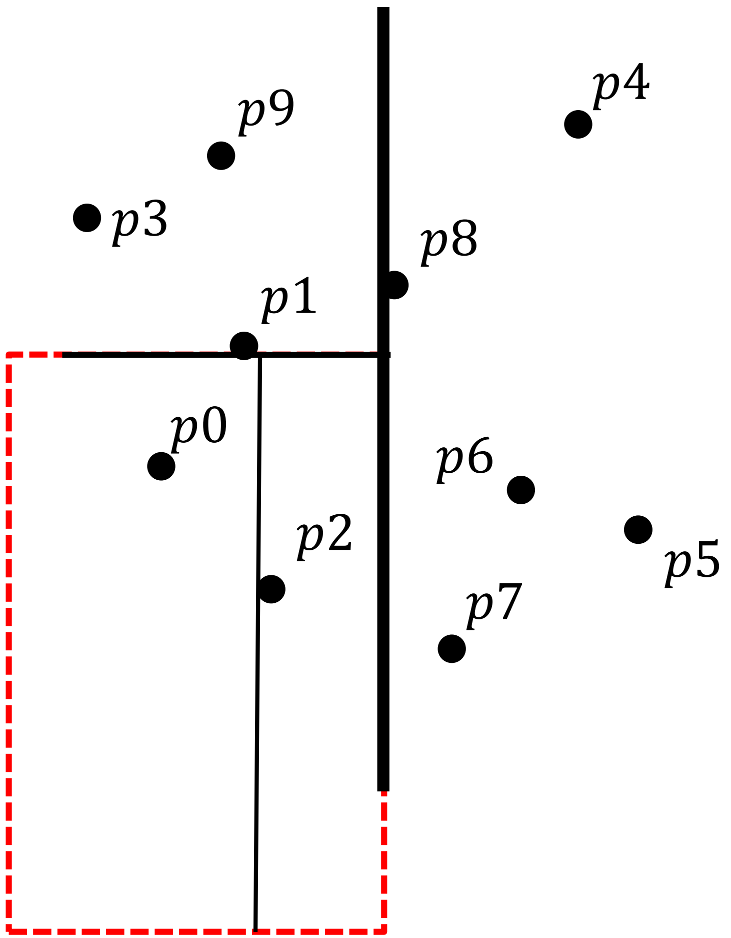

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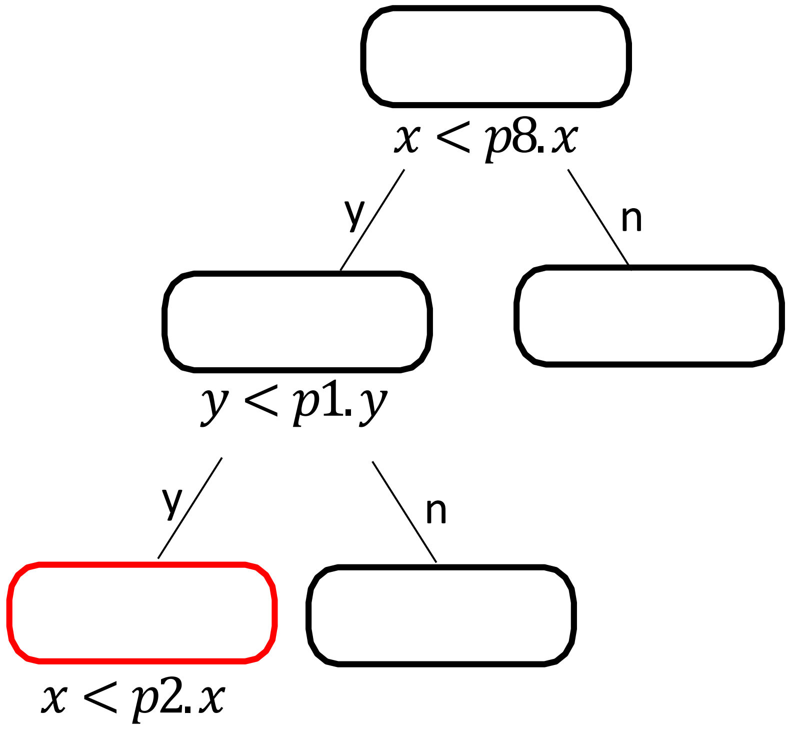

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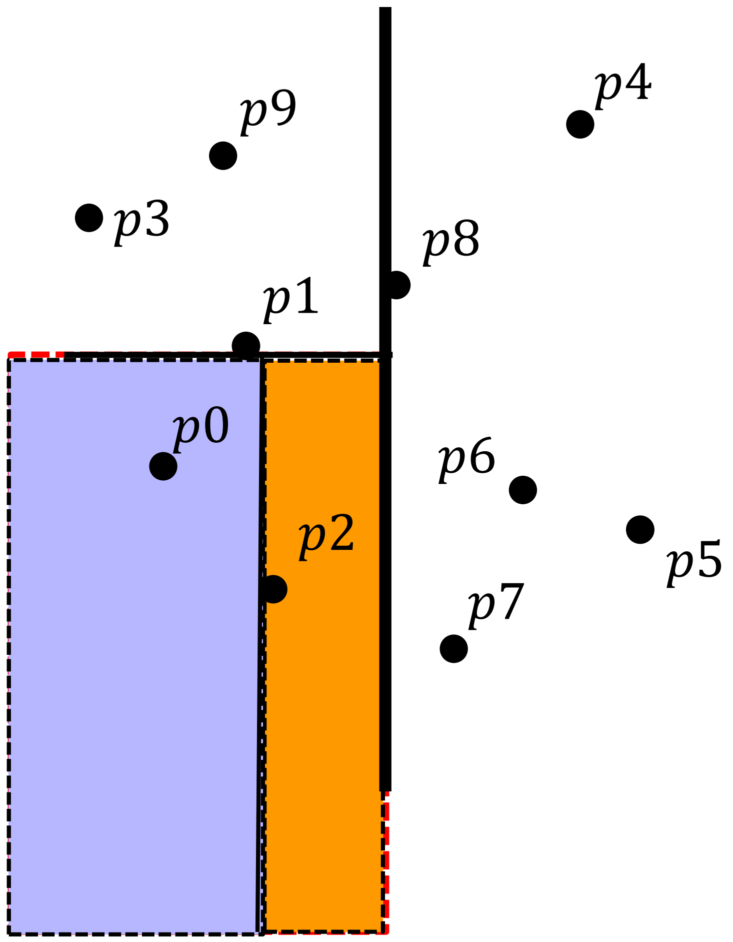

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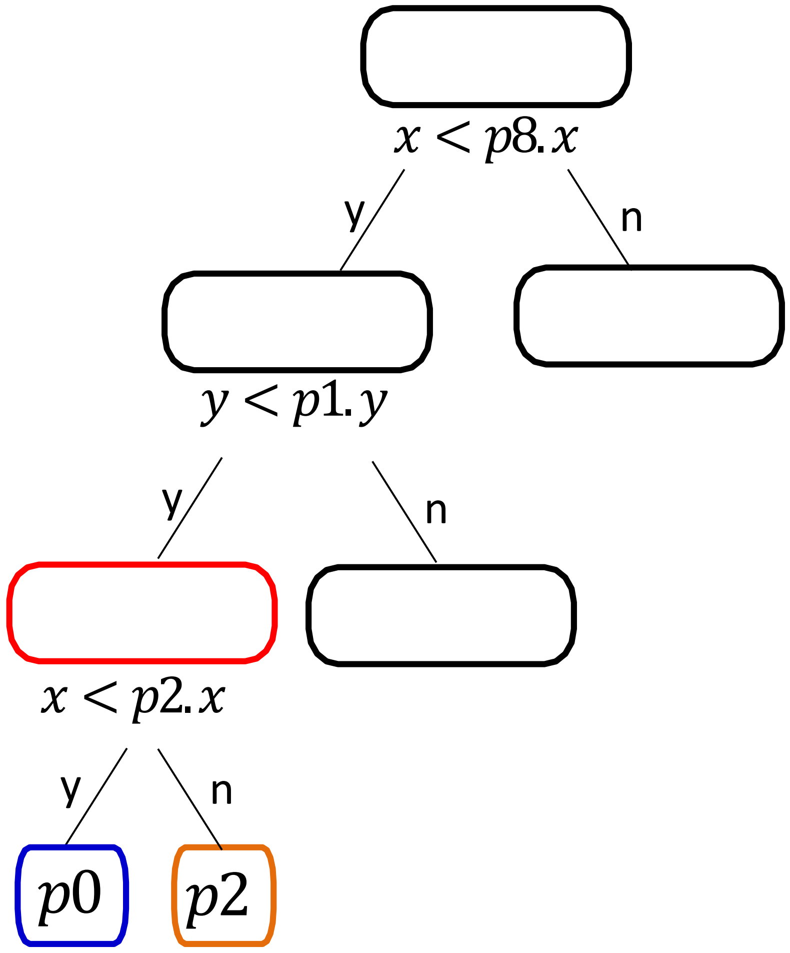

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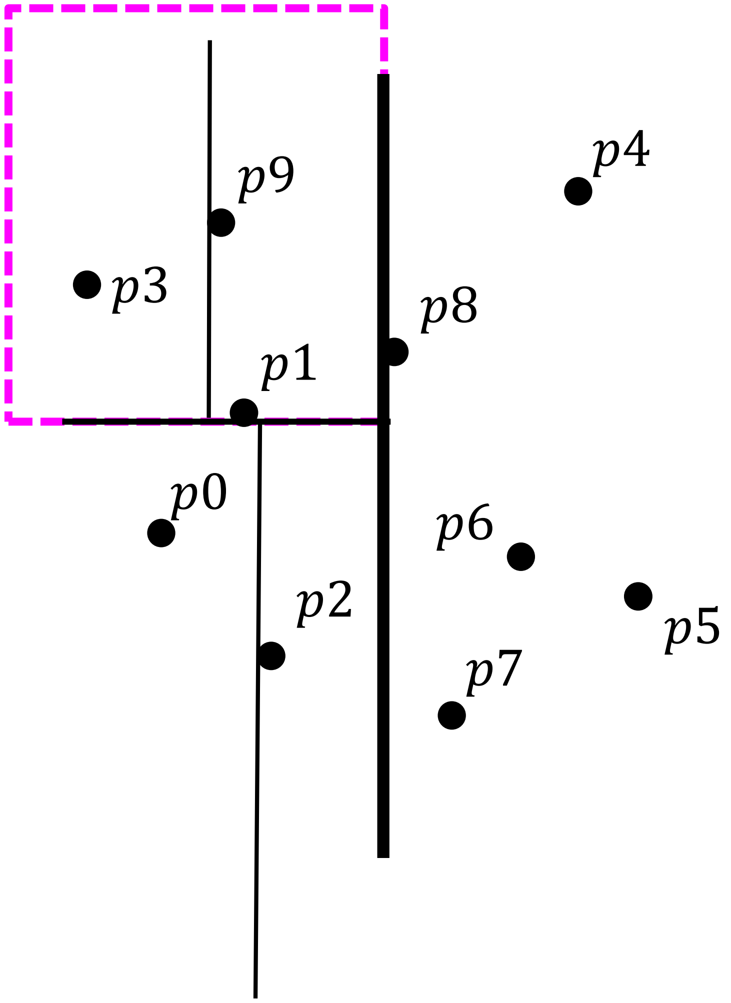

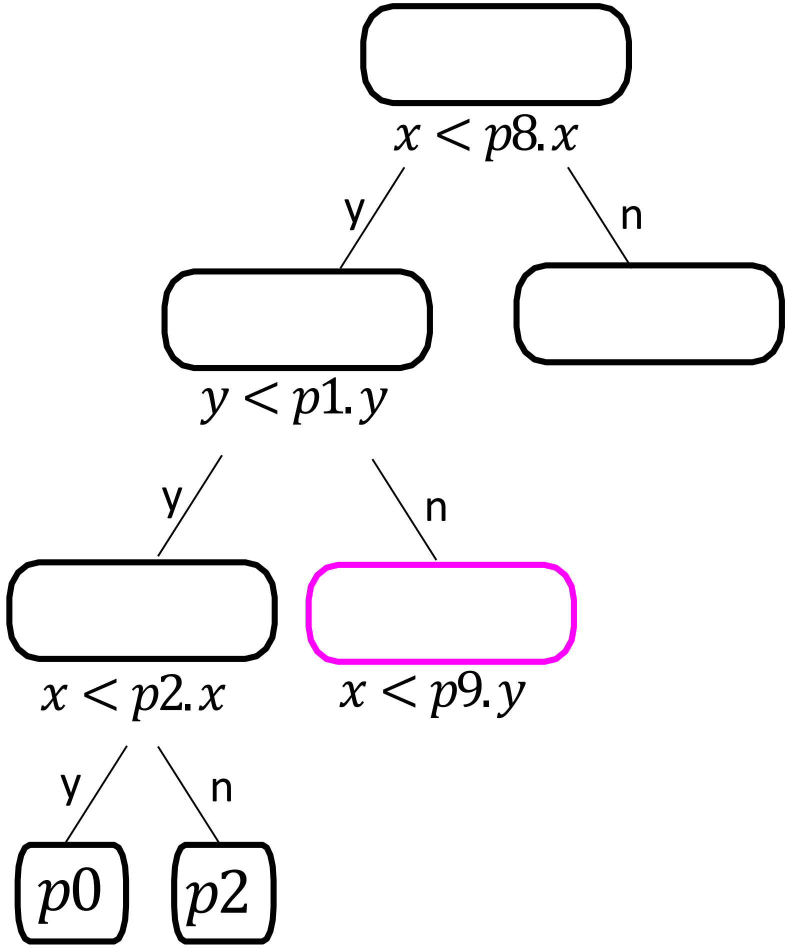

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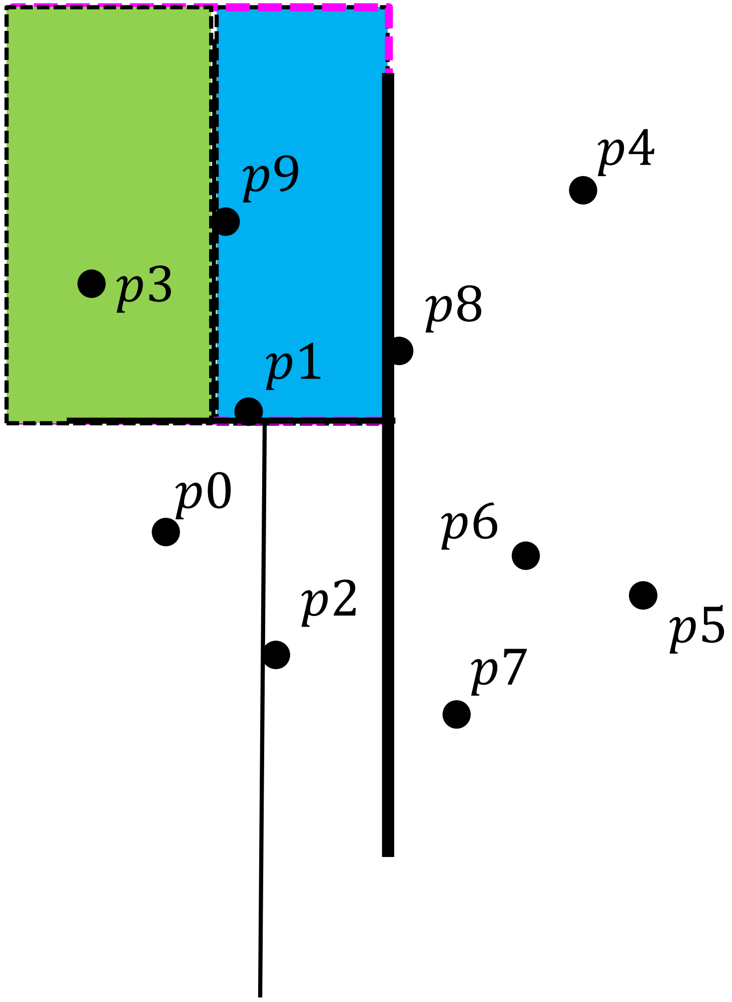

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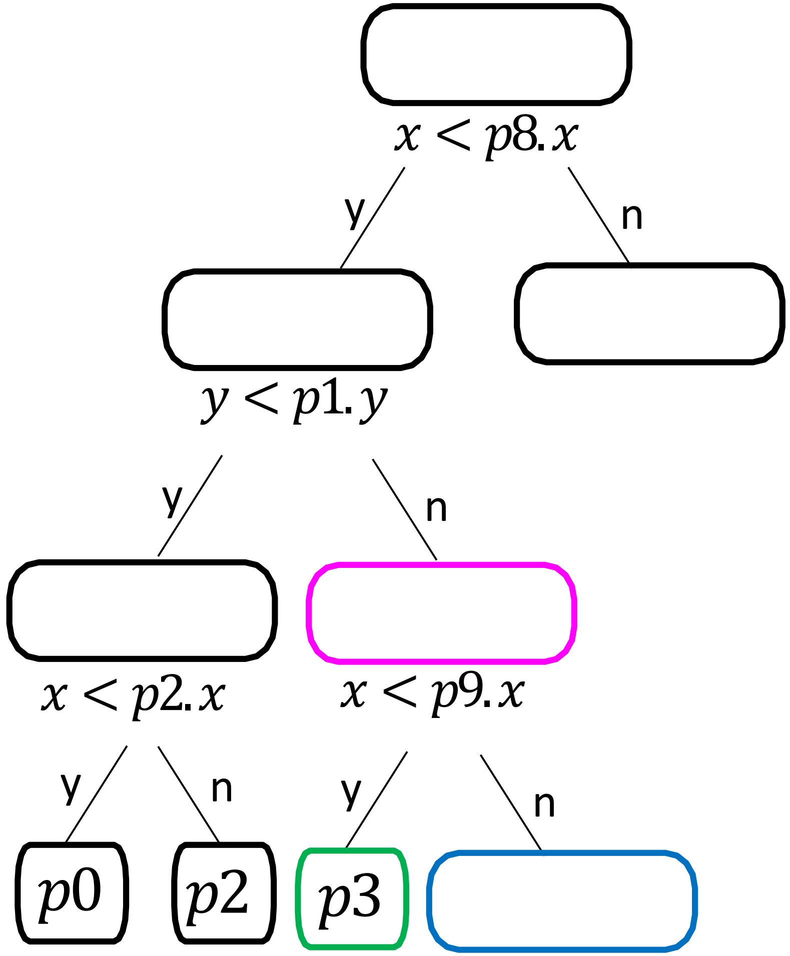

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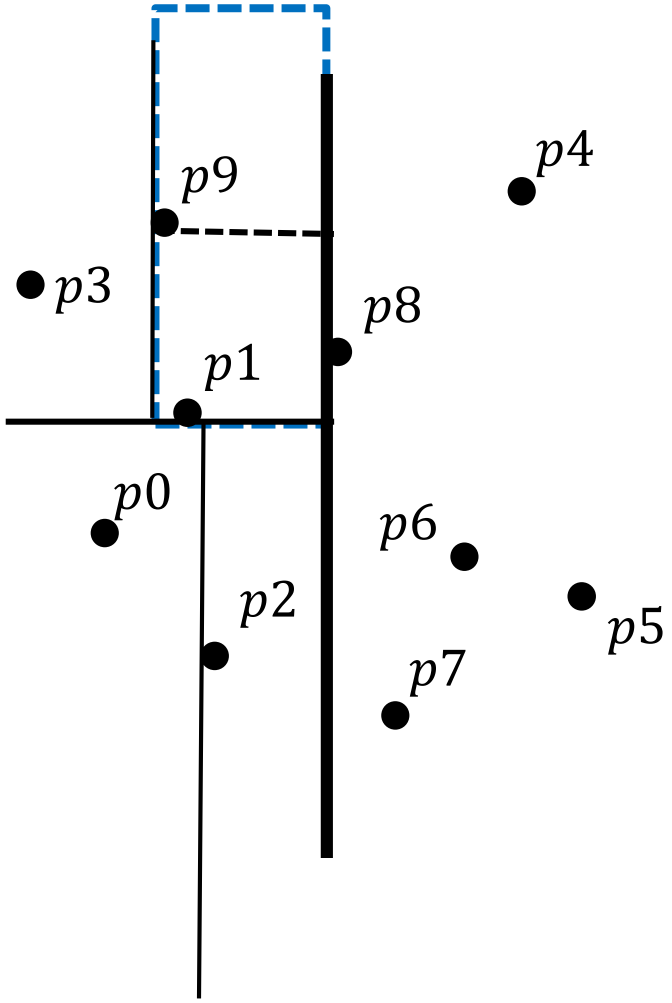

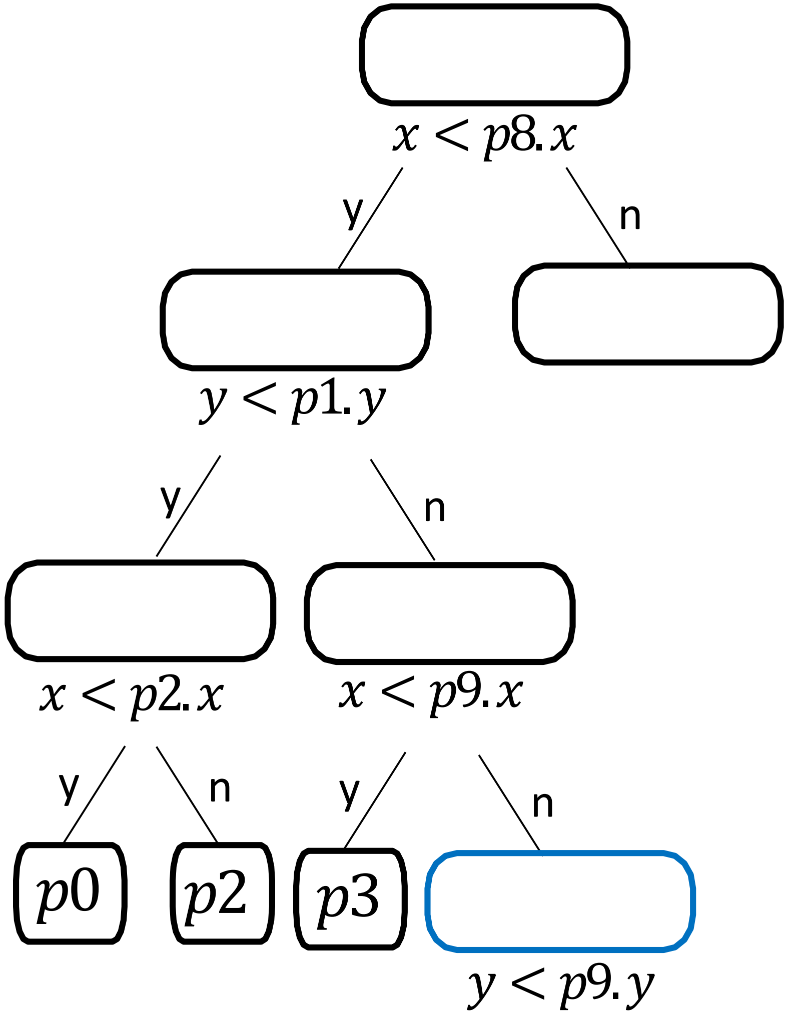

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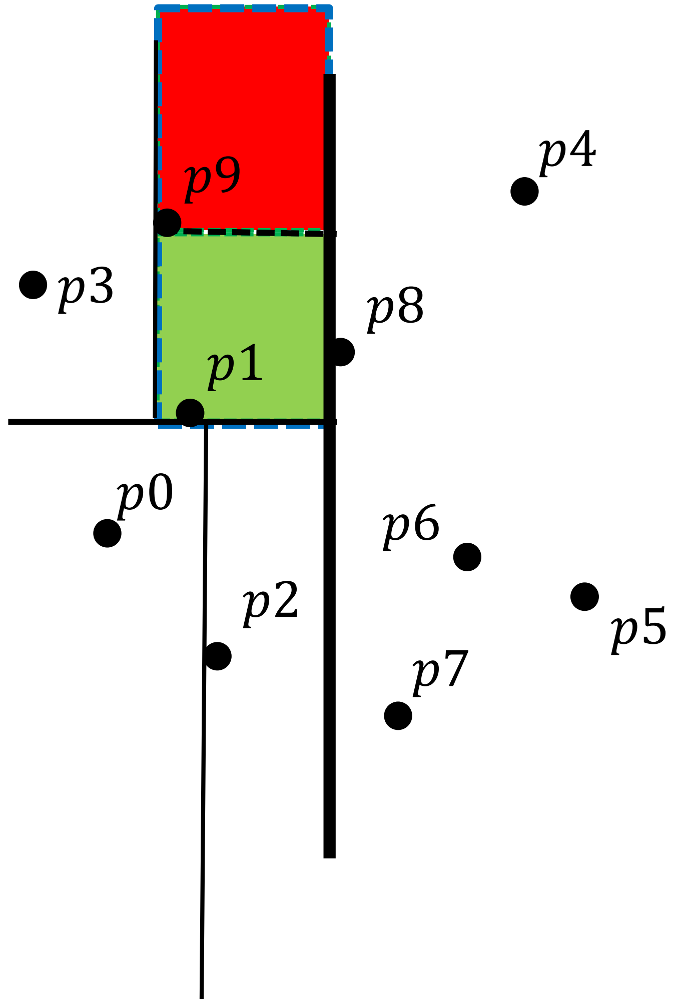

