CS 240 – Data Structures and Data Management

Module 9: String Matching

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Based on lecture notes by many previous cs240 instructors

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Outline

- String Matching
    - Introduction
    - Karp-Rabin Algorithm
    - Knuth-Morris-Pratt algorithm
    - Boyer-Moore Algorithm
    - Suffix Trees
    - Suffix Arrays
    - Conclusion

Pattern Matching Definitions [1]

- Search for a string (pattern) in a large body of text
- $T[0\ldots n-1]$ text (or haystack) being searched
- $P[0 \ldots m-1]$ pattern (or needle) being searched for
- Strings over alphabet Σ
- Return the first occurrence of P in T
- Example

$T =$ L i t t l e p i g l e t s c o o k e d f o r m o t h e r p i g

$7+0$ $7+1$ $7+2$

$P =$ p i g

$n = 36,\ m = 3,\ i = 7$

  - return smallest i such that

$$T[i+j] = P[j] \