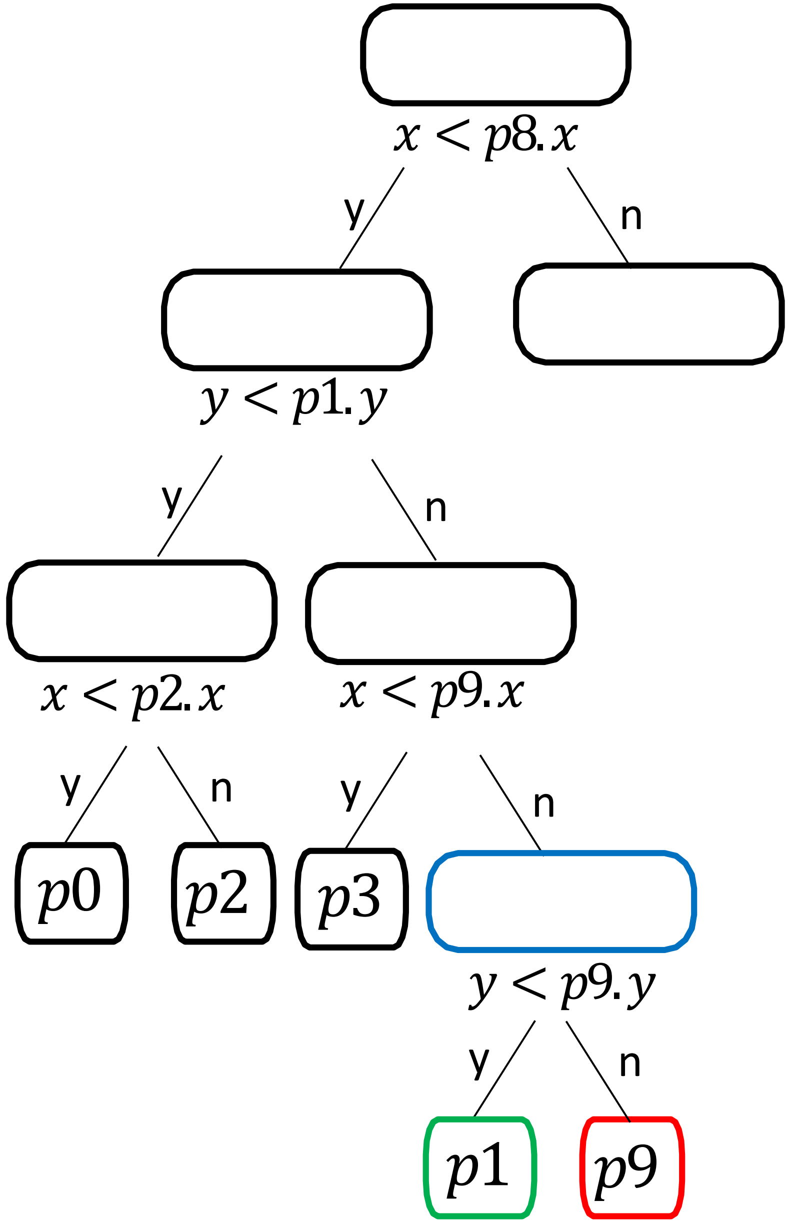

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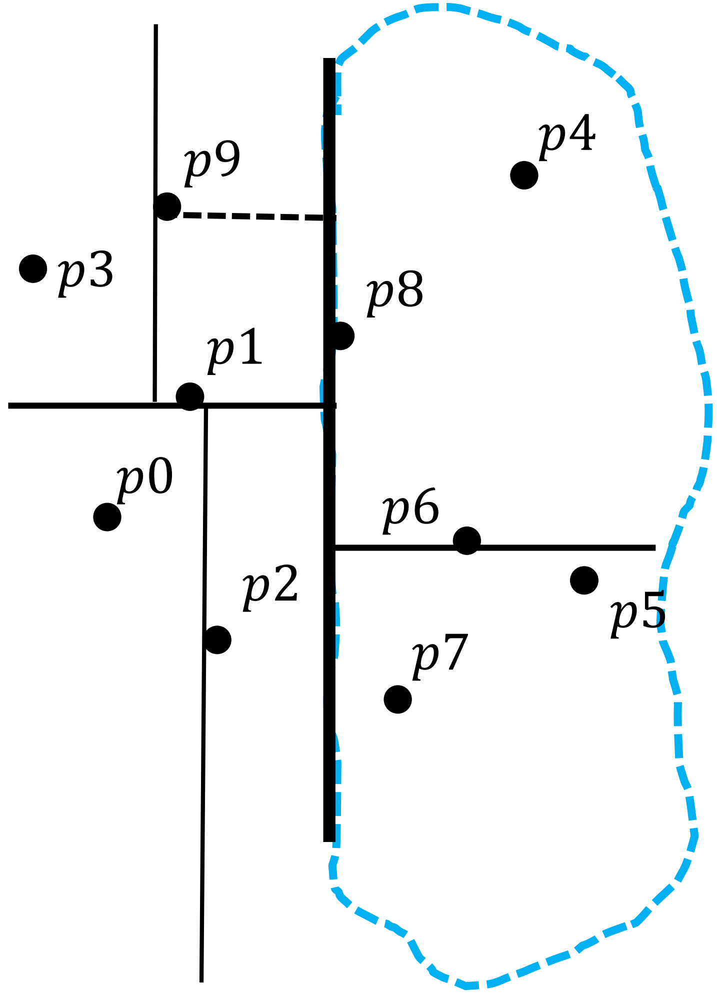

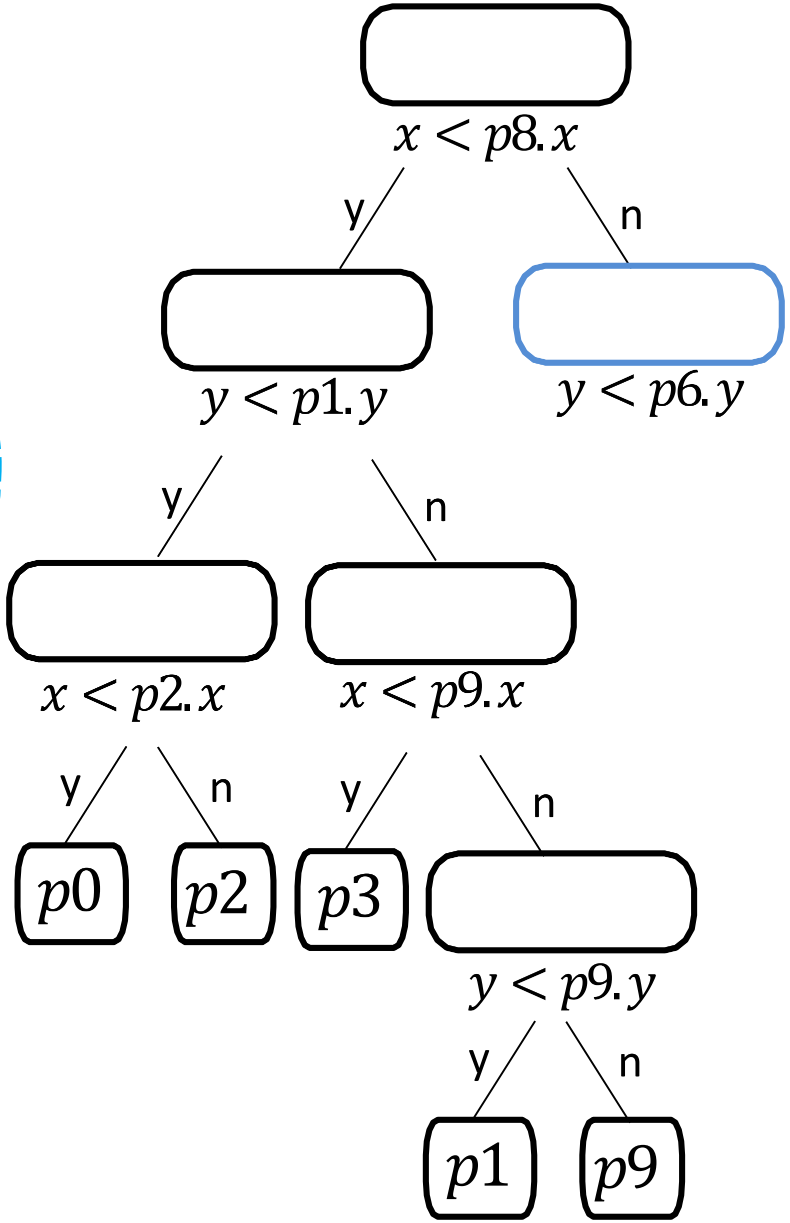

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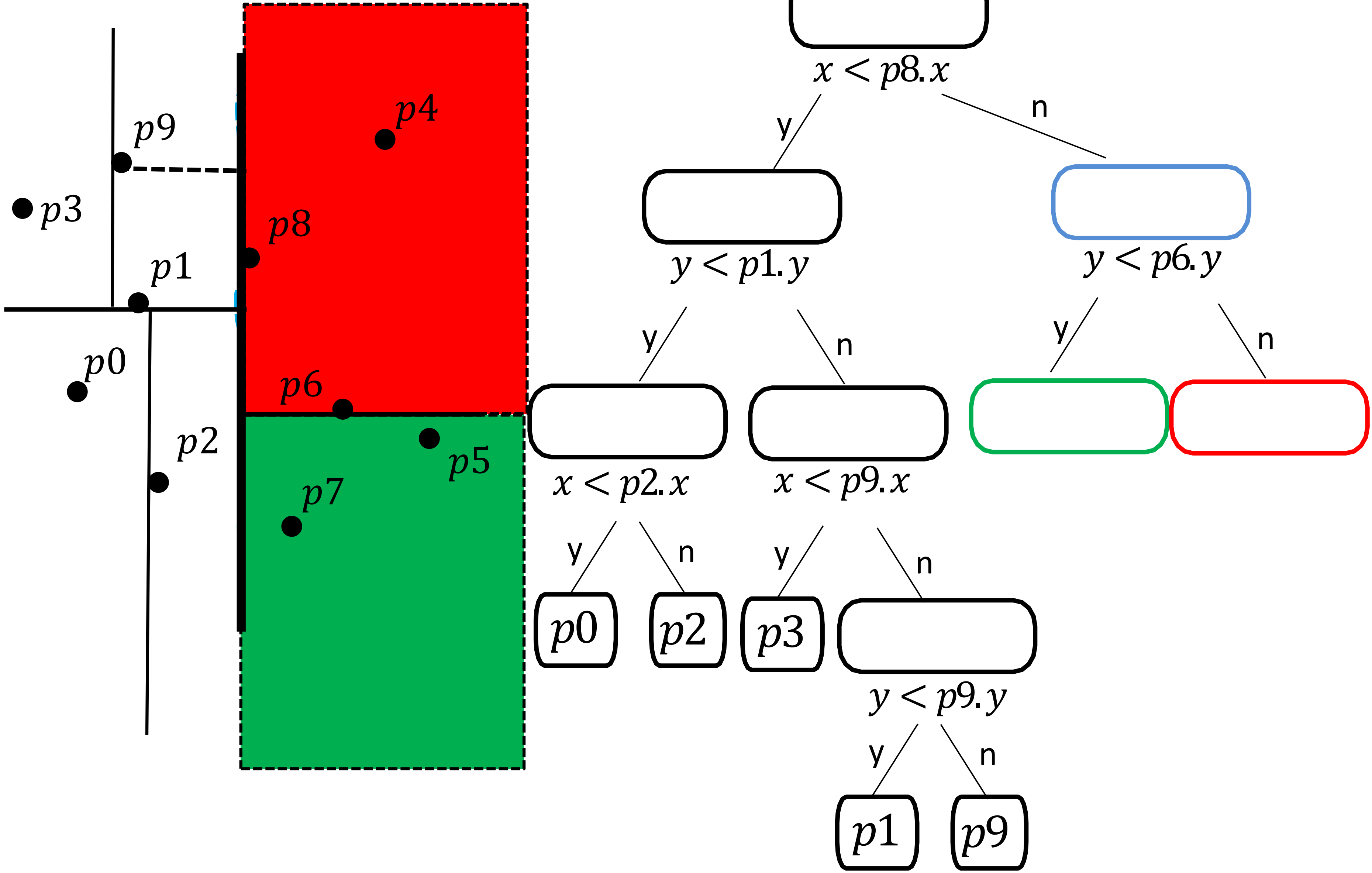
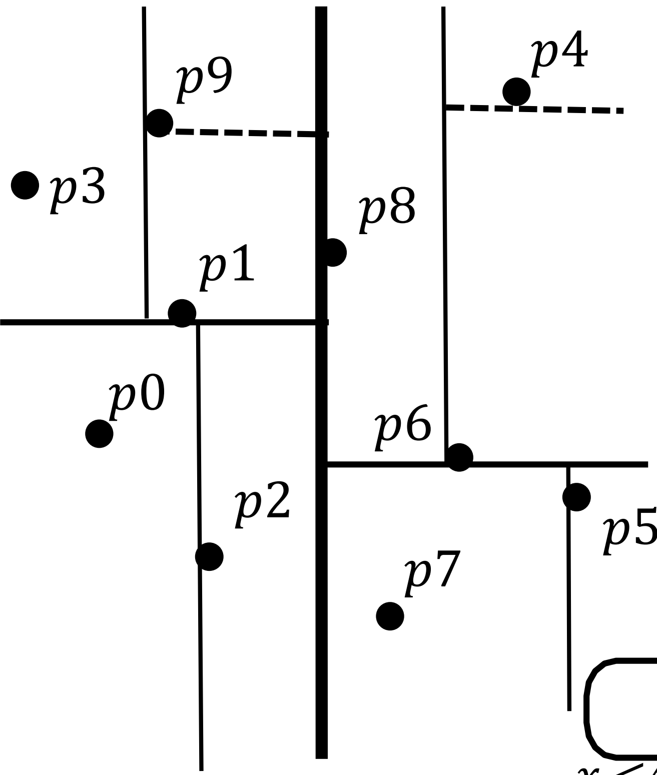
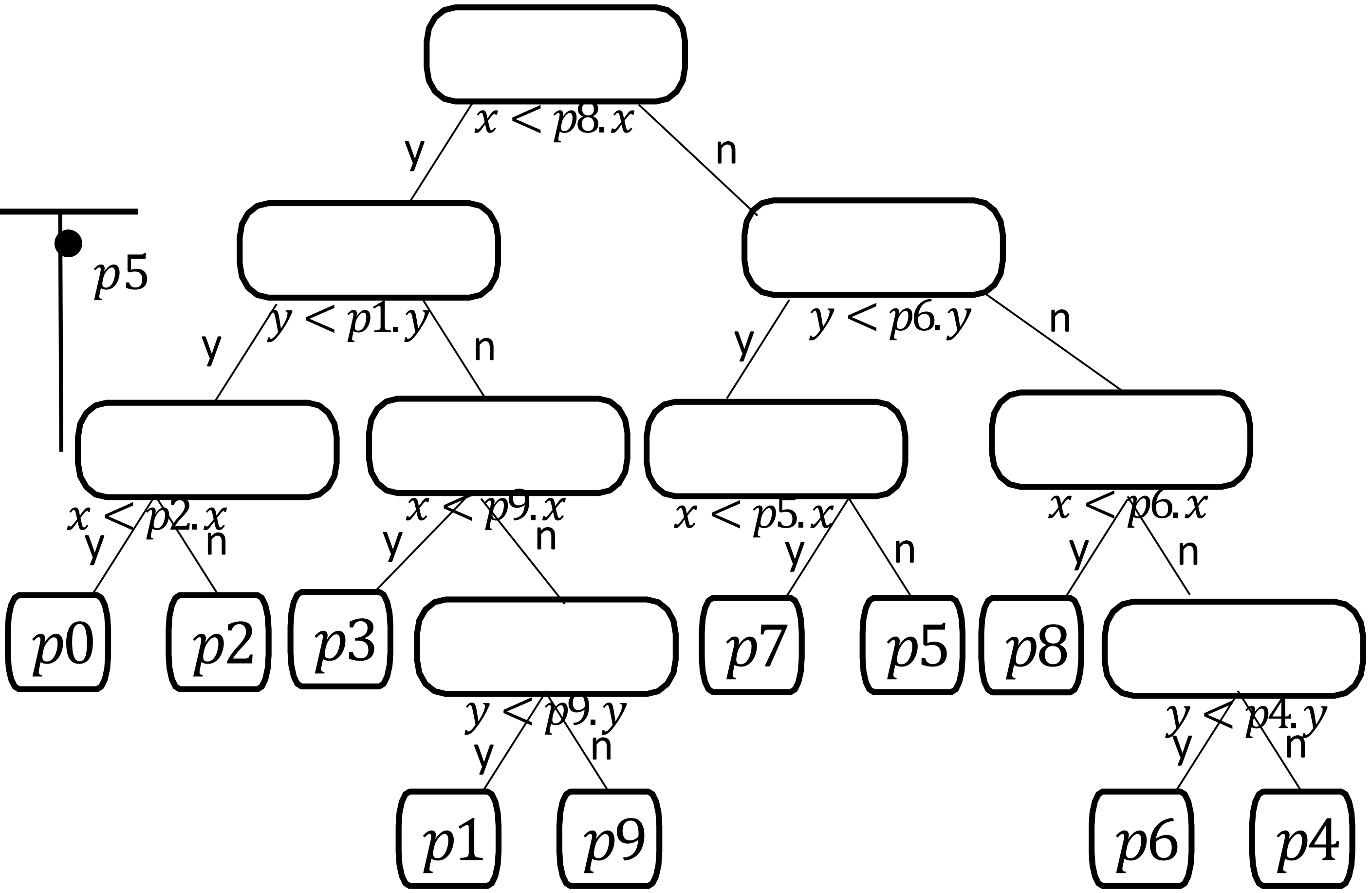


kd-tree example





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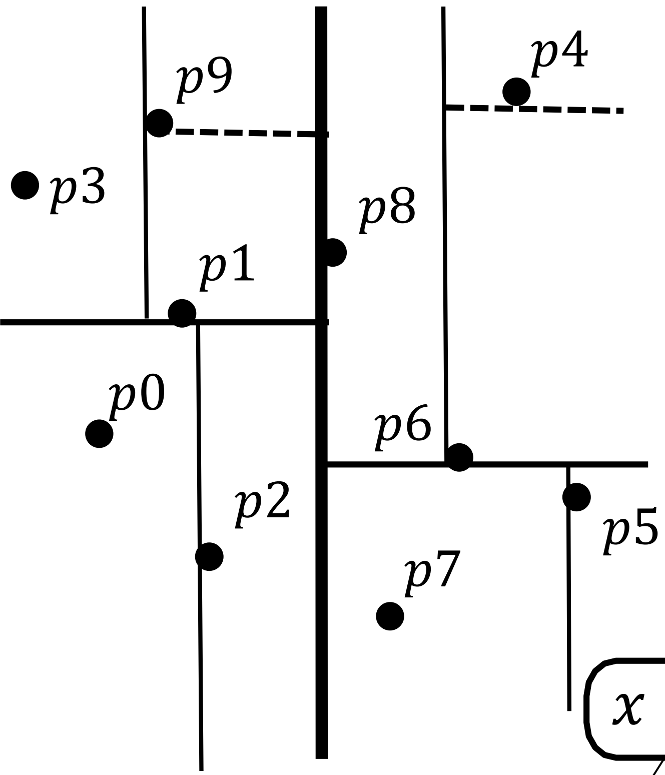
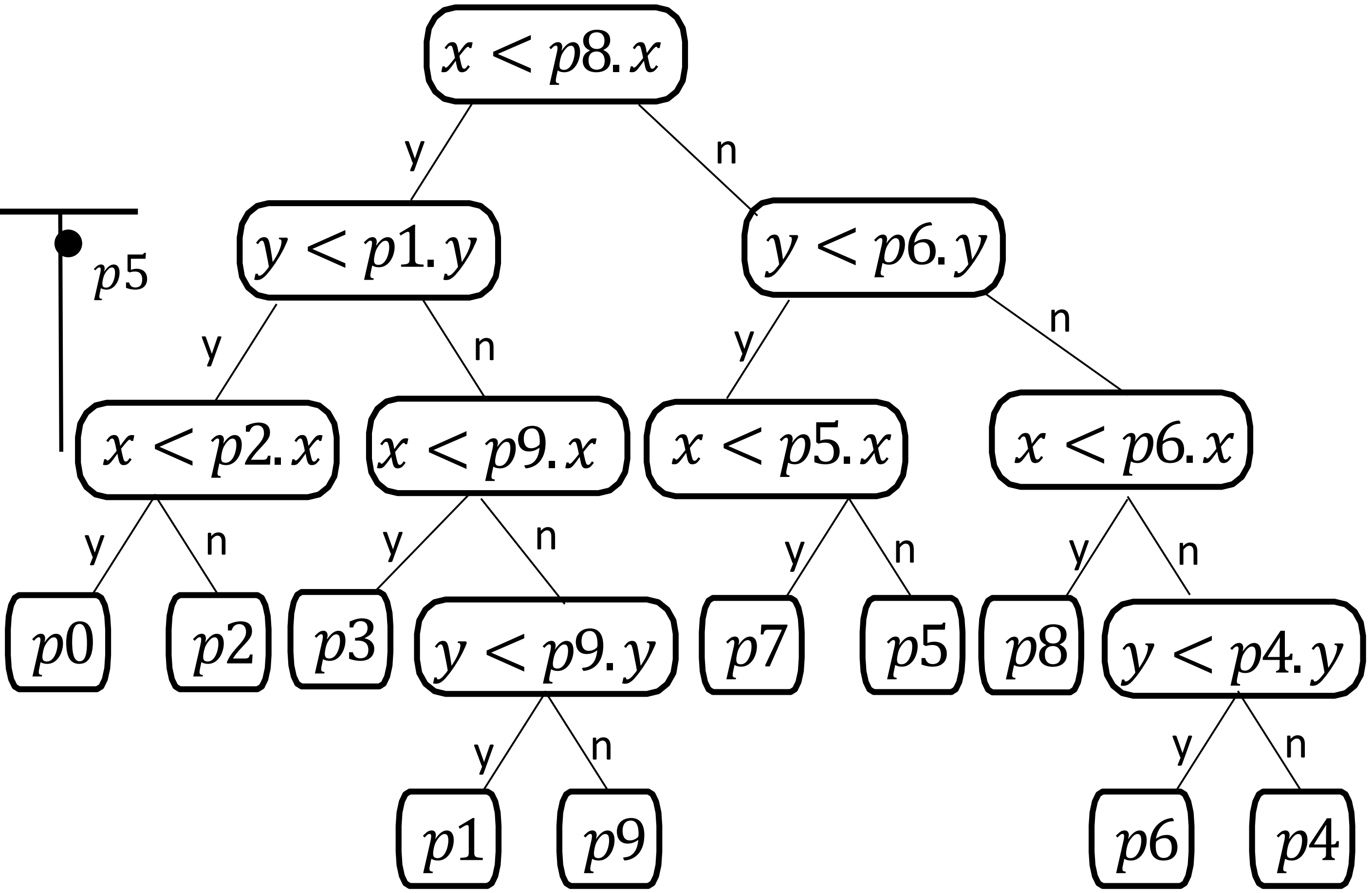




Building kd-trees

- Points $S = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$
- Build kd-tree with initial x-split
  - if $|S| \leq 1$ create a leaf and return
  - else find x-coordinate in position $m = \left\lfloor \frac{n}{2} \right\rfloor$ in sorted list of x -coordinates

    or partition by calling *quickSelect*$(S, \left\lfloor \frac{n}{2} \right\rfloor)$
    - partition S into $S_{x<m}$ and $S_{x\geq m}$ by comparing the x coordinate of a point with m
    - create left subtree recursively (splitting on y) for points $S_{x<m}$
    - create right subtree recursively (splitting on y) for points $S_{x\geq m}$
    - each node keeps track of the splitting line
- Building with initial y-split symmetric
- Points on split lines belong to right/top side

kd-tree Construction Running Time

- Partition S in $\Theta(n)$ expected time with *QuickSelect*
- Both subtrees have $\approx n/2$ points
- Sloppy recurrence
  - $T^{exp}(n) = 2T^{exp}\left(\frac{n}{2}\right) + O(n)$
  - resolves to $\Theta(n \log n)$ expected time
- Running time can be improved to $\Theta(n \log n)$ worst-case by pre-sorting coordinates
- Recurrence inequality for height

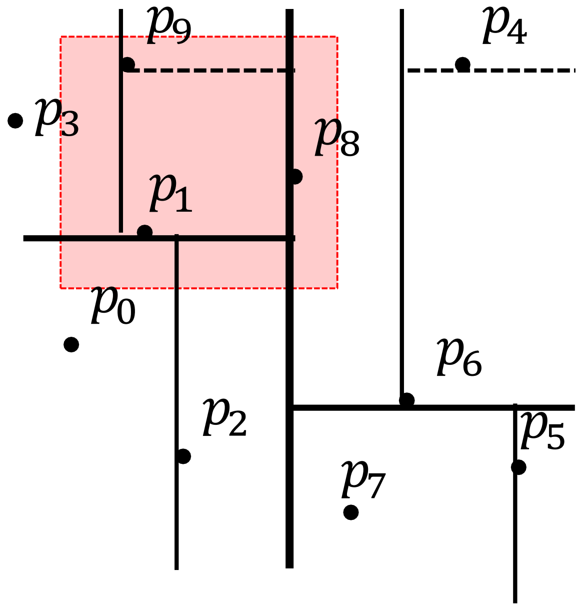
$$h(1) = 0$$
$$h(n) \leq h\left(\left\lceil\frac{n}{2}\right\rceil\right) + 1$$

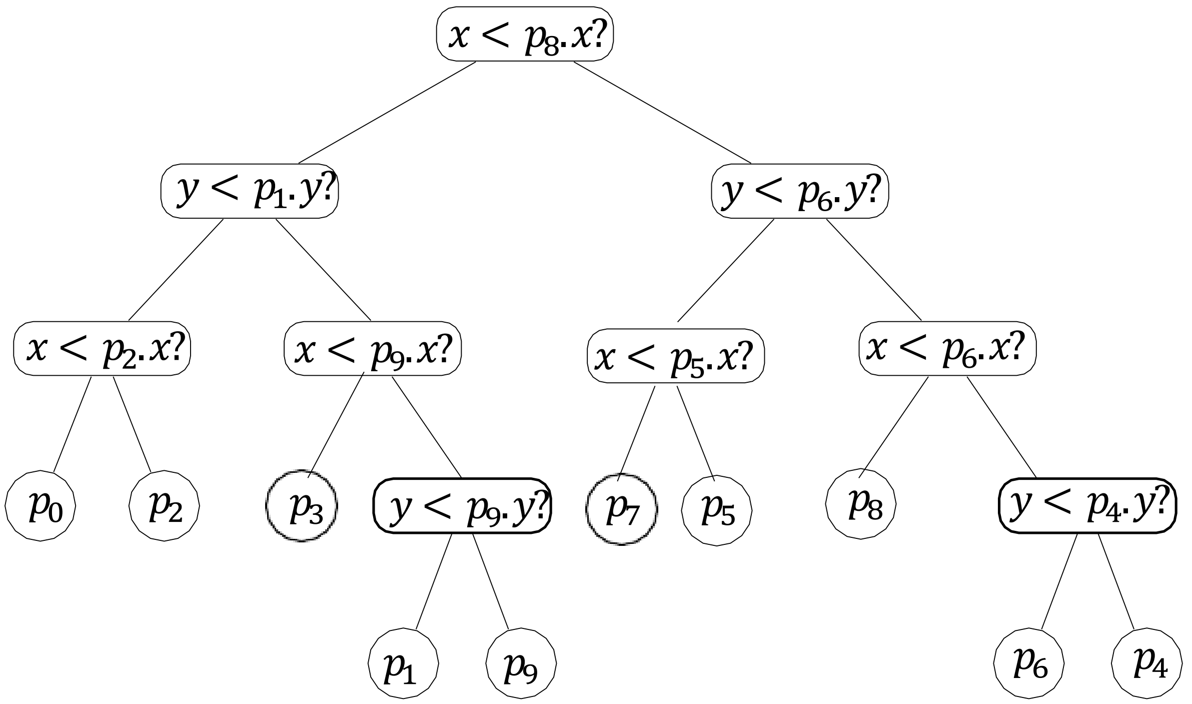
  - resolves to $O(\log n)$, specifically $\lceil \log n \rceil$

kd-tree Dictionary Operations

- *Search* as in binary search tree using indicated coordinate
- *Insert* first search, insert as new leaf
- *Delete* first search, remove leaf and any parent with one child
- **Problem**
    - kd-tree do not handle insertion/delection well
    - after insert or delete, split might no longer be at exact median
    - height is no longer guaranteed to be $O(\log n)$
    - remedy
        - allow a certain imbalance
        - re-building the entire tree when it becomes too unbalanced
        - no details
        - but *rangeSearch* will be slower

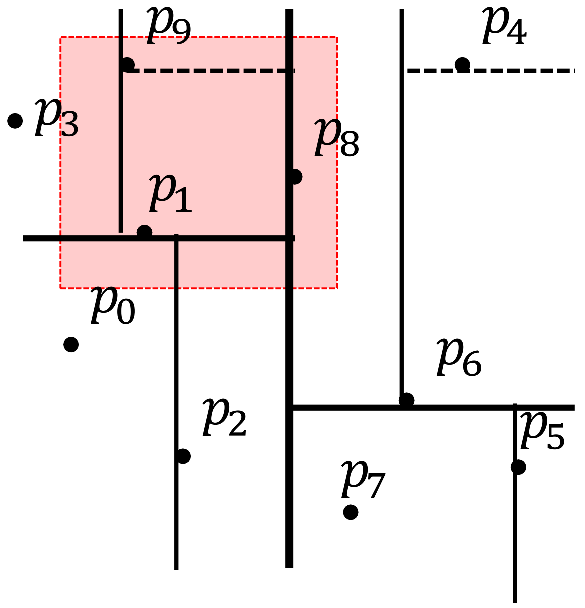
kd-tree: Range Search Example

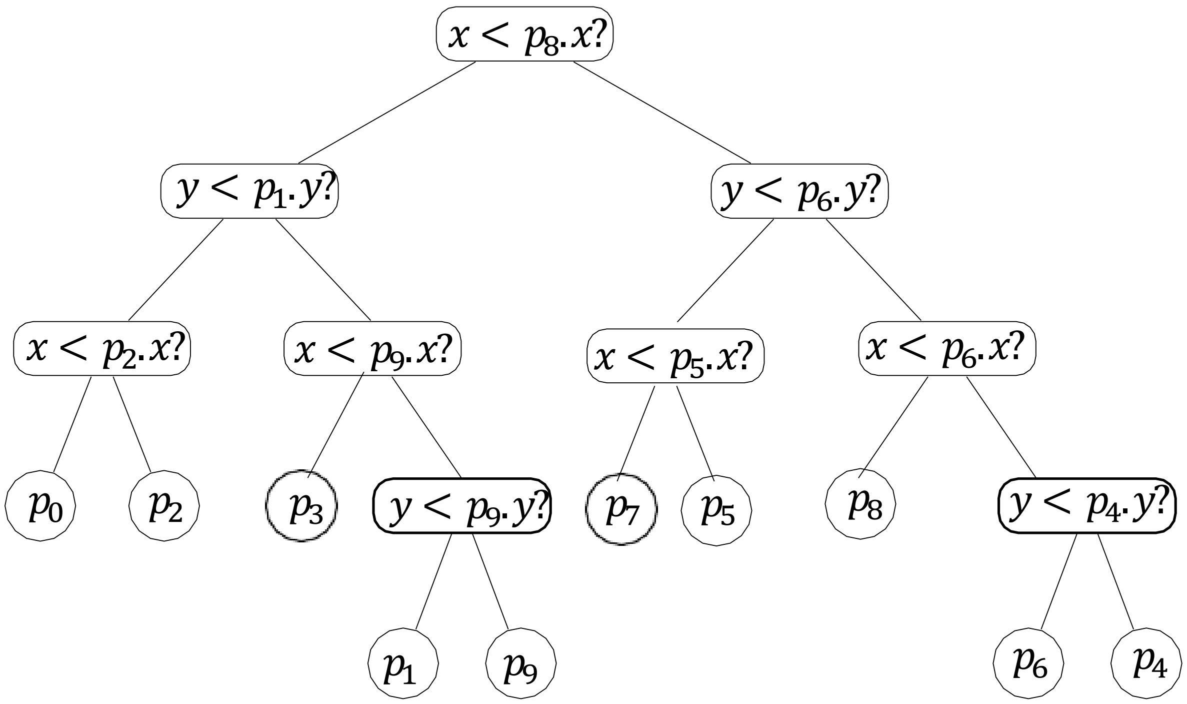




- Every node is associated with a region
    - range search is similar to quadtrees

kd-tree: Range Search Example





- Query rectangle A
- Let R be region associated with current node, have 3 cases
    1. $R \cap A = \emptyset$: red (outside) node, do not search its children
    2. $R \subseteq A$: green (inside) node, no need to search children, report all points in R
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        - if R is a leaf, if it stores point inside A, report it

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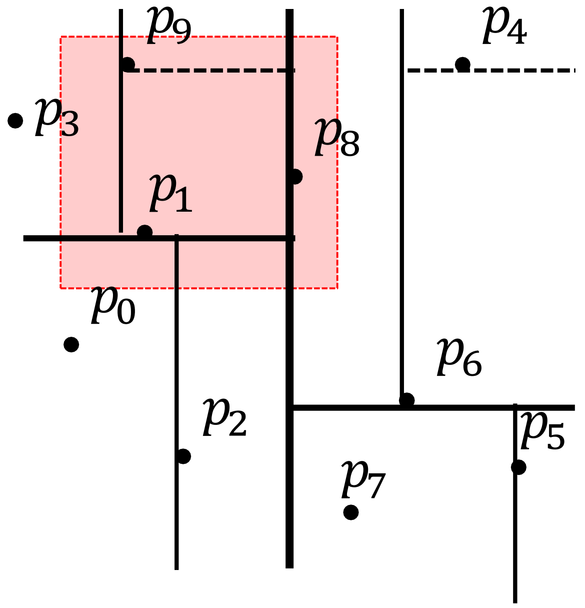

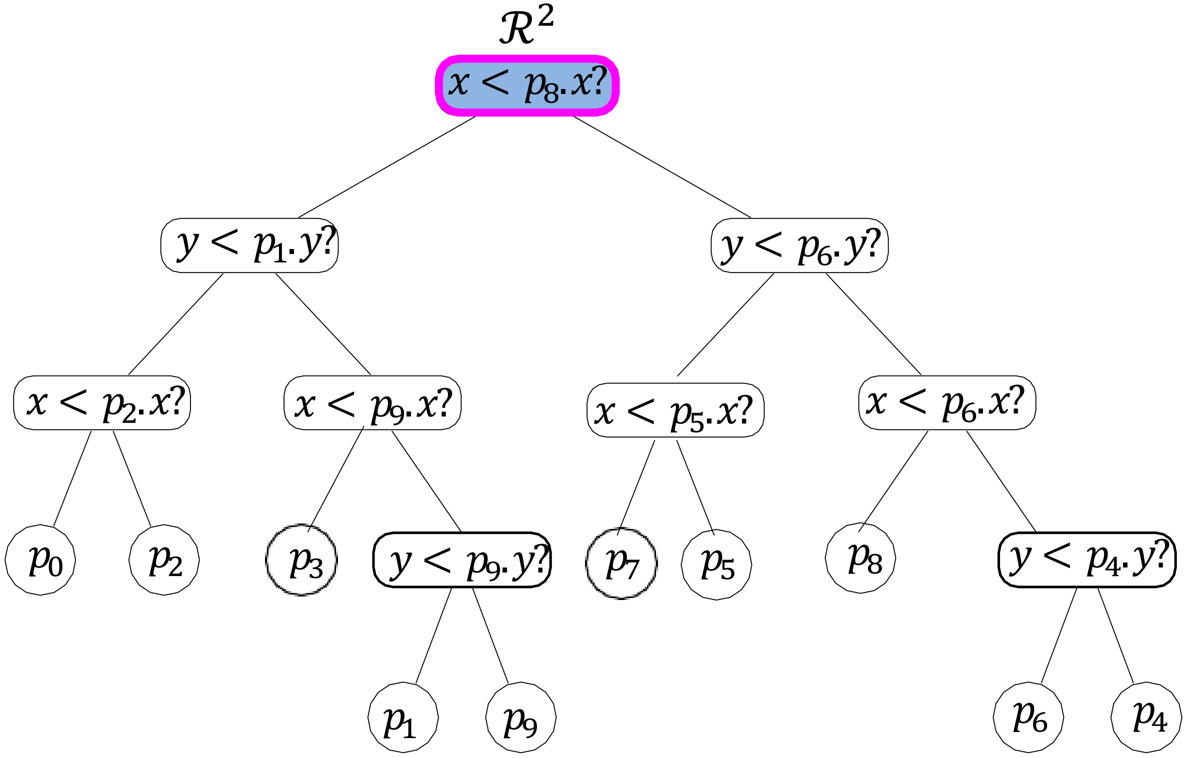

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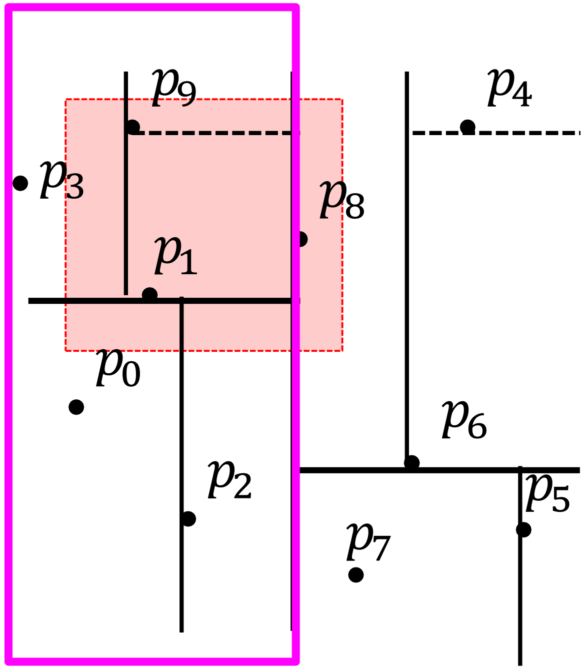

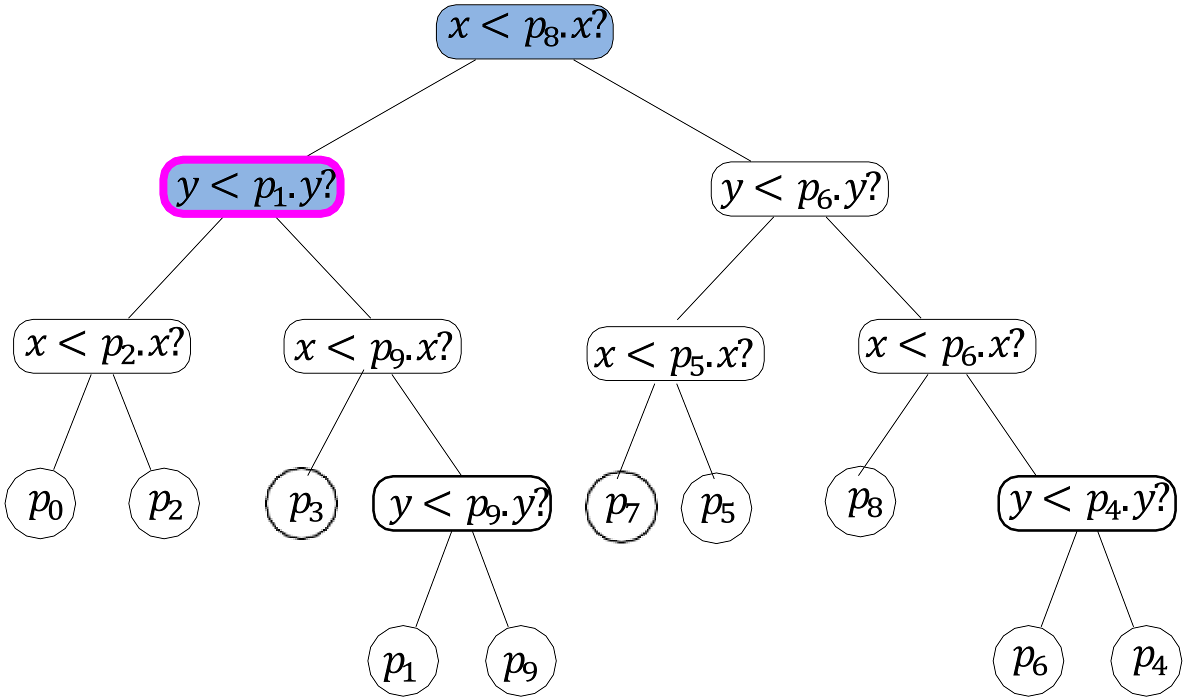

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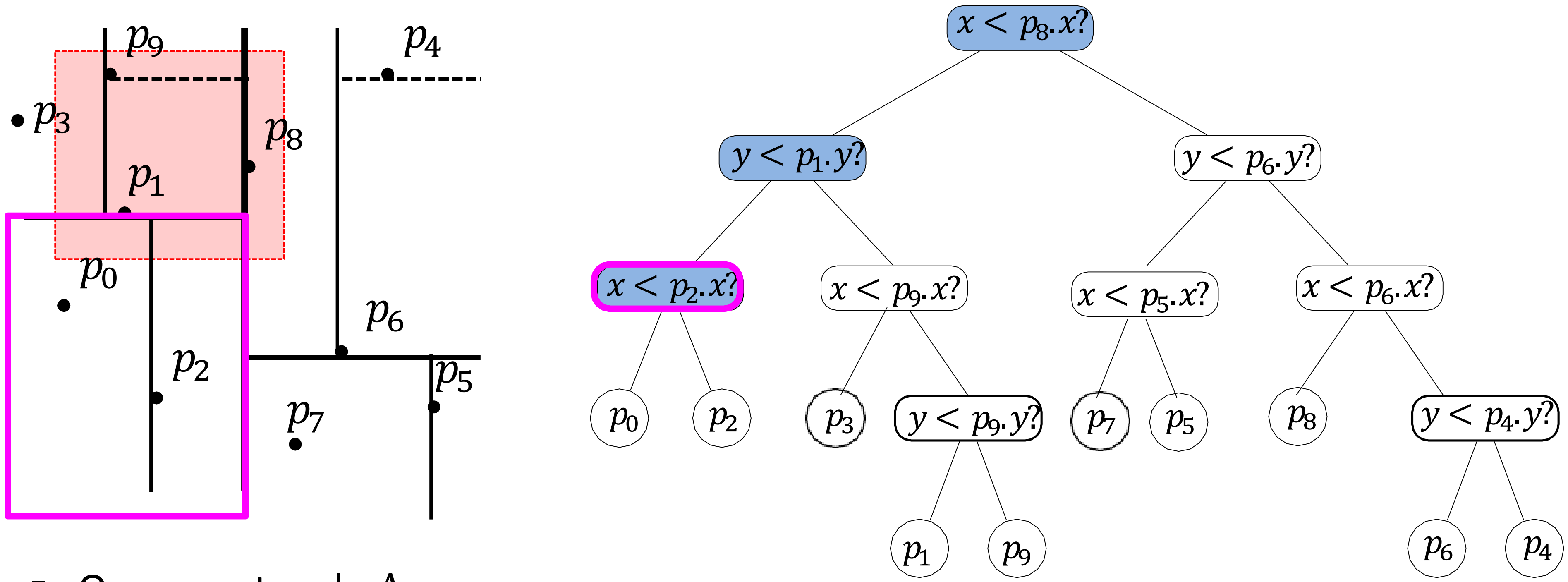

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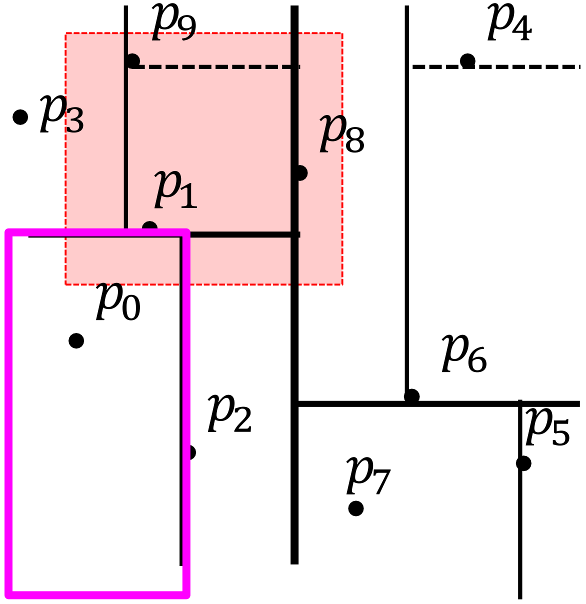

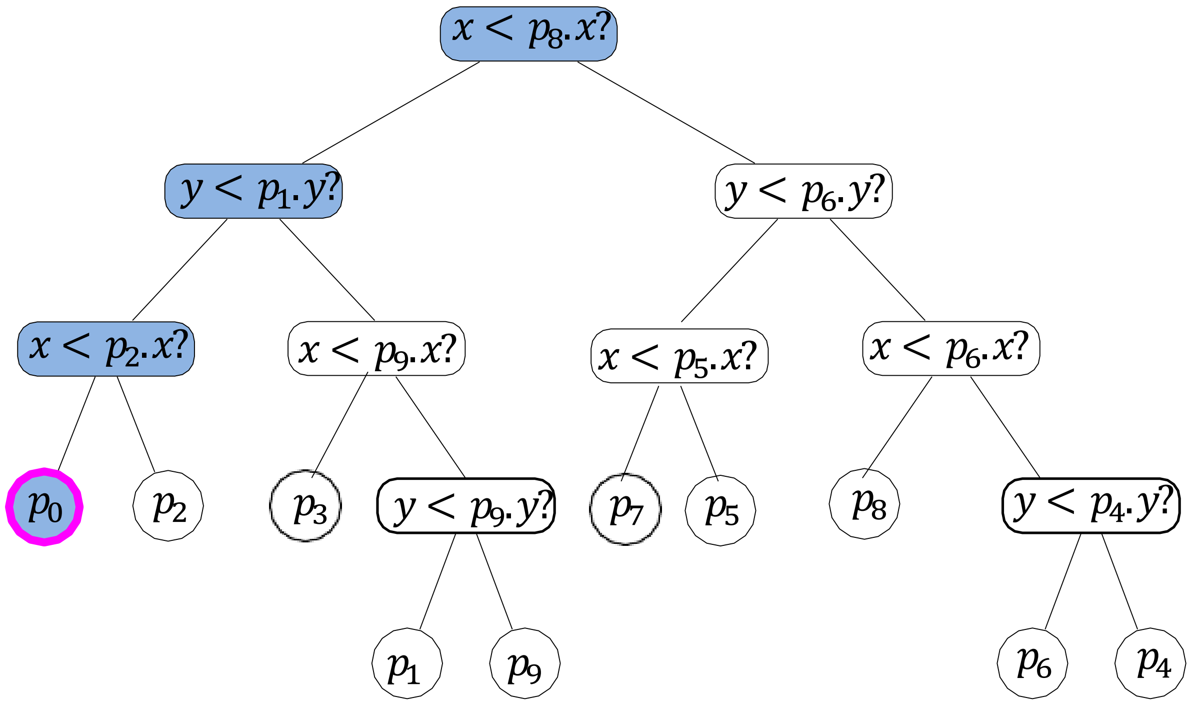

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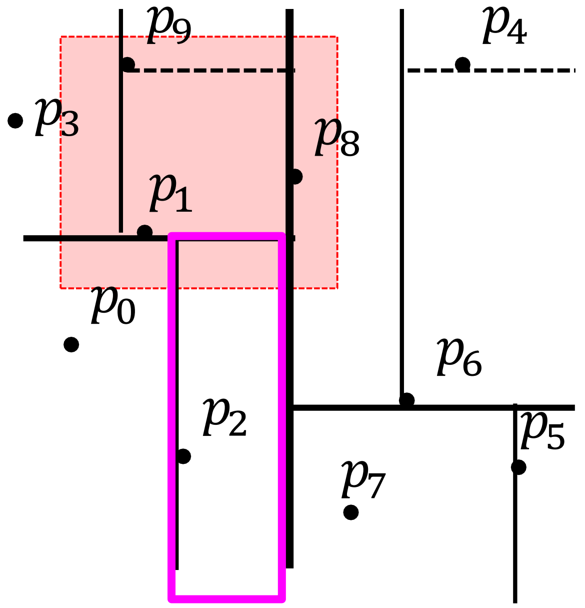

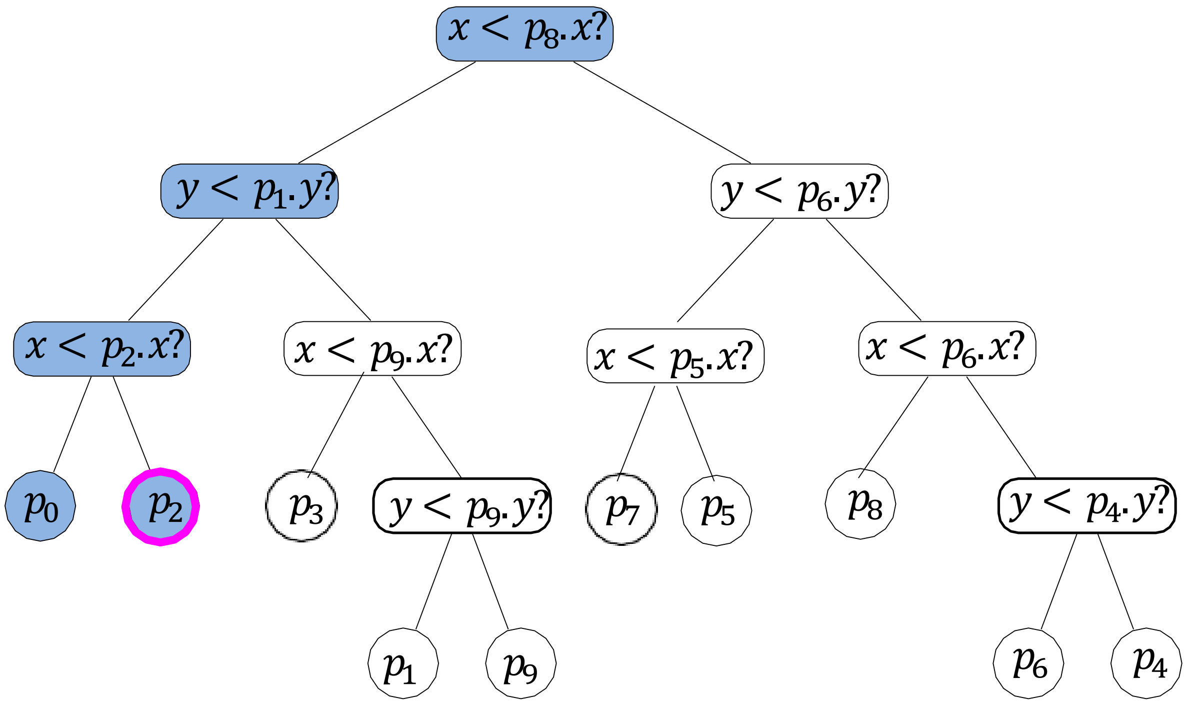

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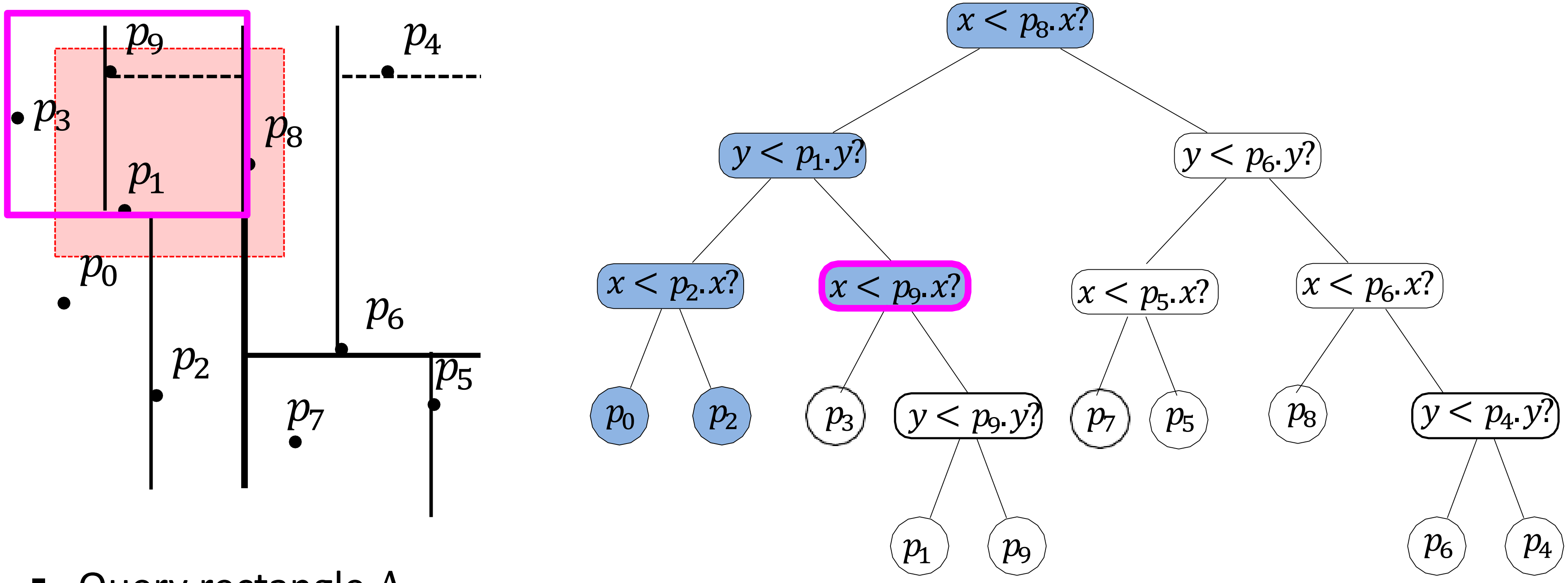

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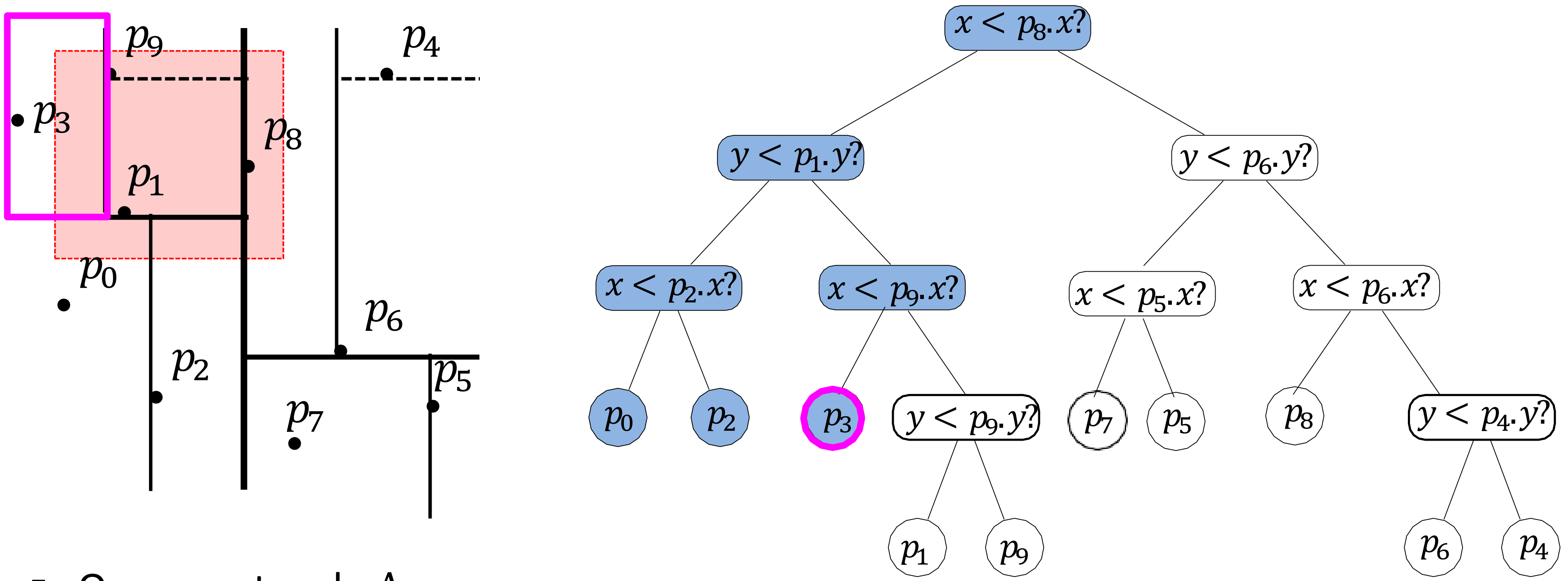

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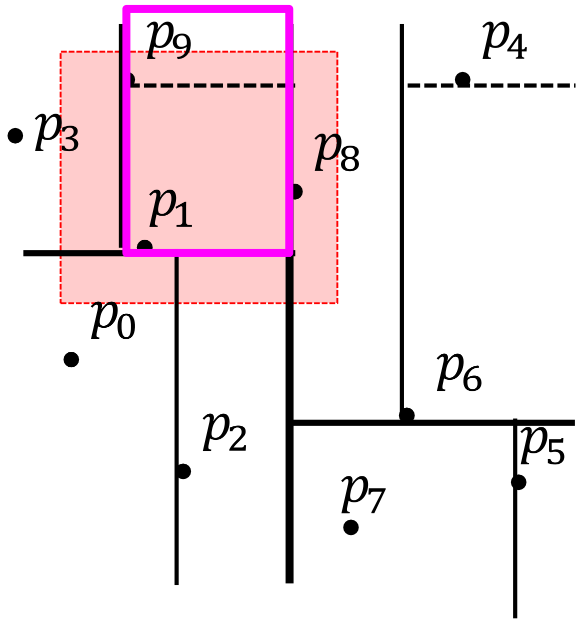

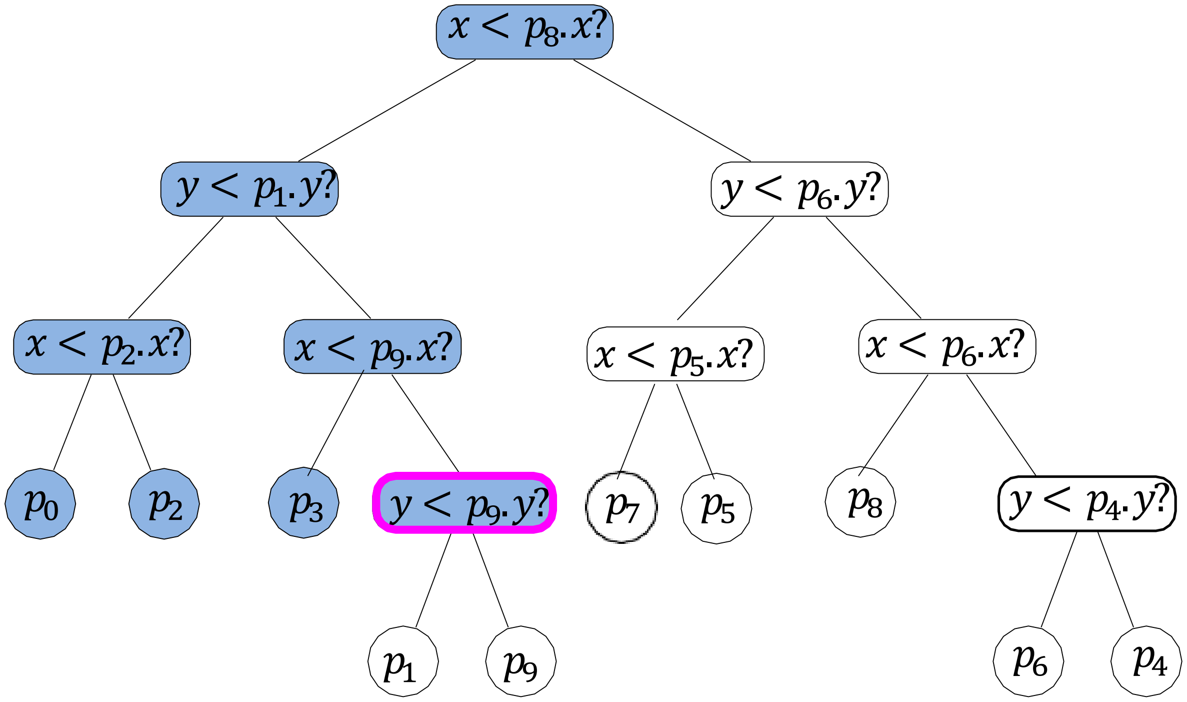

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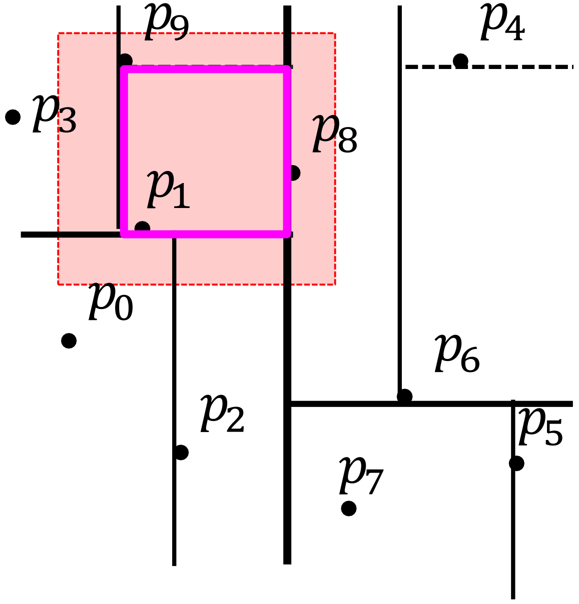

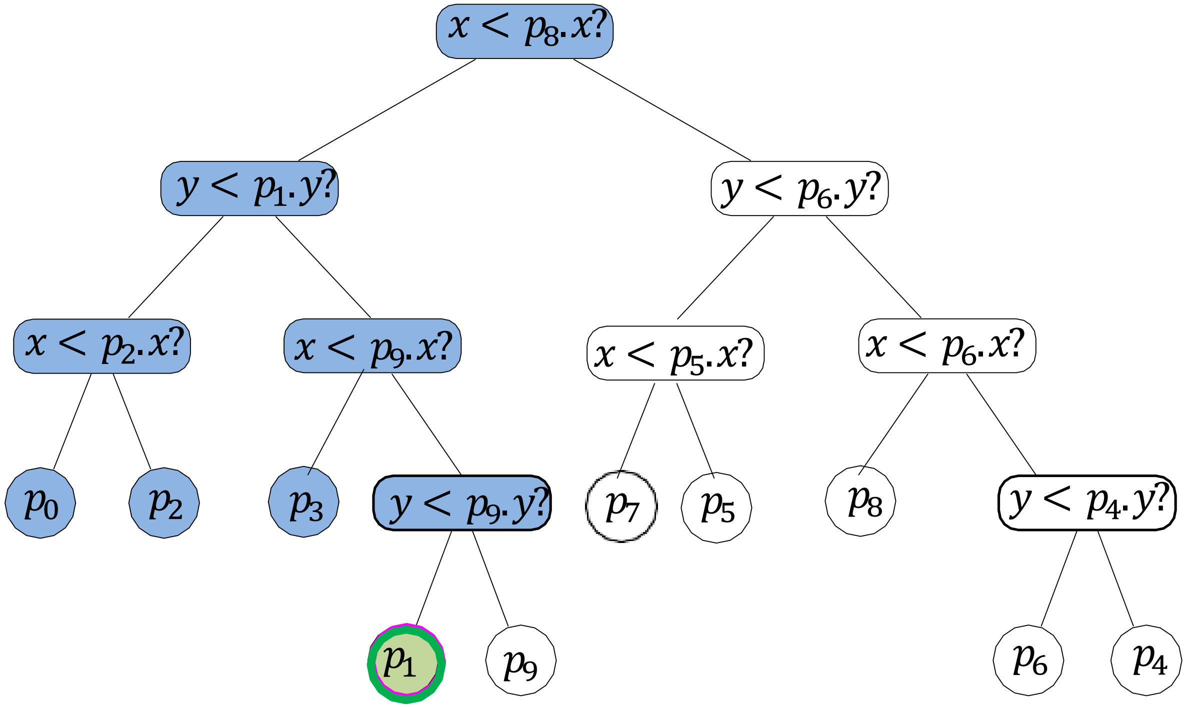

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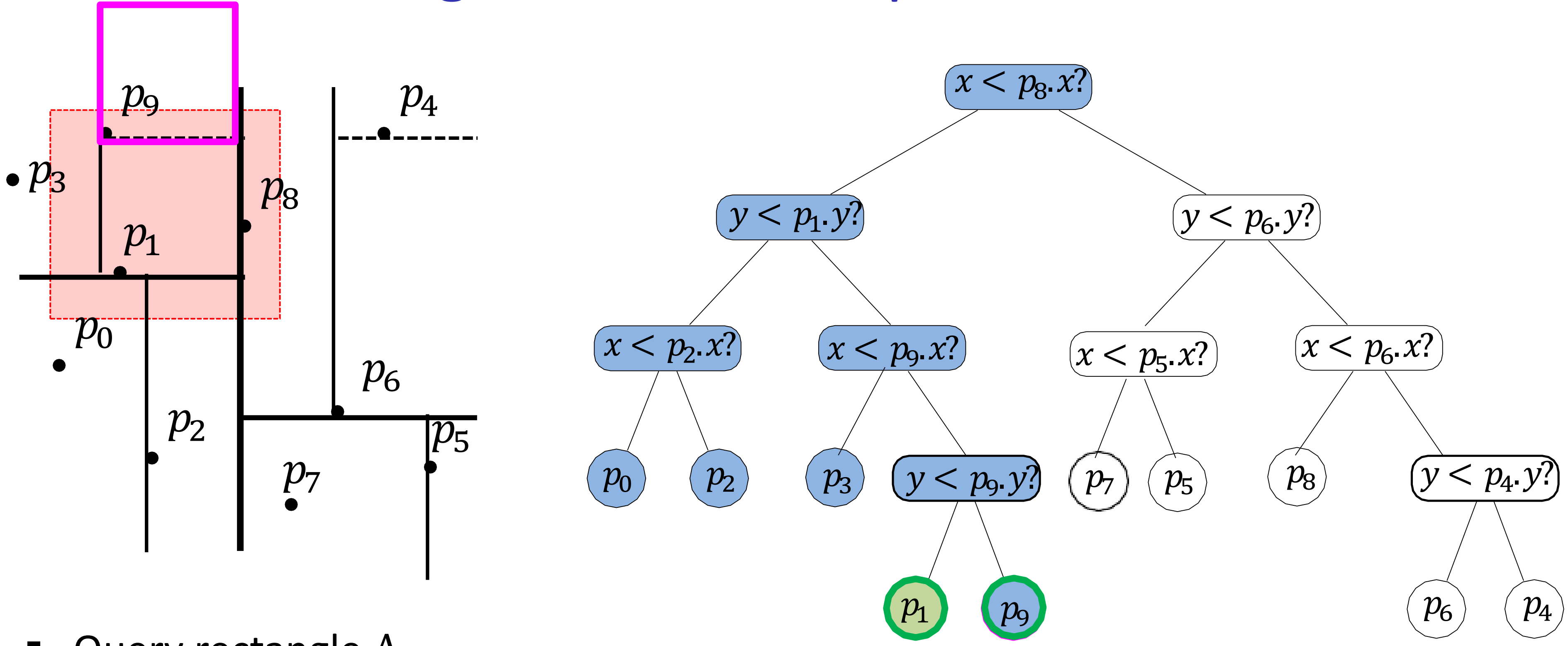

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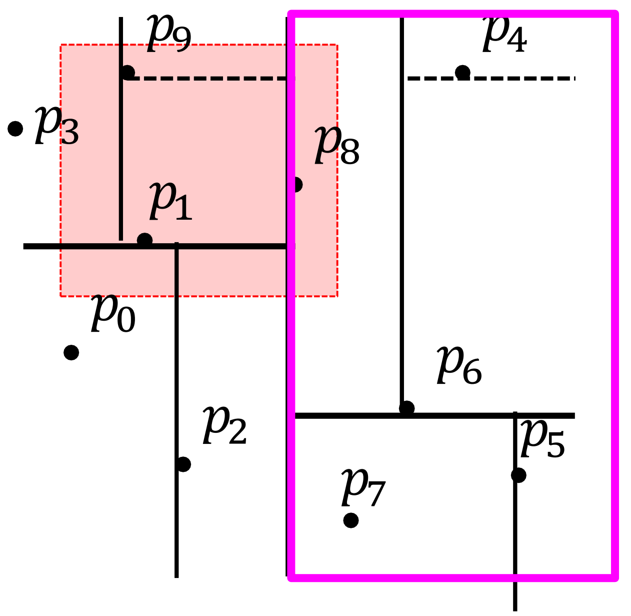

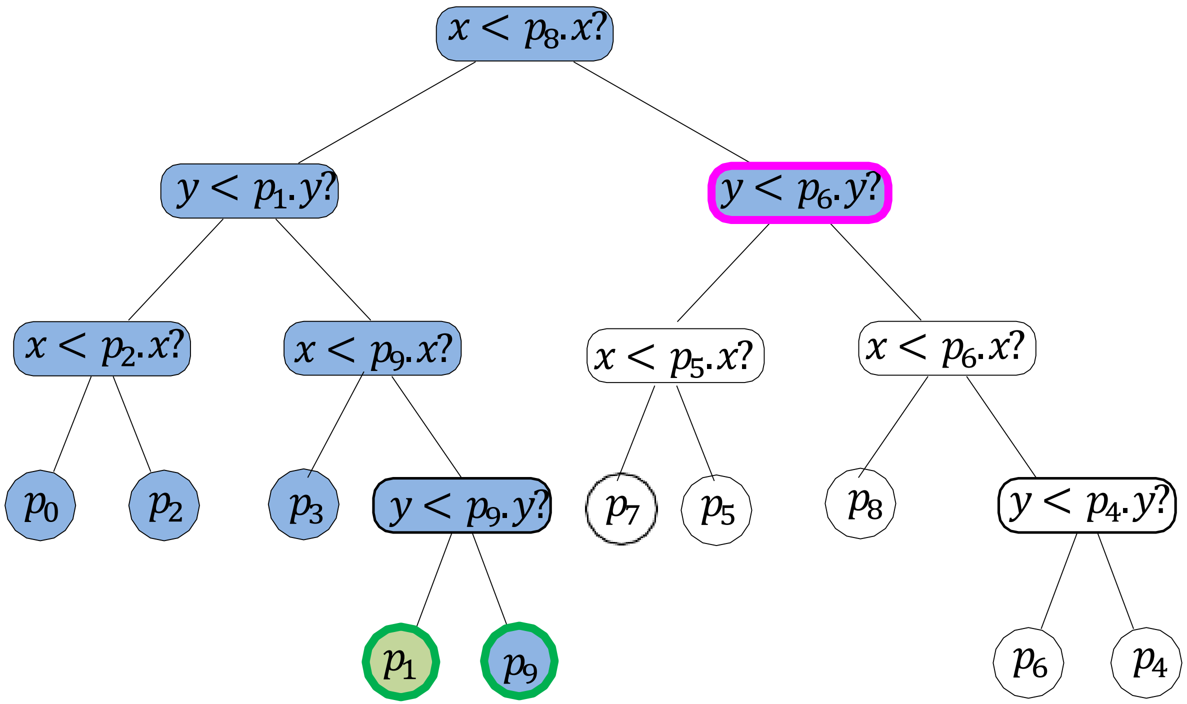

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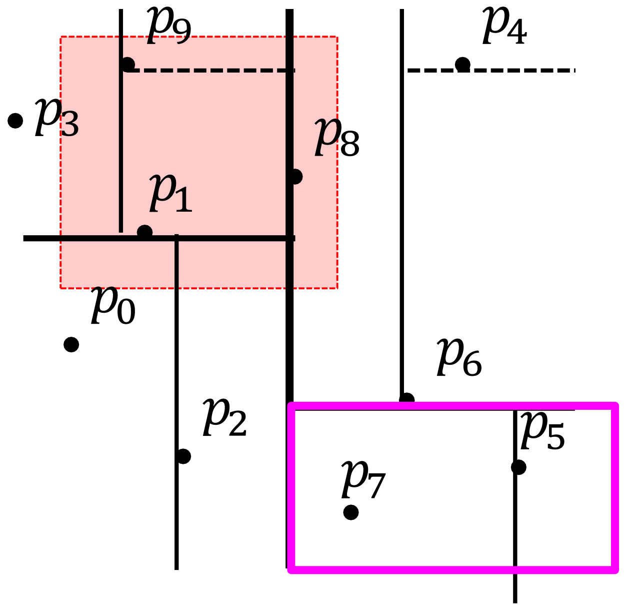

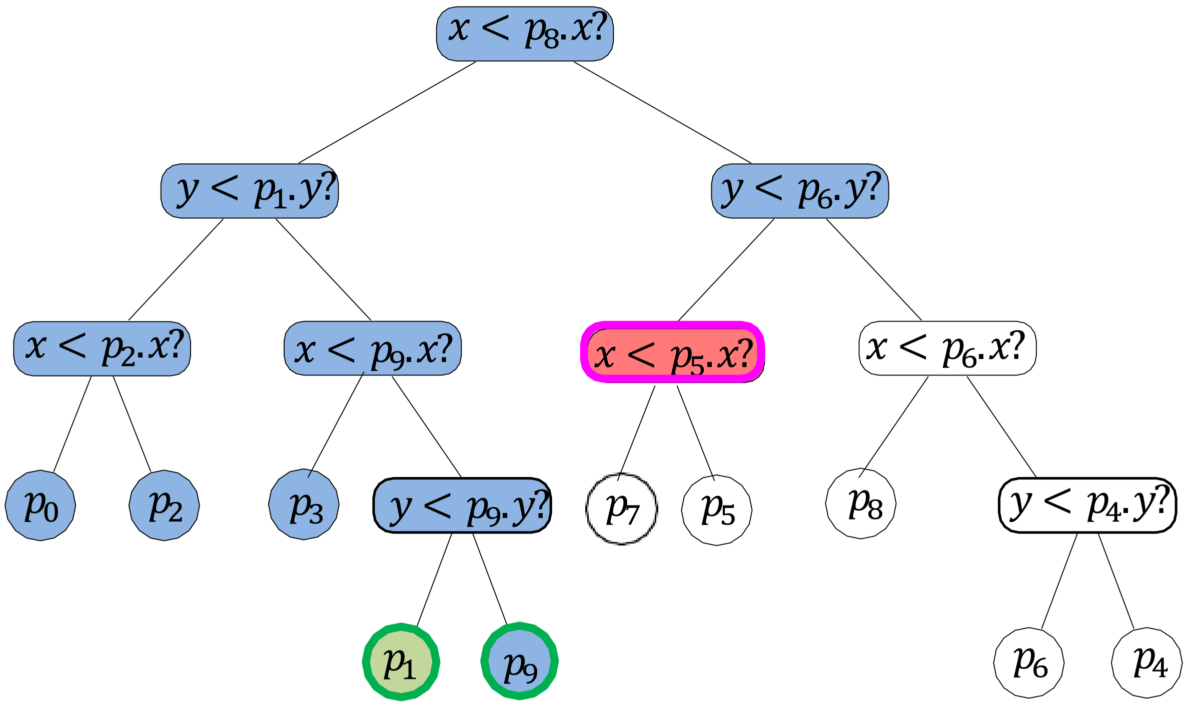

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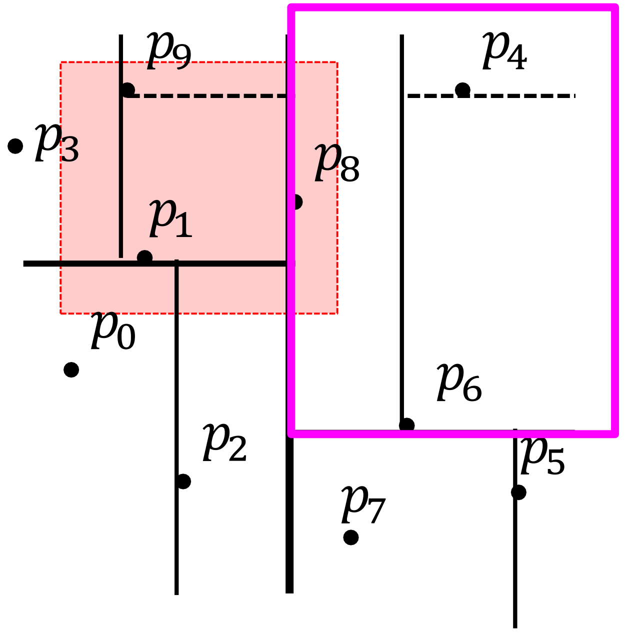

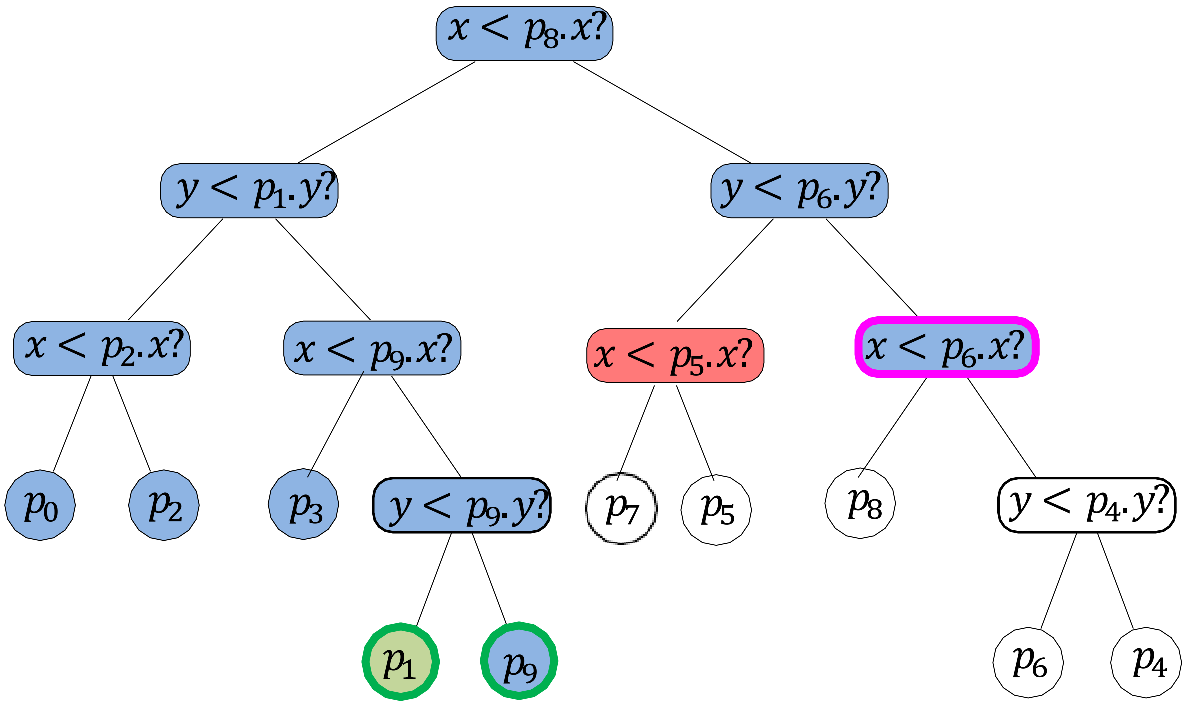

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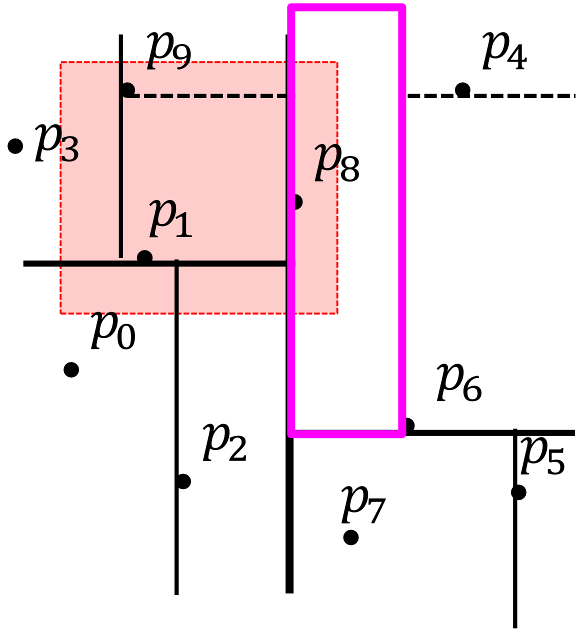

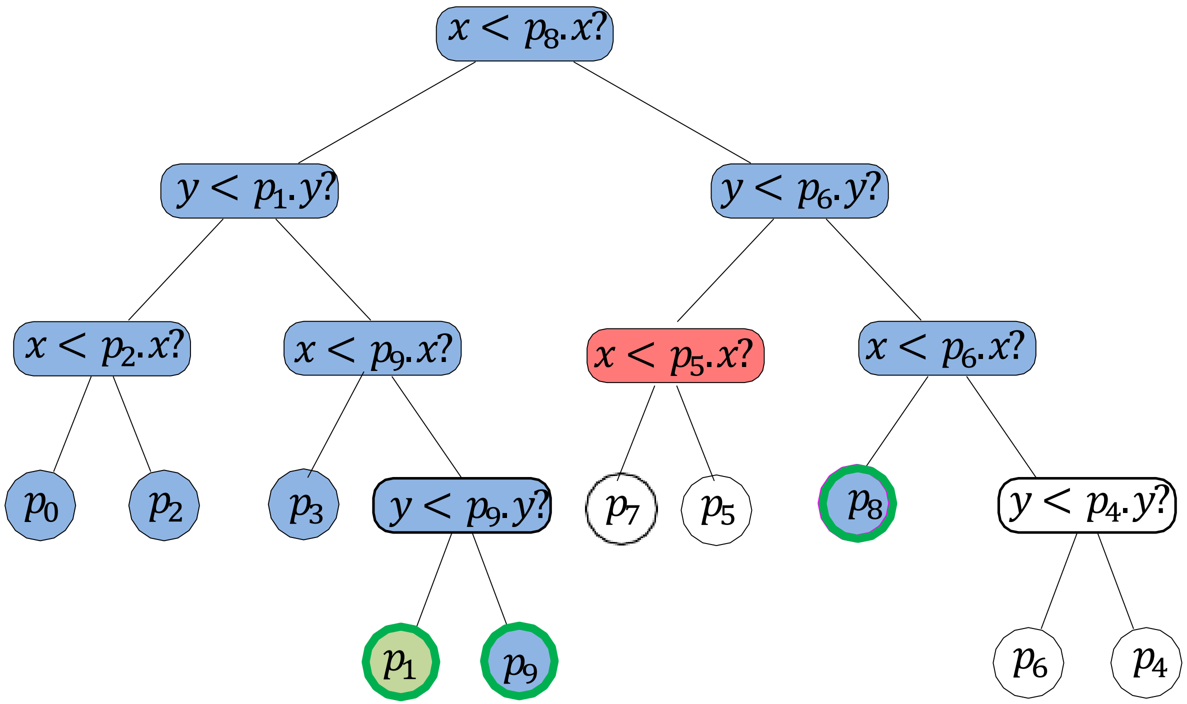

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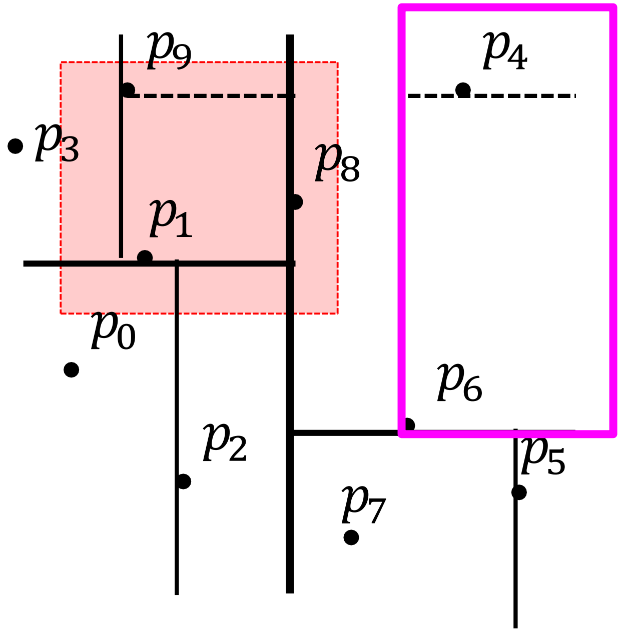

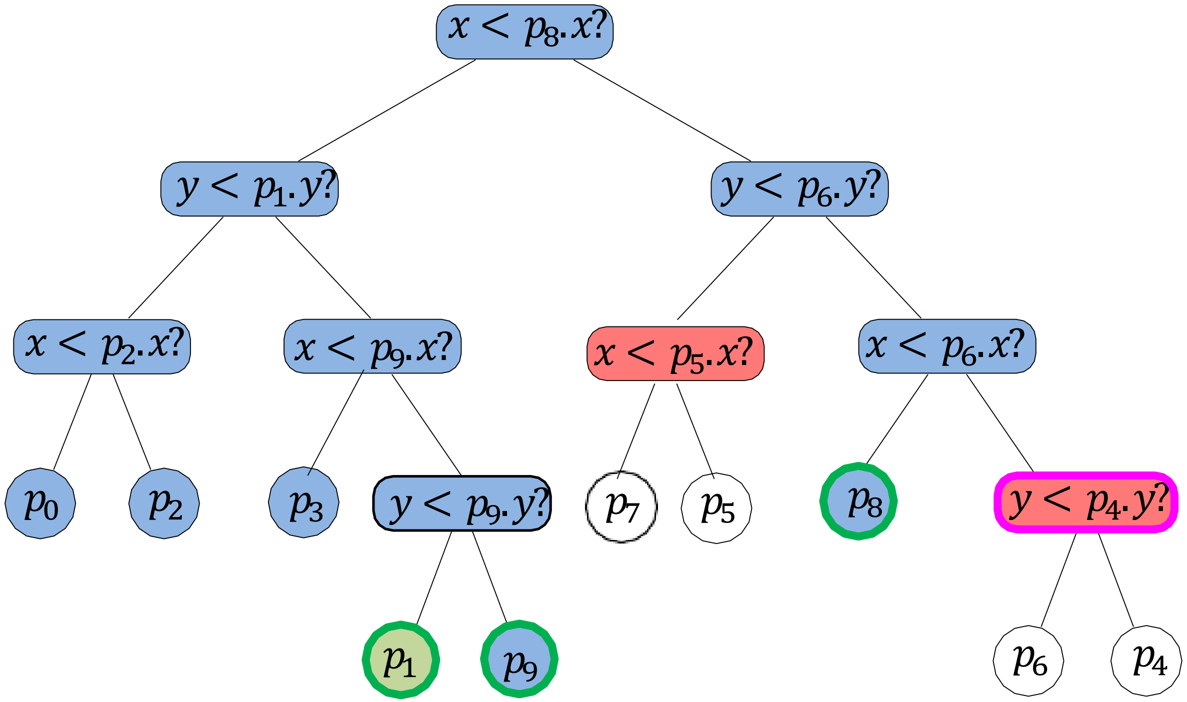

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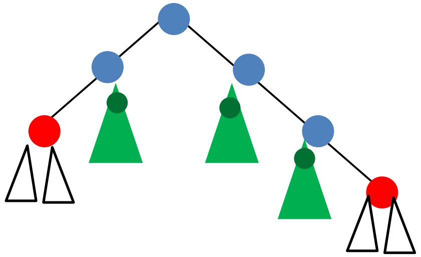
kd-tree Range Search

```
kdTree::RangeSearch($r \leftarrow$ root, $A$)
$r$ : root of kd-tree, $A$: query rectangle
        $R \leftarrow$ region associated with node $r$
        if $R \subseteq A$ then
                report all points below $r$
                return
        if $R \cap A = \emptyset$ then return
        if $r$ is a leaf then
              $p \leftarrow$ point stored at $r$
              if $p \in A$ return $p$
              else return
        for each child $v$ of $r$ do
              kdTree::RangeSearch($v$, $A$)
```

- We assume that each node stores its associated region
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line

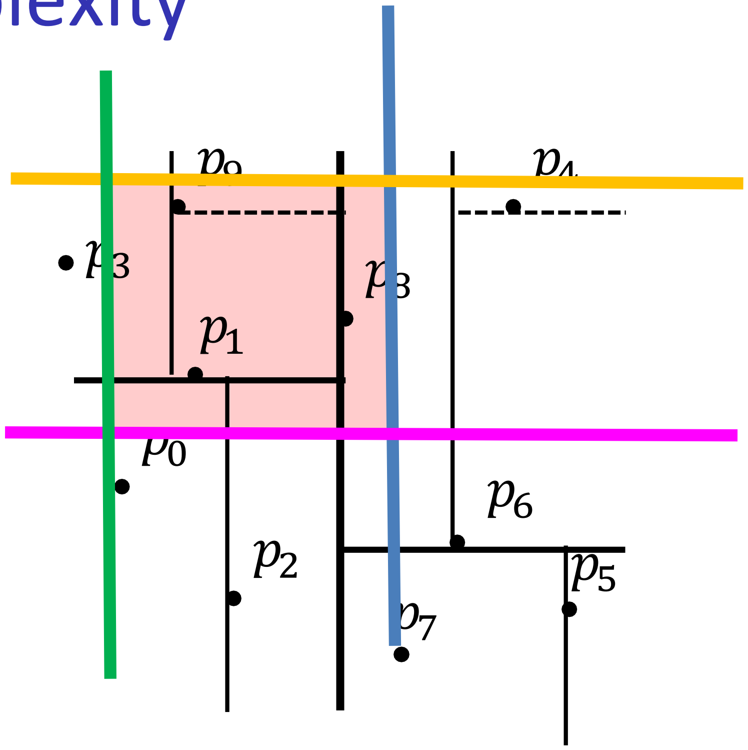
kd-tree: Range Search Complexity

- We visit blue and red nodes and also green nodes
    - at each blue, red and topmost green node do a constant amount of work
    - for each topmost green node v, report points stored at leaves in the subtree rooted at v
        - each node has 2 children → number of internal nodes is less than number of leaves for any subtree
        - at most s leaves over all green subtrees, at most $2s$ nodes over all green subtrees, $O(s)$ work to report points stored in green subtrees
    - topmost green nodes + red nodes $\le 2 \cdot$ blue nodes
        - each topmost green and red node has a blue parent
    - for running time, enough to count blue nodes and add $O(s)$



- Let $Q(n)$ is the number of blue nodes visited
    - neither $R \cap A = \emptyset$ nor $R \subseteq A$
    - these are regions that intersect A but not completely inside A
- Can show that $Q(n)$ satisfies $Q(n) \le 2Q\left(\frac{n}{4}\right) + O(1)$
    - resolves to $Q(n) \in O(\sqrt{n})$
- Therefore, running time of range search is $O(s + \sqrt{n})$

kd-tree: Range Search Complexity

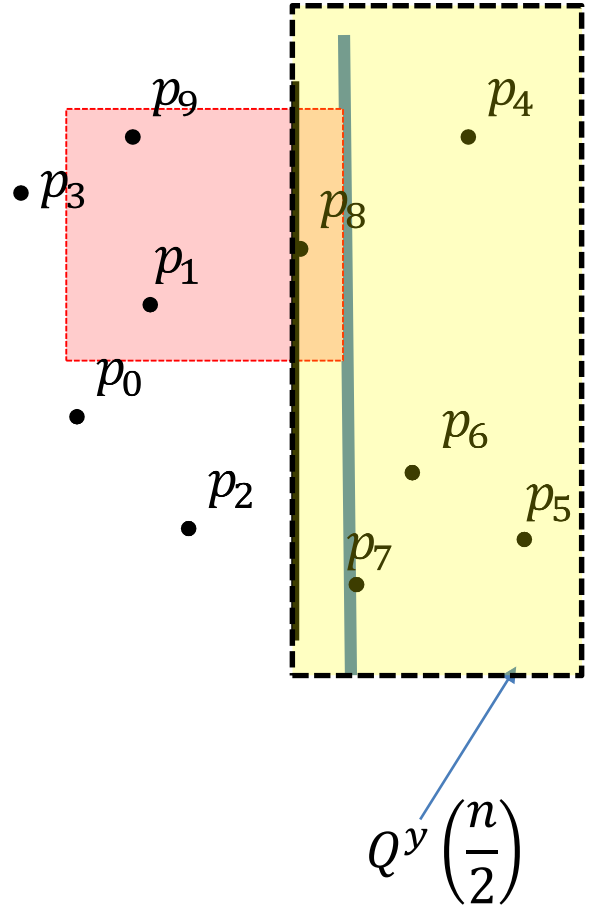


- search rectangle A
- $Q(n) =$ # regions intersecting A but not completely inside A
- $Q(n) \leq$ # regions intersecting 🟦
  $+$ # regions intersecting 🟩
  $+$ # regions intersecting 🟪
  $+$ # regions intersecting 🟨
- Will look at # regions intersecting 🟦
- Other cases are handled similarly

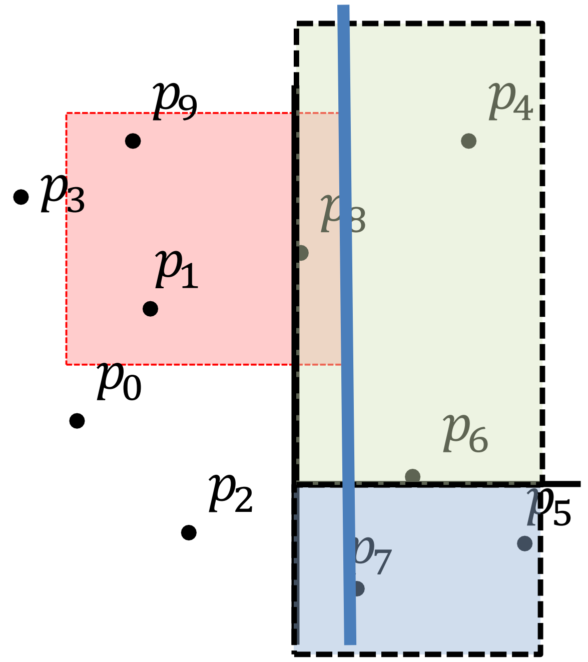
kd-tree: Range Search Complexity

- $Q^x(n)$ = # regions intersected by ▌, if tree root split by x coordinate

- $Q^x(n) = 1 + Q^y\left(\frac{n}{2}\right)$
    - 1 for the root region R
    - root region is split in 2 by vertical line
    - ▌ can intersect only one of these regions



kd-tree: Range Search Complexity



- $Q^x(n)$ = # regions intersected by |, if tree root split by x coordinate

- $Q^x(n) = 1 + Q^y\left(\frac{n}{2}\right)$
    - 1 for the root region
    - root region is split in 2 by vertical line
    - | can intersect only one of these regions

- Next, $Q^y\left(\frac{n}{2}\right) = 1 + 2Q^x\left(\frac{n}{4}\right)$
    - 1 for the root region
    - root region is split in 2 by horizontal line
    - | can intersect both of these regions

- Combining, get recurrence $Q^x(n) = 2 + 2Q^x\left(\frac{n}{4}\right)$

- Resolves to $Q^x(n) \in O(\sqrt{n}\,)$

kd-tree: Higher Dimensions

- kd-trees for d-dimensional space
  - at depth 0 (the root) partition is based on the 1st coordinate
  - at depth 1 partition is based on the 2nd coordinate
  - …
  - at depth $d-1$ the partition is based on the last coordinate
  - at depth d start all over again, partitioning on 1st coordinate
- Storage $O(n)$
- Height $O(\log n)$
- Construction time $O(n \log n)$
- Range query time $O(s + n^{1-\frac{1}{d}})$
  - assumes that d is a constant

Outline

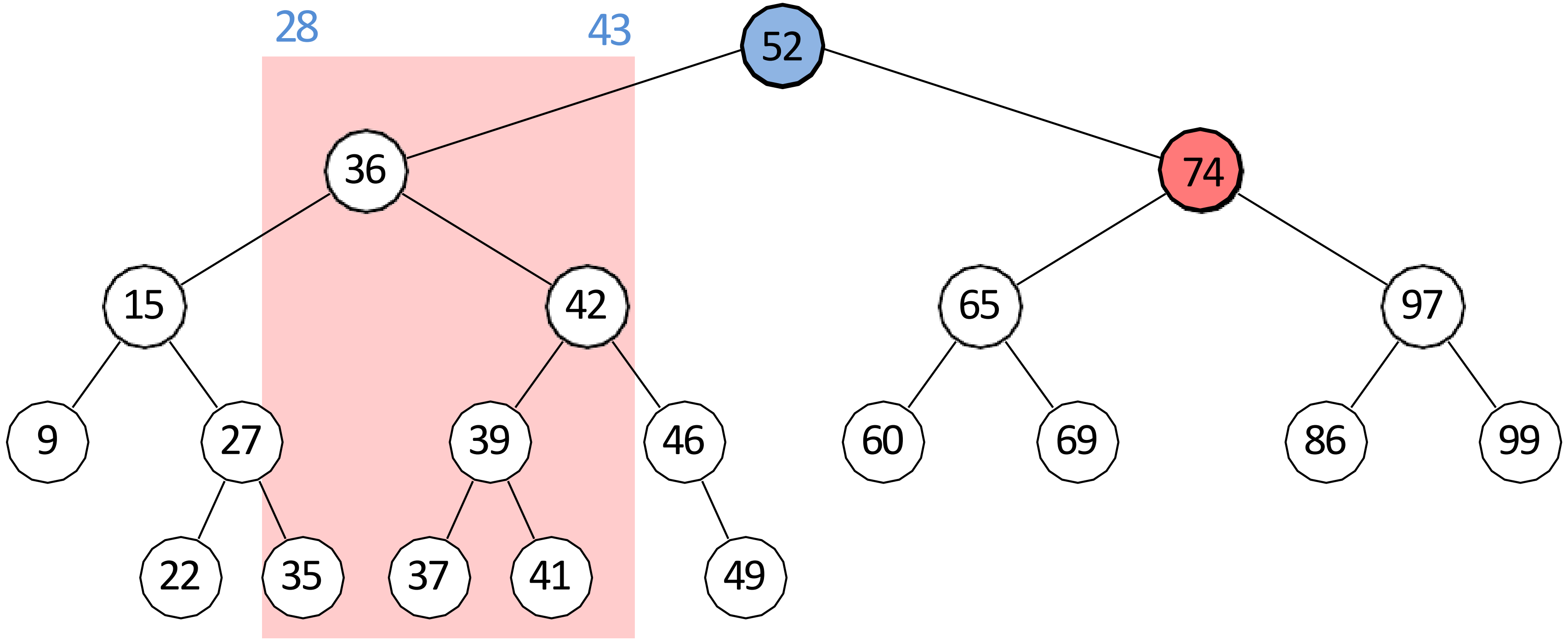
- Range-Searching in Dictionaries for Points
  - Range Search
  - Multi-Dimensional Data
  - Quadtrees
  - kd-Trees
  - Range Trees
  - Conclusion

Towards Range Trees

- Quadtrees and kd-trees
  - intuitive and simple
  - but both may be slow for range searches
  - quadtrees are also potentially wasteful in space
- Consider BST/AVL trees
  - efficient for one-dimensional dictionaries, if balanced
    - range search is also efficient
  - can we use ideas from BST/AVL trees for multi dimensional dictionaries?
- First let us consider range search in BST

BST Range Search example

BST::RangeSearch-recursive$(T, 28, 43)$



- blue node: recurse either to the left, or to the right, or both (according to the key value)
    - boundary node, one or both subtrees may intersect range query
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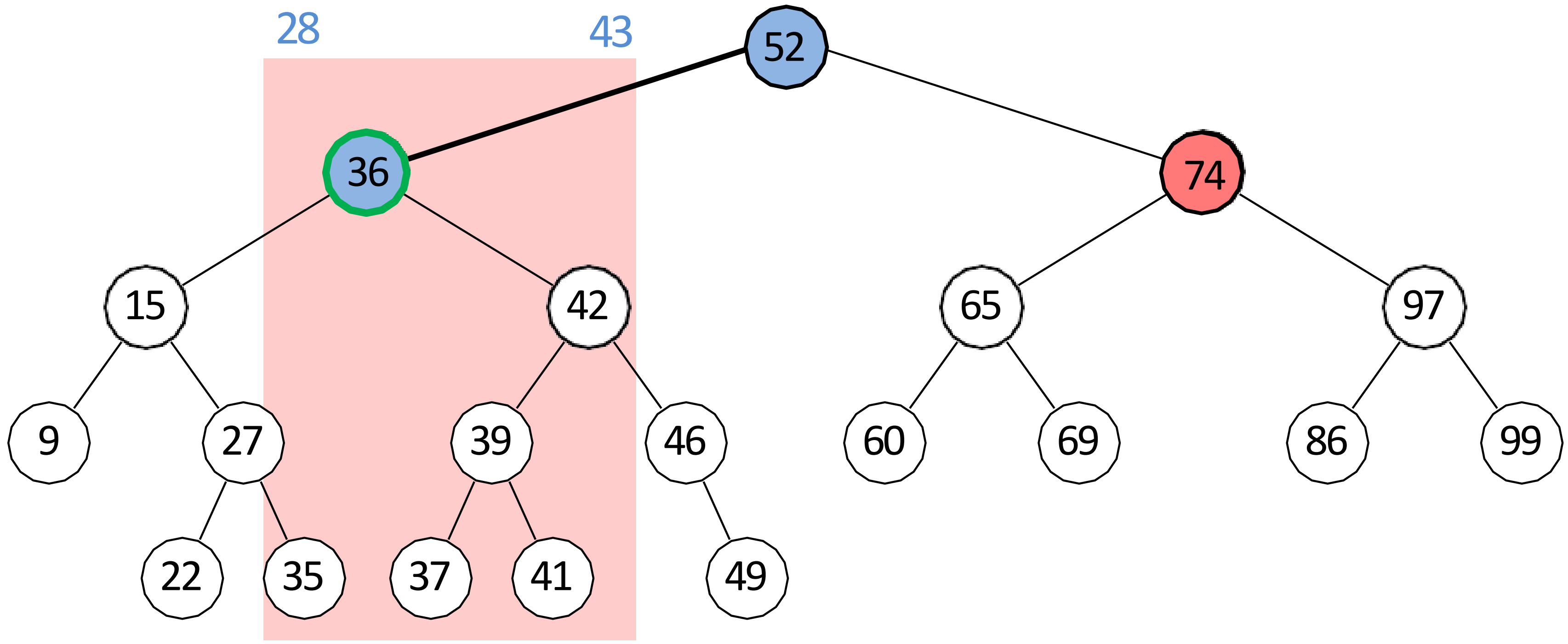
    - outside node, subtree does not intersect range query
- green node : all the keys in the subtree are in the range
    - inside node, subtree completely inside range query

BST Range Search example

BST::RangeSearch-recursive$(T, 28, 43)$



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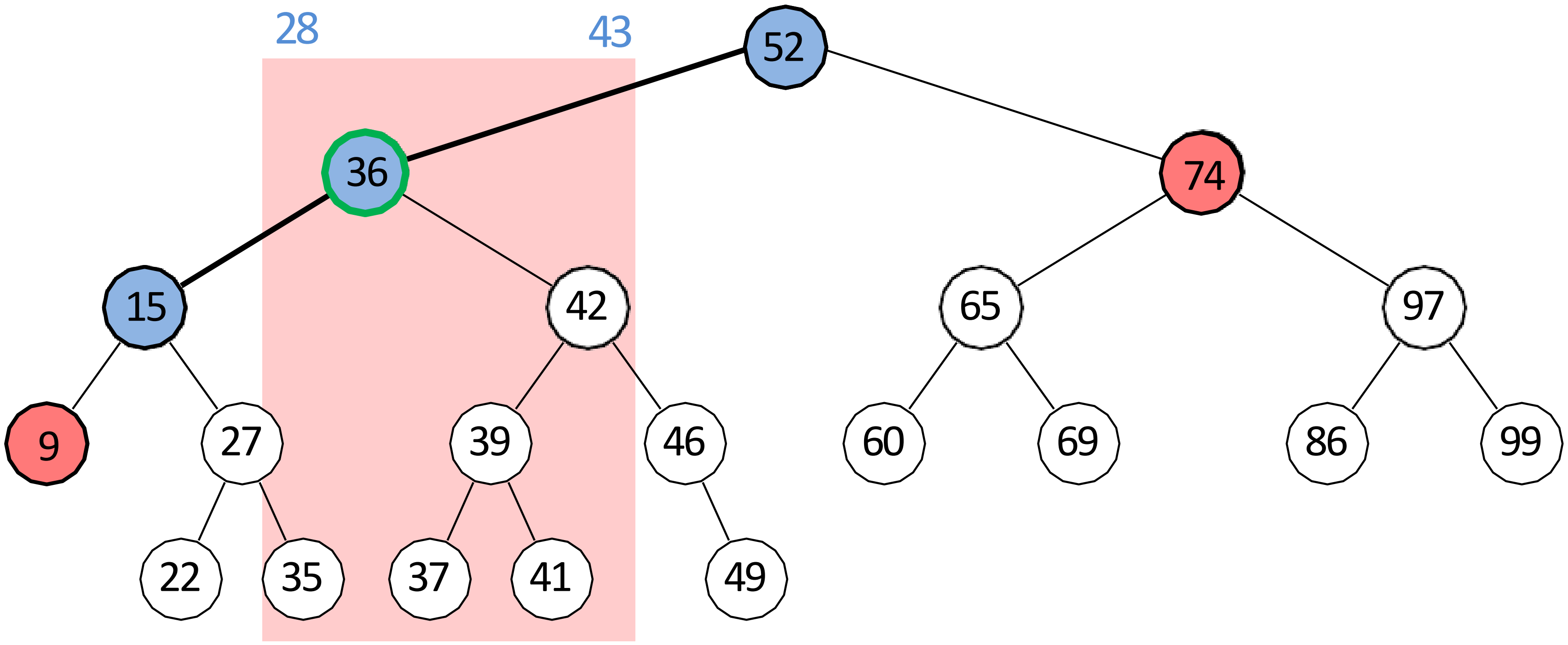

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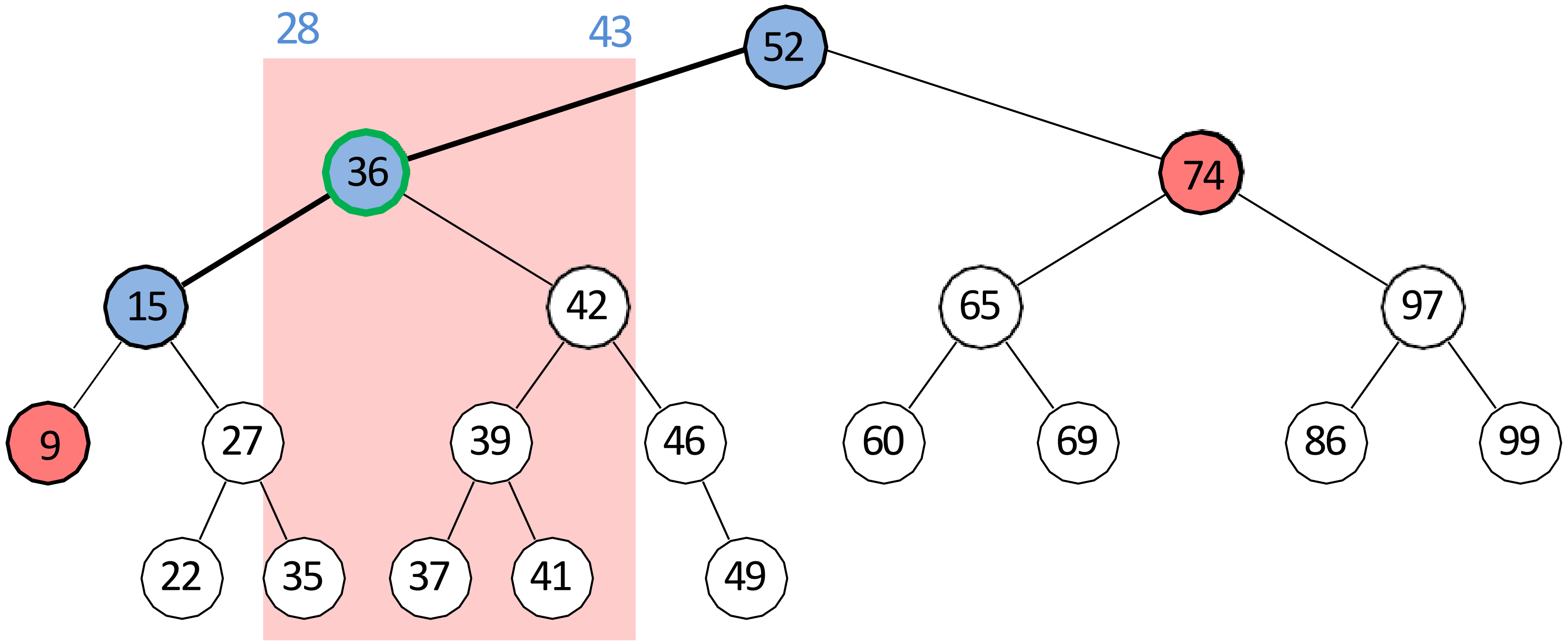

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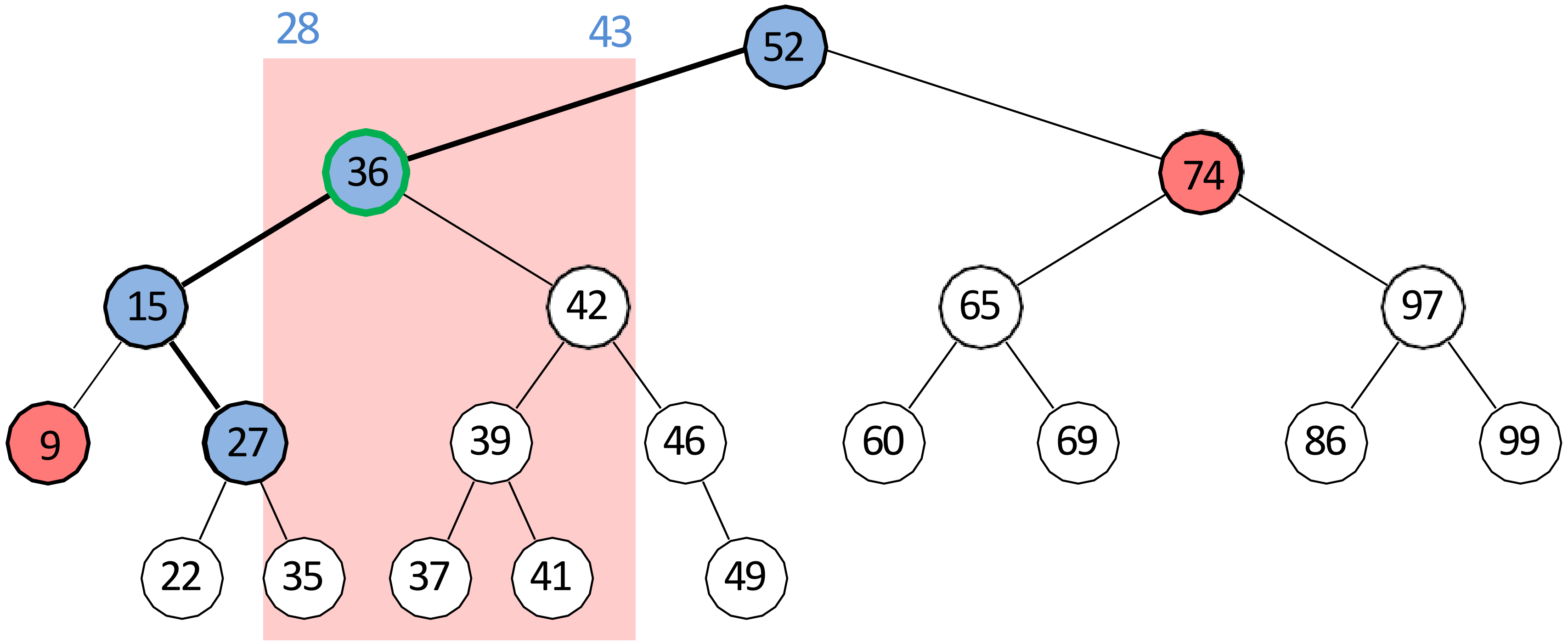

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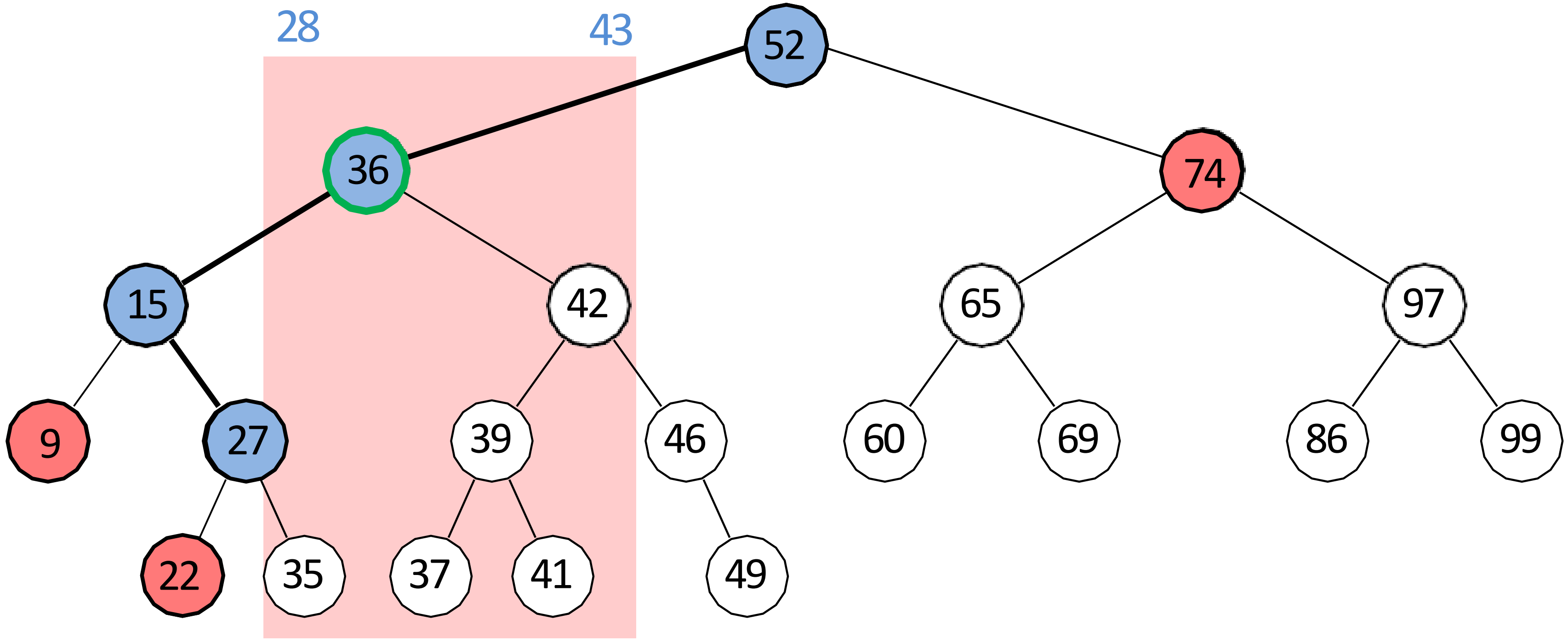

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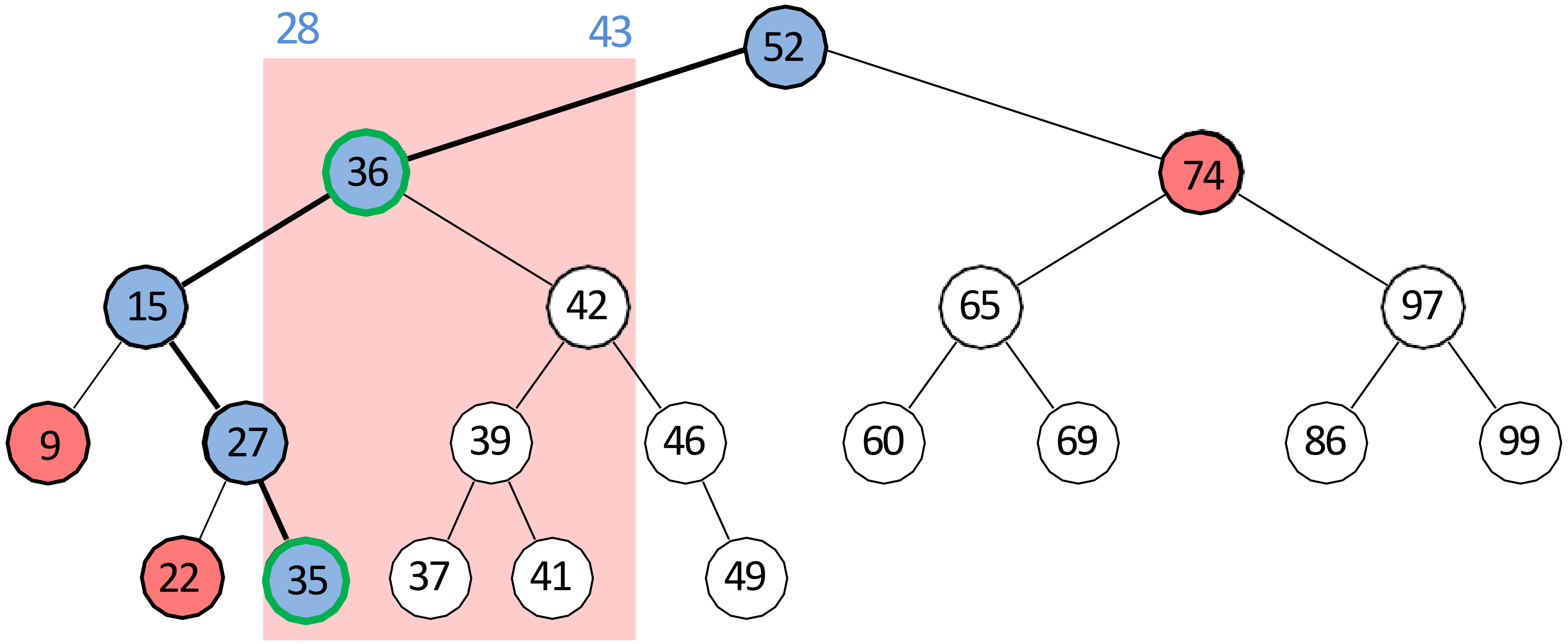

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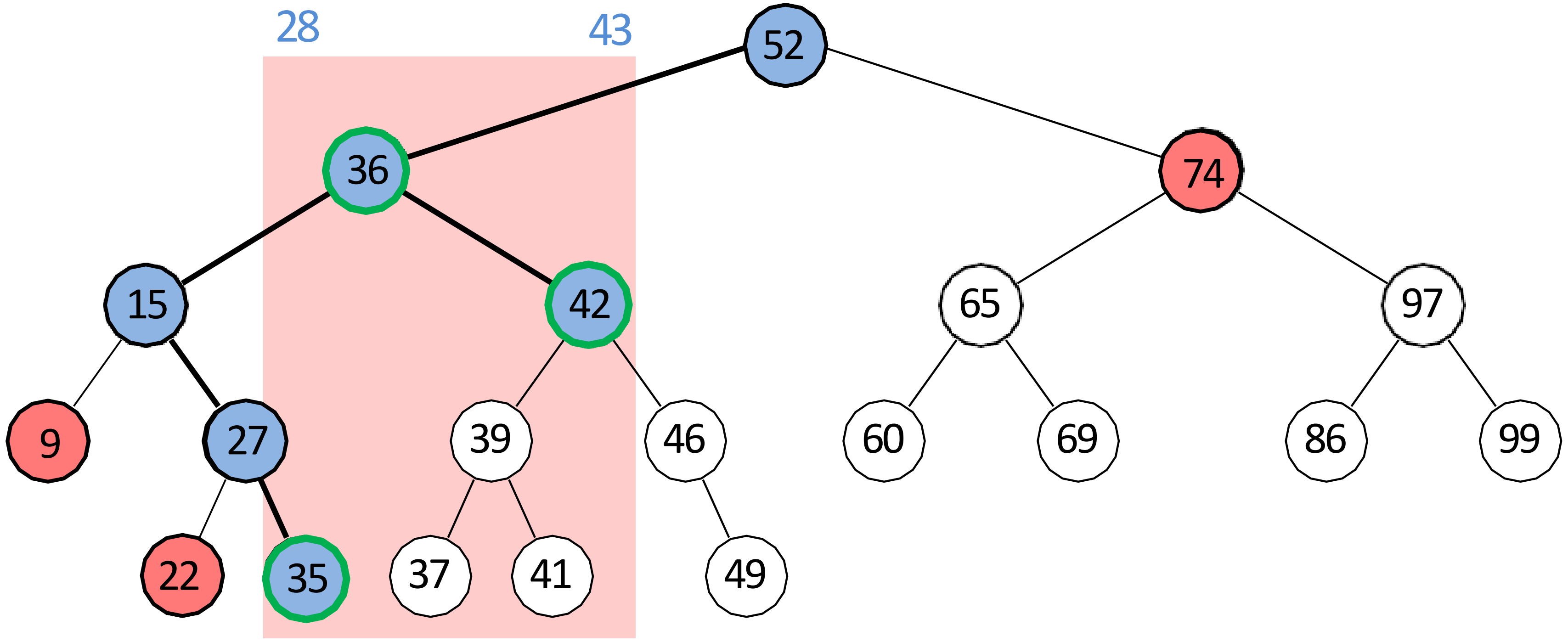

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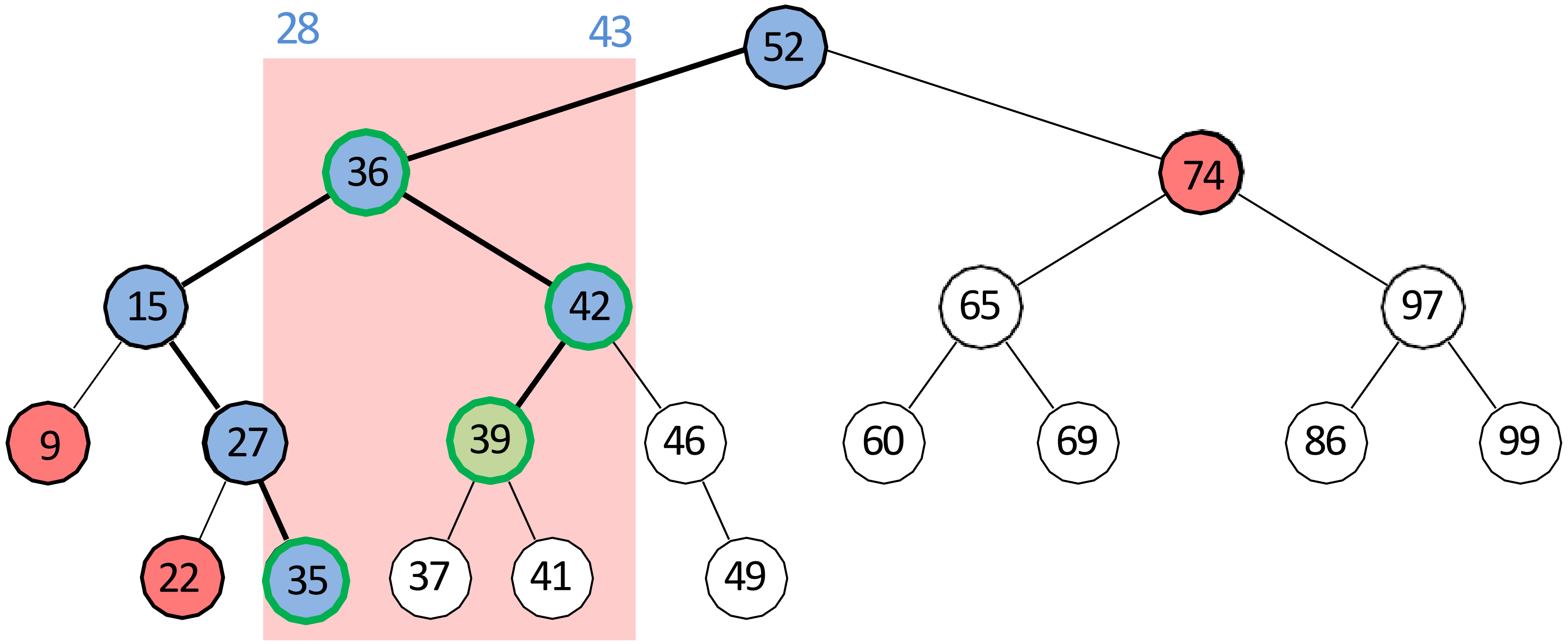

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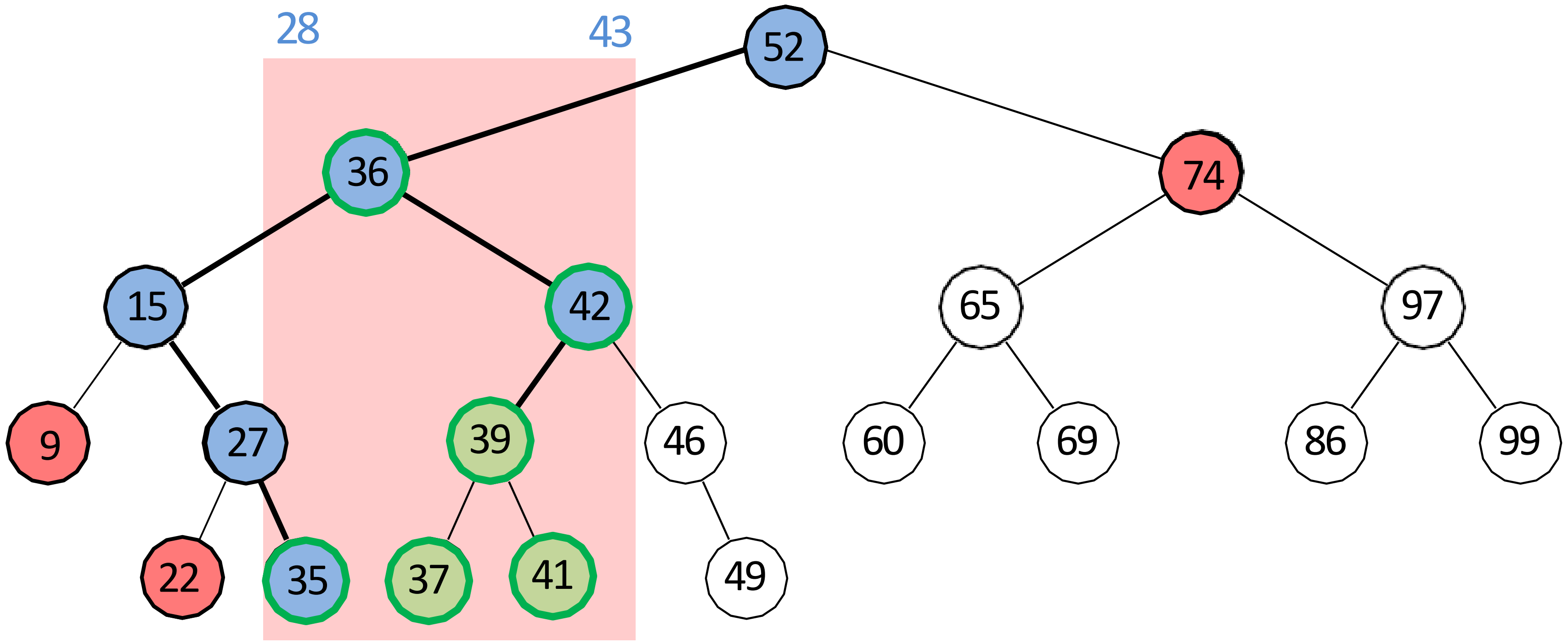

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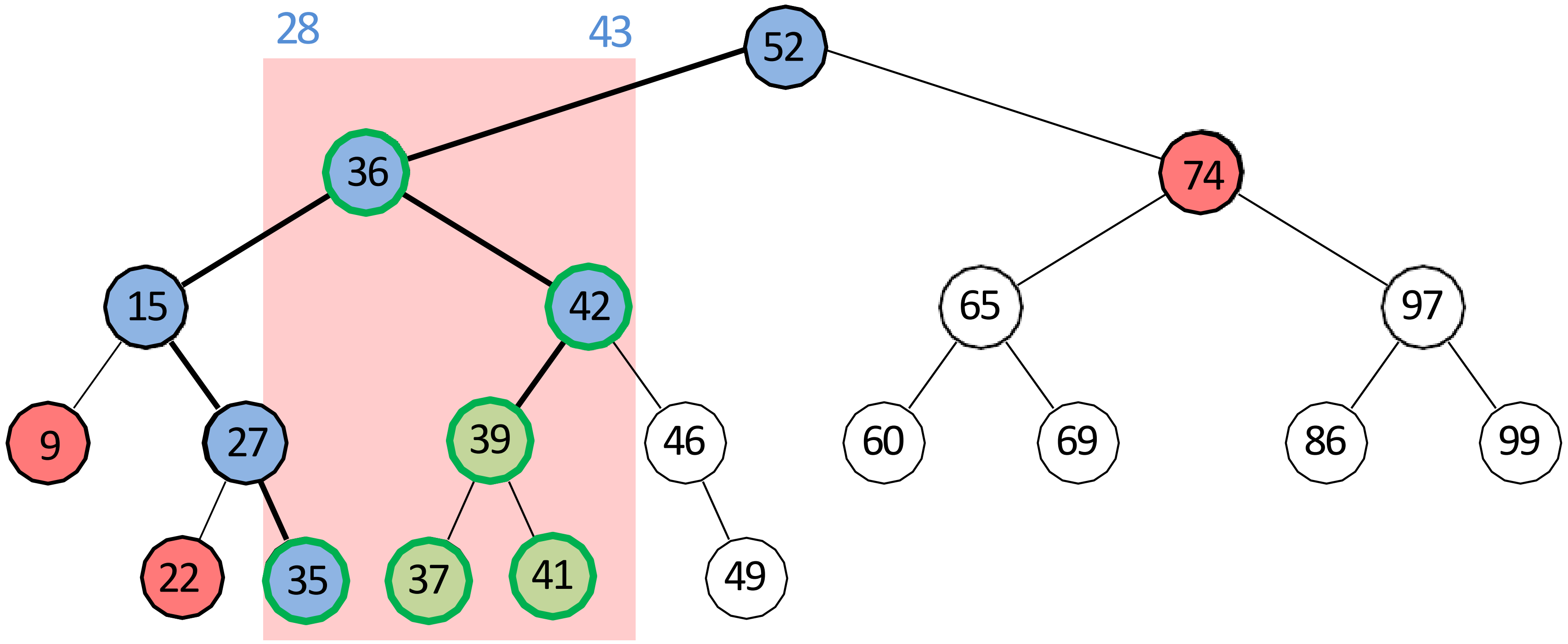

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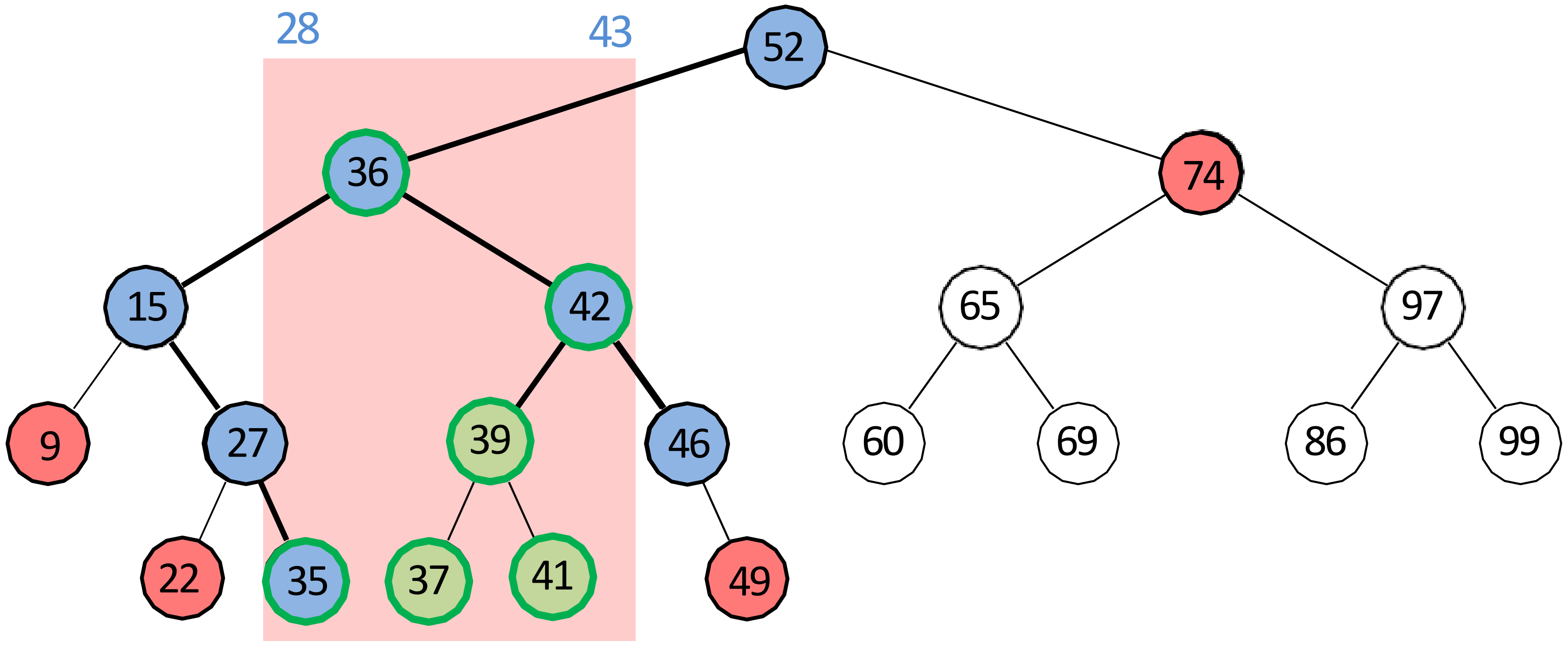

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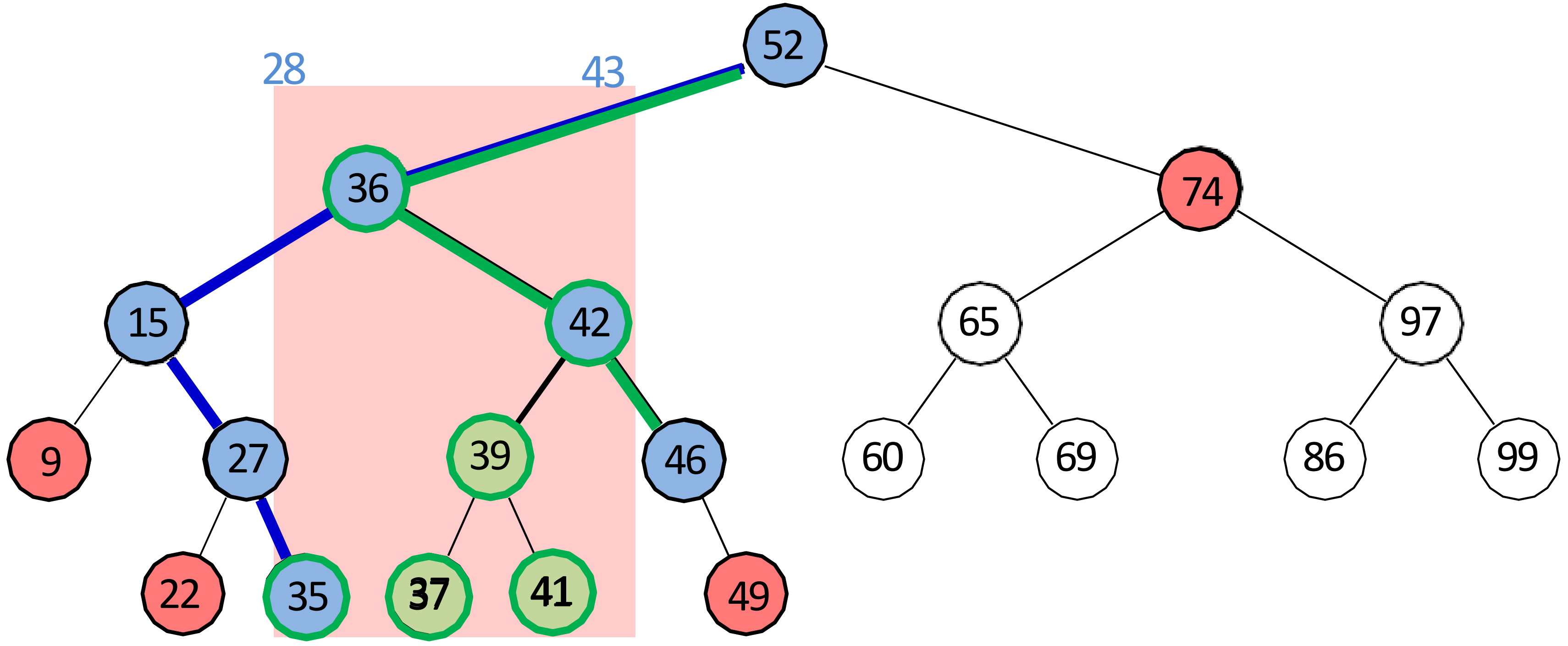
BST Range Search

```
*BST::RangeSearch-recursive*$(r \leftarrow root, k_1, k_2)$
$r$:  root of a binary search tree, $k_1, k_2$: search keys
Returns keys in subtree at $r$ that are in range $[k_1, k_2]$
 **if** $r = NIL$  **then return**
 **if** $k_1 \leq r.key \leq k_2$ **then**
        $L \leftarrow$ *BST::RangeSearch-recursive*$(r.left, k_1, k_2)$
        $R \leftarrow$ *BST::RangeSearch-recursive*$(l.right, k_1, k_2)$
        **return** $L \cup \{r.key\} \cup R$
 **if** $r.key < k_1$ **then**
        return *BST::RangeSearch-recusive*$(r.right, k_1, k_2)$
 **if** $r.key > k_2$ **then**
        return *BST-RangeSearch-recursive*$(r.left, k_1, k_2)$
```

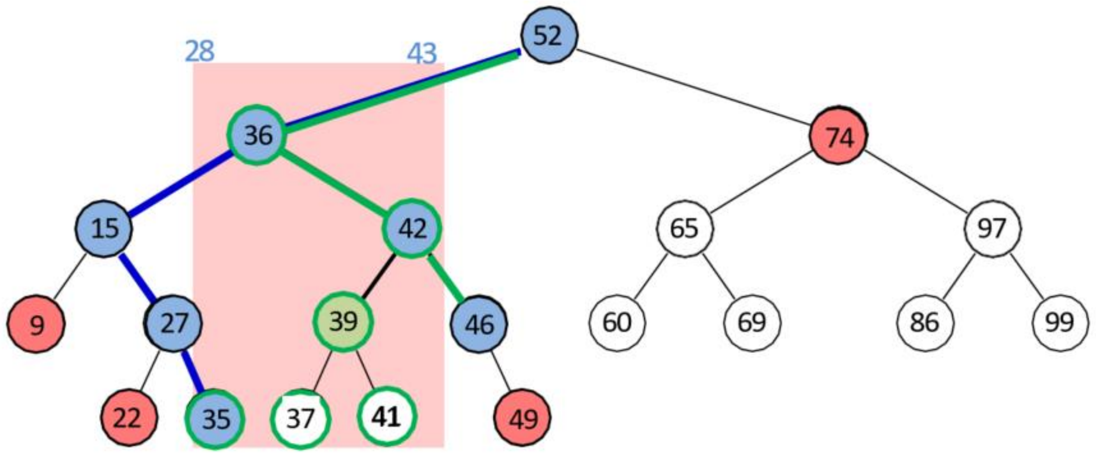
- Keys returned in sorted order

Modified BST Range Search



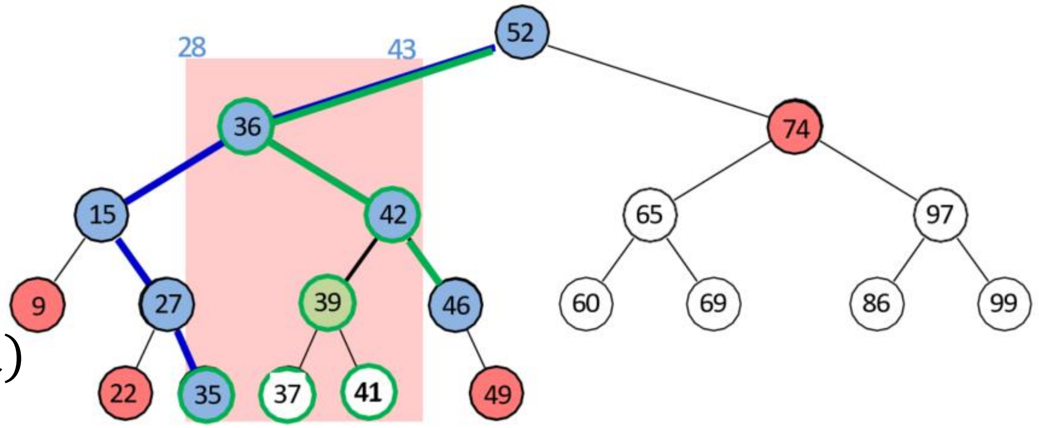
- Search for left boundary k_1 : this gives path ***P*₁**
- Search for right boundary k_2 : this gives path ***P*₂**
- Boundary (blue nodes) are exactly all the nodes on paths ***P*₁** and ***P*₂**
- Nodes are partitioned into three groups: boundary, outside, inside

Modified BST Range Search



- Boundary nodes: nodes in P_1 and P_2
    - check if boundary nodes are in the search range
- Outside nodes: nodes that are left of P_1 or right of P_2
    - outside nodes are not in the search range
    - range search is never called on an outside node
- Inside nodes: nodes that are right of P_1 and left of P_2
    - **we will stop the search at the topmost inside node**
    - all descendants of such node are in the range, just report them without search
    - this is not more efficient for BST range search, but will be efficient when we move to 2D search in ***range trees***

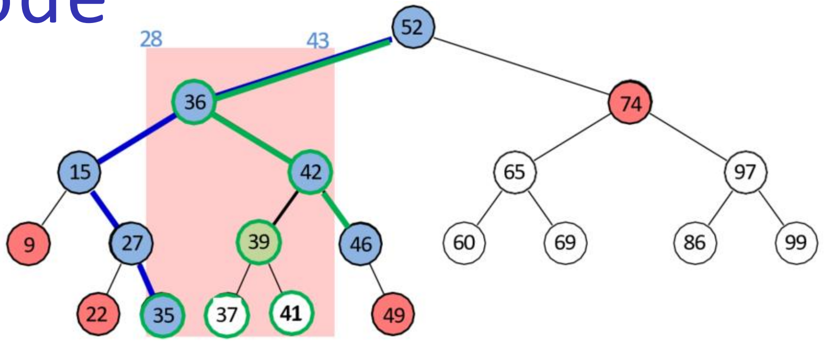
Modified BST Range Search Analysis

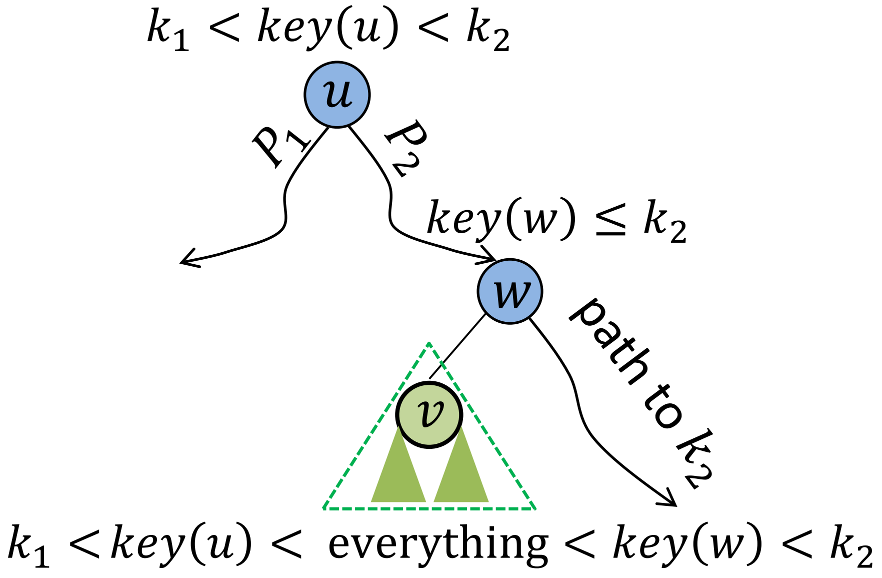


- Assume balanced BST
- Running time consists of
  1. search for path ***P*₁**
     - $O(\log n)$
  2. search for path ***P*₂** is $O(\log n)$
     - $O(\log n)$
  3. check if boundary nodes in the range
     - $O(1)$ at each boundary node, there are $O(\log n)$ of them, $O(\log n)$ total time
  4. spend $O(1)$ at each topmost inside node
     - since each topmost inside node is a child of boundary node, there are at most $O(\log n)$ topmost inside nodes, so total time $O(\log n)$
  5. report descendants in subtrees of all topmost inside nodes
     - topmost nodes are disjoint, so #descendants for inside topmost nodes is at most s, output size $$\sum_{\text{topmost inside node } v} \#\text{descendants of } v \le s$$
- Total time $O(s + \log n)$

How to Find Top Inside Node

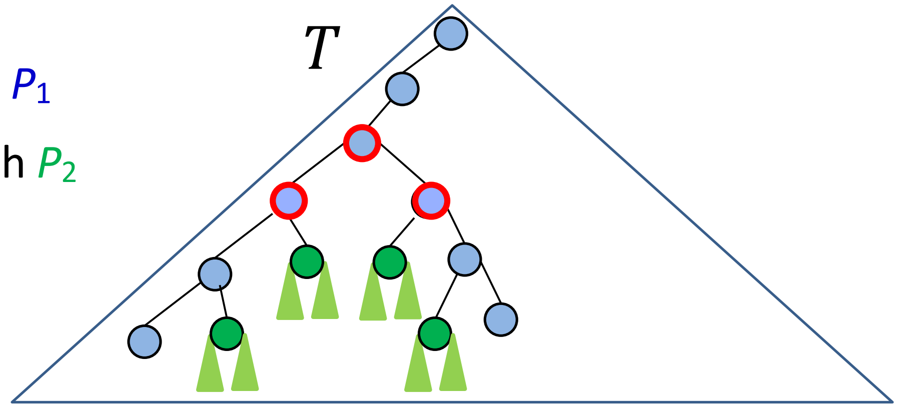
- v is a top inside node if
  - v is not is in P_1 or P_2
  - parent of v is in P_1 or P_2 (but not both)
  - if parent is in P_1, then v is right child
  - if parent is in P_2, then v is left child





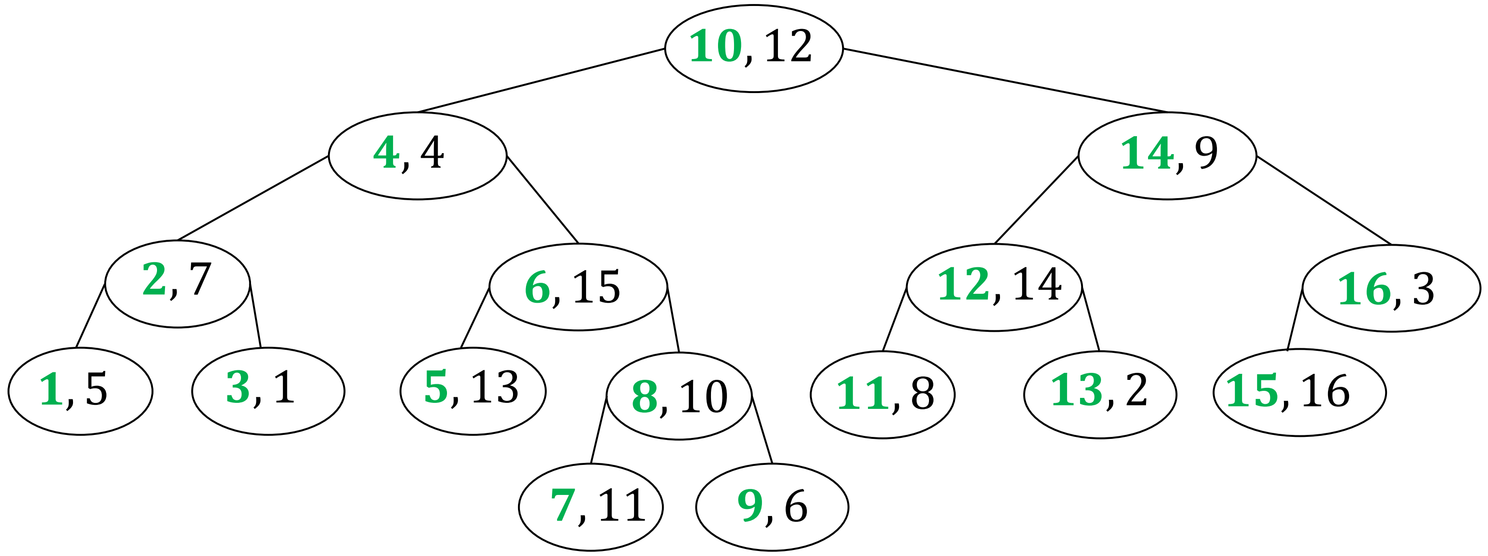
- Thus for each top inside node can report all descendants, no need for search
  - BST range search does not become not faster overall, but top inside nodes are important for $2d$ range search efficiency
  - also important if need to just count the number of points in the search range

Modified BST Range Search Summary



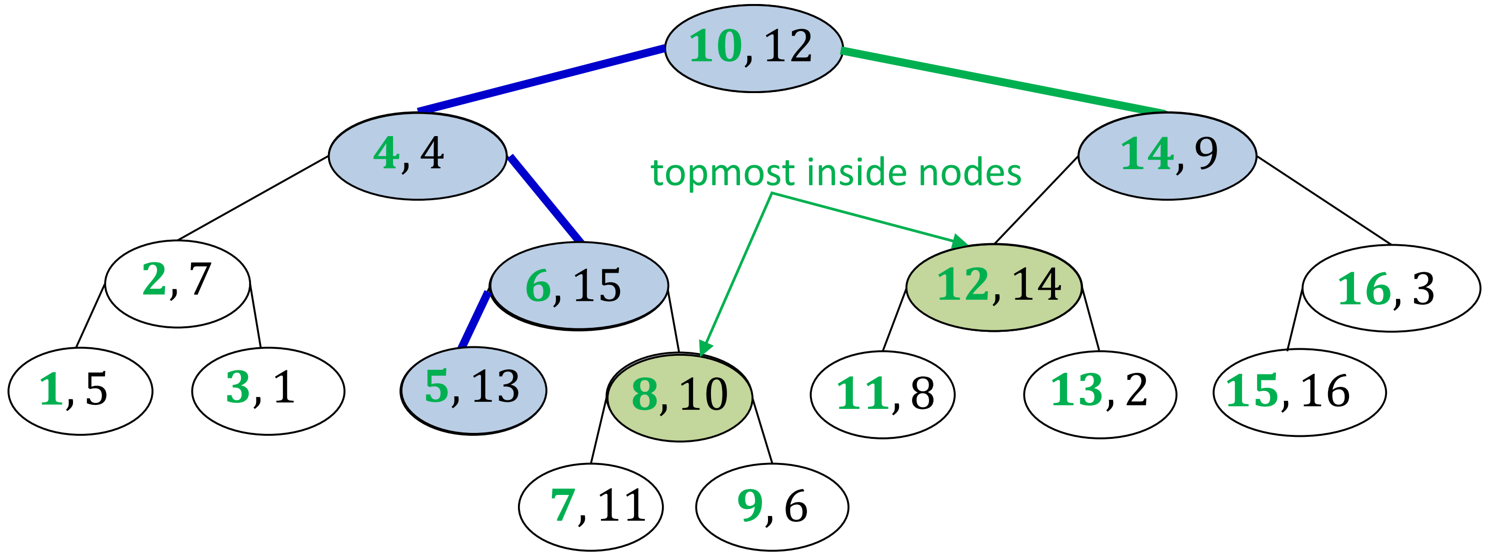
- Search for k_1: this gives left boundary path P_1
- Search for k_2: this gives right boundary path P_2
- Find all topmost inside nodes
  - not in P_1 or P_2
  - left children of nodes in P_2
  - right children of nodes in P_1
- Inside node (which is not a topmost inside) is in a subtree of some topmost inside node
- Set of inside nodes = union disjoint subtrees rooted at topmost inside nodes
- To output nodes in the search range
  - test each node in P_1 , P_2 and report if in range
  - go over all topmost inside nodes and report all nodes in their subtree

2D Range Tree Motivation



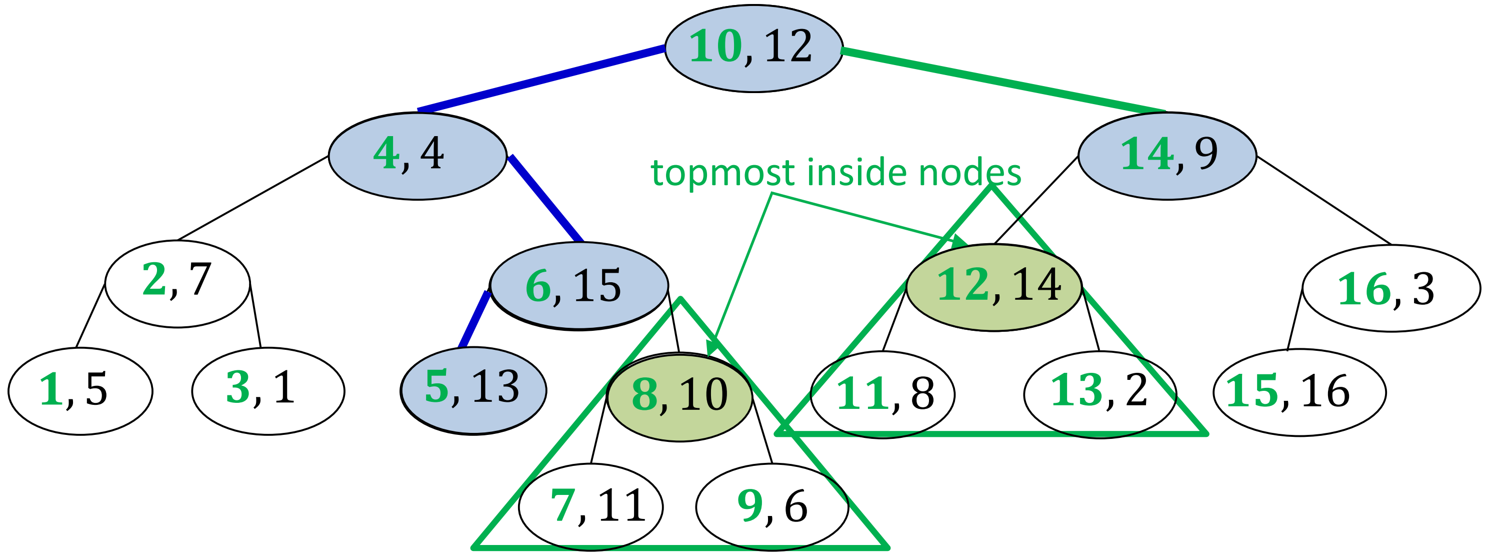
- Have a set of 2D points
  - $S = \{(1,5), (2,7), (3,1), (4,4), (5,13), (6,15)(7,11), (8,10), (9,6), (10,12), (11,8), (12,14), (13,2), (14,9), (15,16), (16,3)\}$
- Example of 2D range search
- *BST-RangeSearch*$(T, 5, 14, 5, 9)$
  - find all points with $5 \leq x \leq 14$ and $5 \leq y \leq 9$
- Construct BST with x-coordinate key
  - recall that points are in general positon, so all x-keys are distinct
    - for any (x_1, y_1) and (x_2, y_2) in our set of points, $x_1 \neq x_2$
  - can search efficiently based only on x-coordinate

2D Range Tree Motivation



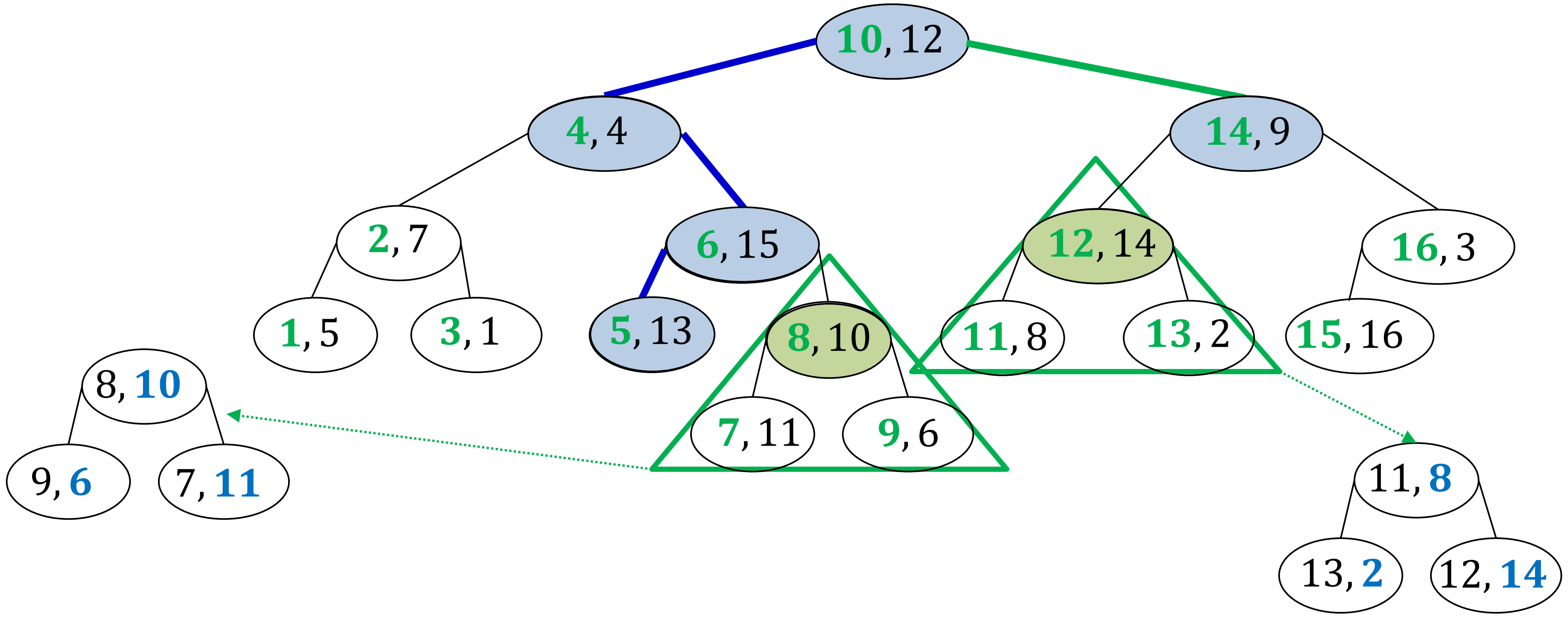
- Consider $2D$ range search *BST-RangeSearch*$(T, 5, 14, 5, 9)$
- First perform *BST-RangeSearch*$(T, 5, 14)$
  - let A be the set of nodes *BST-RangeSearch*$(T, 5, 14)$ returns
    - $A = \{(10{,}12), (6{,}15), (5{,}13), (14{,}9), (8{,}10), (7{,}11), (9{,}6), (12{,}14), (11{,}8), (13{,}2)\}$
  - let B be the set of nodes *BST-RangeSearch*$(T, 5, 14, 5, 9)$ should return
    - $B \subseteq A$
  - Need to go over all nodes in A and check if their y-coordinate is in valid range, $O(|A|)$
    - could be very inefficient
    - for example, $|A|$ can be, say $\Theta(n)$ and $|B|$ could be $O(1)$
      - $O(n)$, as bad as exhaustive search and worse than kd-trees search, $O(|B| + \sqrt{n})$

2D Range Tree Motivation



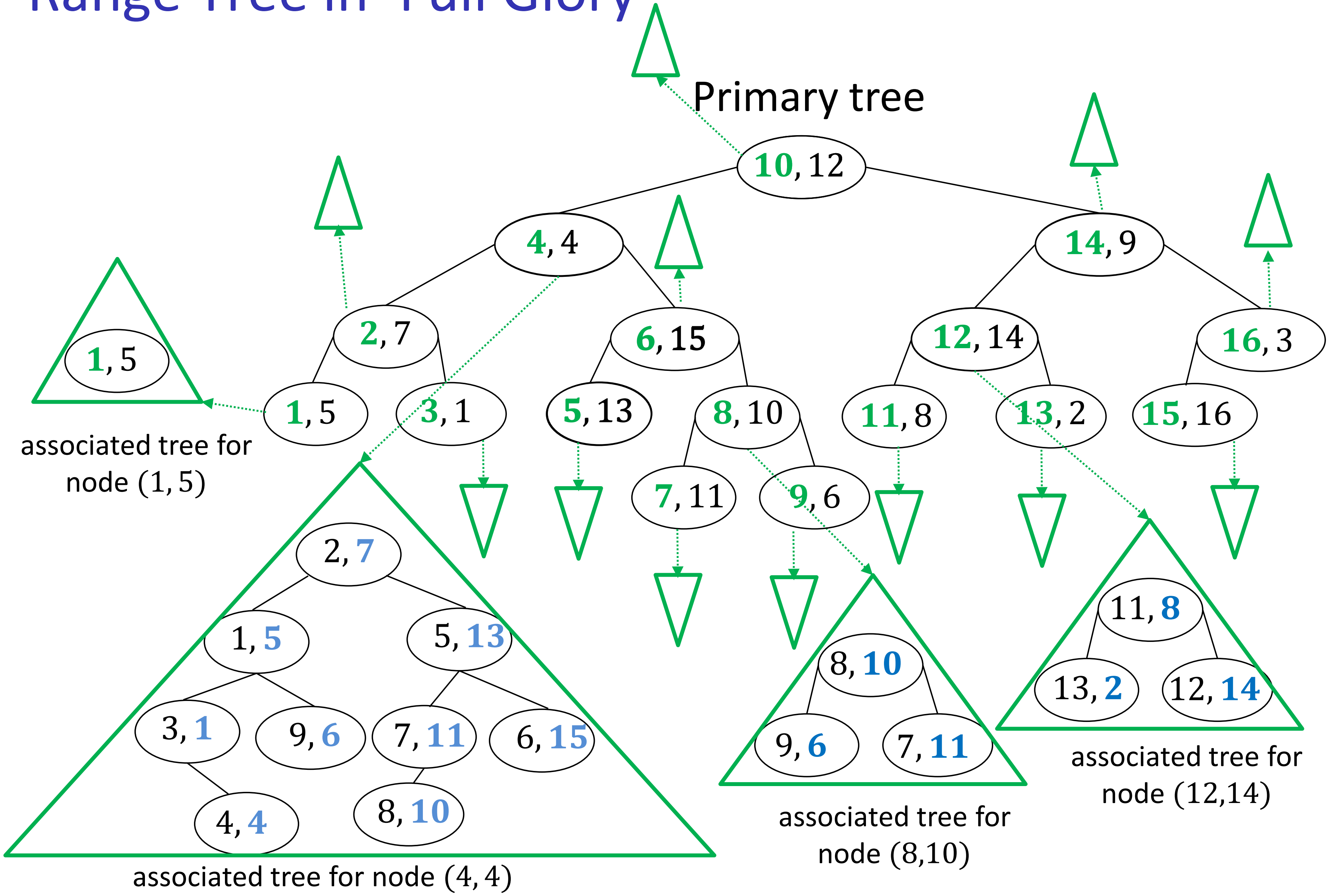
- Consider $2D$ range search *BST-RangeSearch*$(T, 5, 14, 5, 9)$
- First perform only **partial** *BST-RangeSearch*$(T, 5, 14)$
  - find boundary and topmost inside nodes, takes $O(\log n)$ time
- Next
  - for boundary nodes, check if **both** x and y coordinates are in the range, takes $O(\log n)$ time as there are $O(\log n)$ boundary nodes
  - inside nodes are stored in $O(\log n)$ subtrees, with a topmost inside node as a root of each subtree
    - if we could search these subtrees, time would be very efficient
    - however these subtrees do not support efficient search by y coordinate

2D Range Tree Motivation

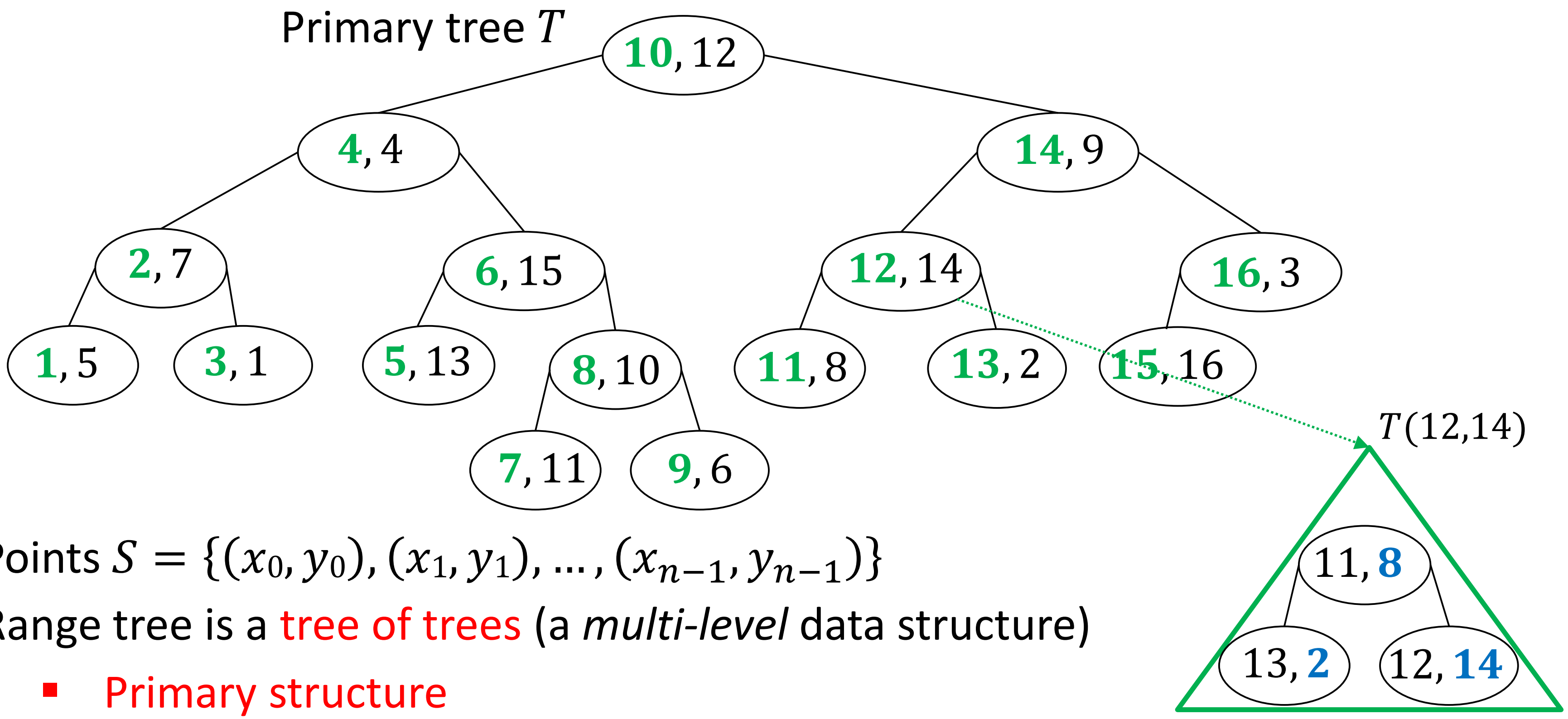


- Need to search subtrees by y-coordinate, but they are x-coordinate based
- Brute-force solution
  - create an associate balanced BST tree for each node v
  - stores the same items as the main (primary) subtree rooted at node v
  - but key is y-coordinate

Range Tree in 'Full Glory'

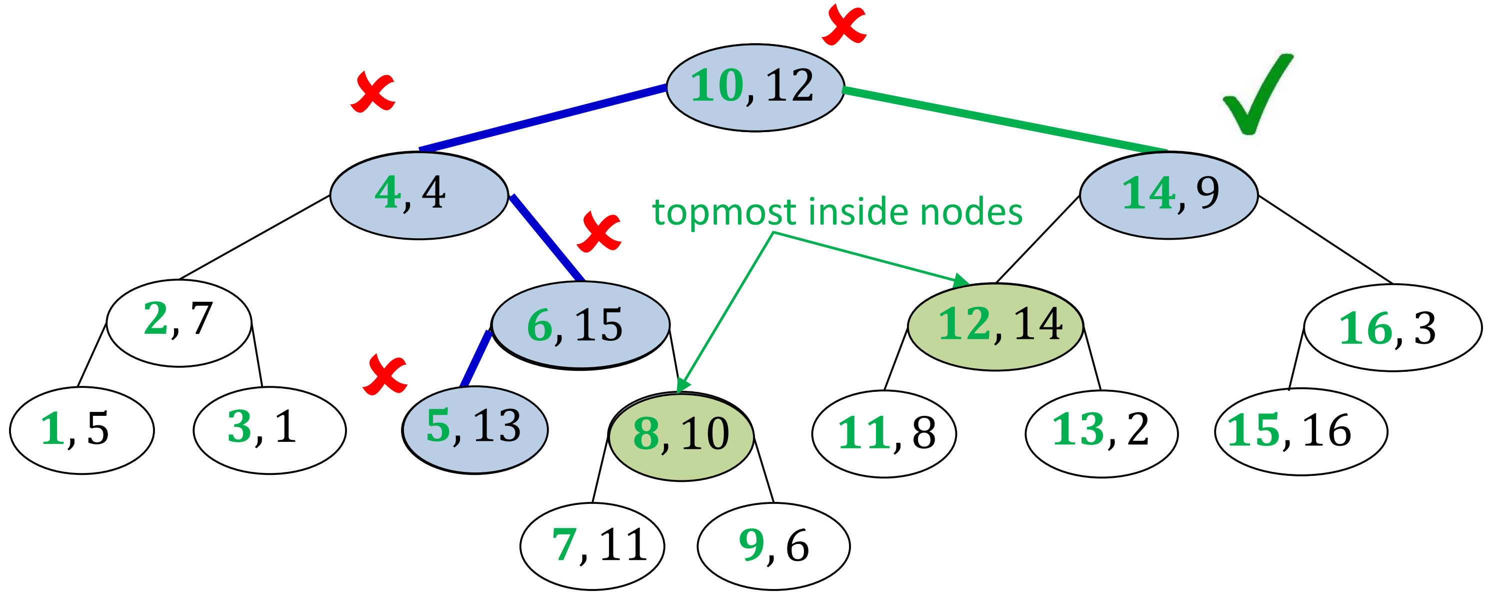


2-dimensional Range Trees Full Definition



- Points $S = \{(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})\}$
- Range tree is a tree of trees (a *multi-level* data structure)
    - Primary structure
        - balanced BST T storing S and uses x-coordinates as keys
        - assume T is balanced, so height is $O(\log n)$
    - Each node v of T stores an associated tree $T(v)$, which is a balanced BST
        - let $S(v)$ be all descendants of v in T, including v
        - $T(v)$ stores $S(v)$ in BST, using y-coordinates as key
            - note that v is not necessarily the root of $T(v)$

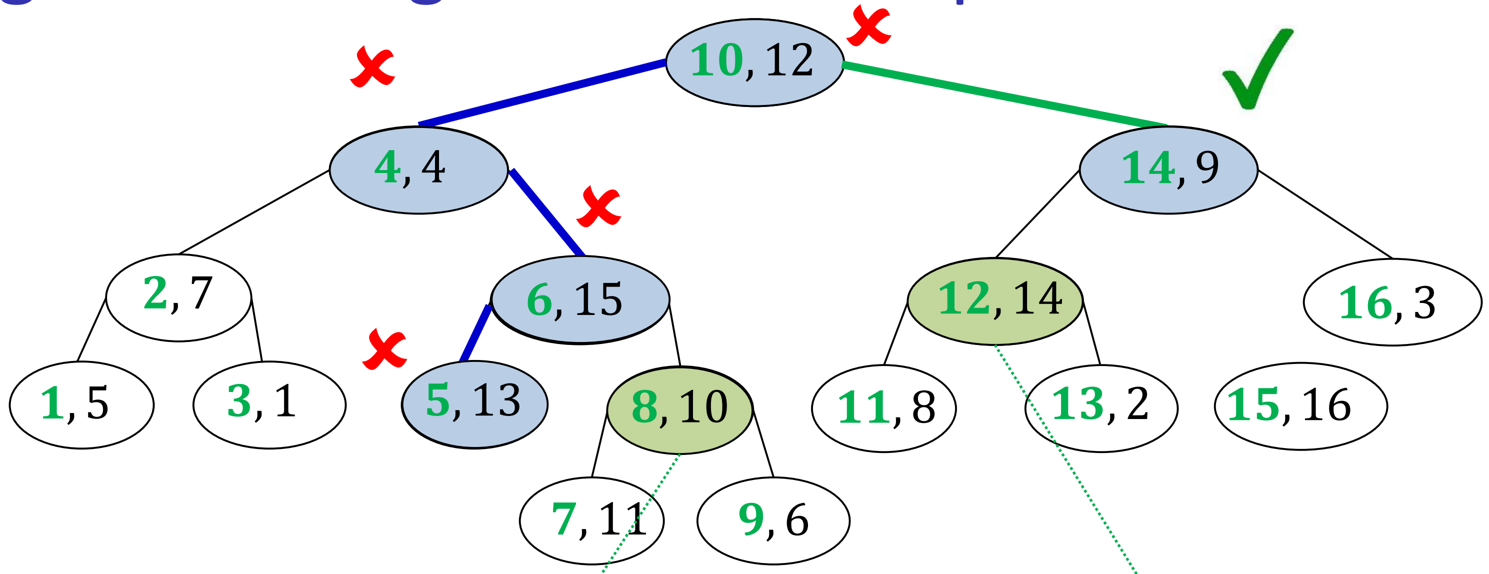
Range search in 2D Range Tree Overview



- *RangeTree::RangeSearch*(T, x_1, x_2, y_1, y_2)
  - *RangeTree::RangeSearch*$(T, 5, 14, 5, 9)$

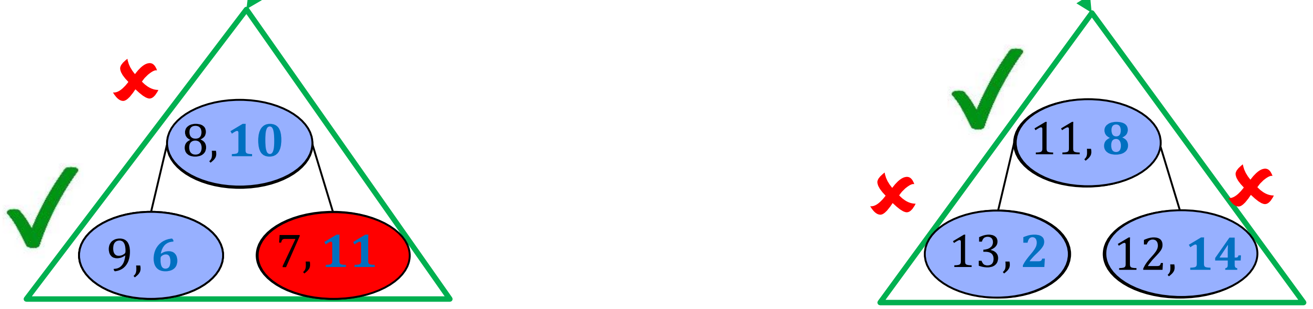
1. Perform *modified BST-RangeSearch*$(T, 5, 14)$
   - find boundary and topmost inside nodes, but do not go through the inside subtrees
   - modified version takes $O(\log n)$ time
     - does not visit all the nodes in valid range for *BST-RangeSearch*$(T, 5, 14)$
2. Check if boundary nodes have valid x-coordinate **and** valid y-coordinate
3. For every topmost inside node v, search in associated tree *BST::RangeSearch*$(T(v), 5, 9)$

Range Tree Range Search Example Finished

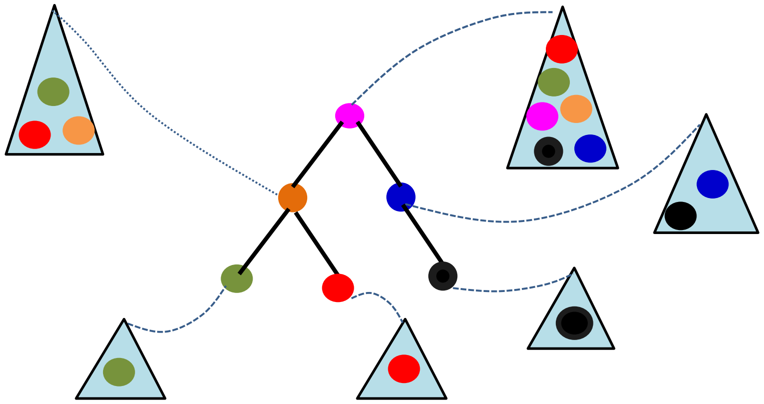


- $RangeTree::RangeSearch(T, 5, 14, 5, 9)$
- For every topmost inside node v, search in associated tree $BST\text{-}RangeSearch(T(v), 5, 9)$

$BST\text{-}rangeSearch(T(8{,}10), 5{,}9)$ $BST\text{-}RangeSearch(T(12{,}14), 5{,}9)$



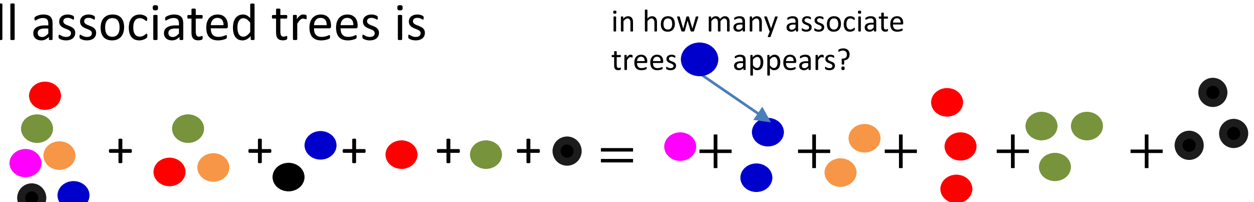
Range Tree Space Analysis



- Primary tree T uses $O(n)$ space
- For each v, associated tree $T(v)$ uses $O(|T(v)|)$ space

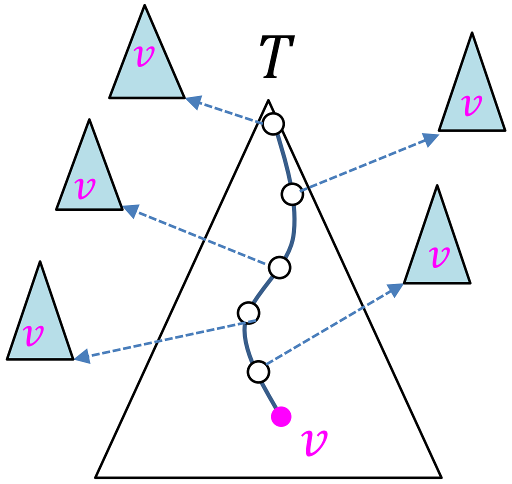
- Space for all associated trees is

$$\sum_{v\in T}|T(v)| =$$



$$= \sum_{v\in T} \underbrace{\text{\#of ancestors of } v}_{\le c\log n}$$

$$\le \sum_{v\in T} c\log n = cn\log n$$



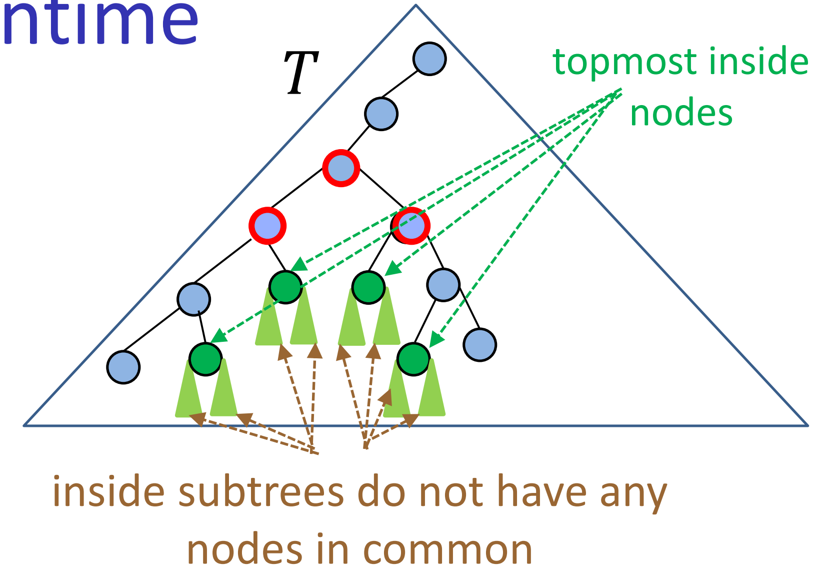
\#of ancestors of v

- Space is $O(n\log n)$
    - in the worst case, have $n/2$ leaves at the last level, and space needed is $\Theta(n\log n)$

Range Trees: Dictionary Operations

- Search(x, y)
  - search by x coordinate in the primary tree T
- Insert(x, y)
  - first, insert point by x-coordinate into the primary tree T
  - then walk up to root and insert point by y-coordinate in *all* $T(v)$ of nodes v on path to root
- Delete
  - analogous to insertion
- **Problem**
  - want binary search trees to be balanced
  - if we use AVL-trees, it makes insert/delete very slow
    - rotation at v changes $S(v)$ and hence requires re-build of $T(v)$
  - instead of rotations, can allow certain imbalance, rebuild entire subtree if violated
    - no details

Range Trees: Range Search Runtime



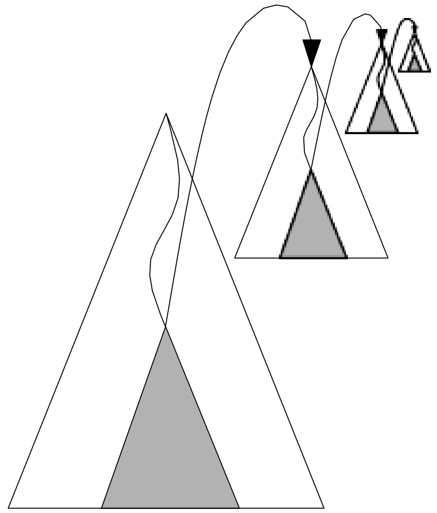
- Find boundary nodes in the primary tree and check if keys are in the range
    - $O(\log n)$
- Find topmost inside nodes in primary tree
    - $O(\log n)$
- For each topmost inside node v, perform range search for y-range in associate tree
    - $O(\log n)$ topmost inside nodes
    - let s_v be #items returned for the subtree of topmost node v
    - running time for one search is $O(\log n + s_v)$

$$\sum_{\substack{\text{topmost inside}\\ \text{node } v}} c(\log n + s_v) = \underbrace{\boxed{\sum_{\substack{\text{topmost inside}\\ \text{node } v}} c\log n}}_{O(\log^2 n)} + \underbrace{\boxed{\sum_{\substack{\text{topmost inside}\\ \text{node } v}} cs_v}}_{\le cs}$$

- Time for range search in range tree: $O(s + \log^2 n)$
    - can make this even more efficient, but this is beyond the scope of the course

Range Trees: Higher Dimensions

- Range trees can be generalized to *d* -dimensional space
    - space $O(n\,(\log n)^{d-1})$
    - construction time $O(n\,(\log n)^{d})$
    - range search time $O(s + (\log n)^{d})$
- Note: d is considered to be a constant
- Space-time tradeoff compared to kd trees



Outline

- Range-Searching in Dictionaries for Points
    - Range Search
    - Multi-Dimensional Data
    - Quadtrees
    - kd-Trees
    - Range Trees
    - Conclusion

Range Search Data Structures Summary

- Quadtrees
  - simple, easy to implement insert/delete (i.e. dynamic set of points)
  - work well only if points evenly distributed
  - wastes space for higher dimensions
  - convention: points on split lines belong to the right/top side
- kd-trees
  - linear space
  - range search is $O(s + \sqrt{n})$
  - inserts/deletes destroy balance and range search time
    - fix with occasional rebuilt
  - convention: points on split lines belong to the right/top side
- Range trees
  - fastest range search $O(s + \log^2 n)$
  - wastes some space
  - insert and delete destroy balance, but can fix this with occasional rebuilt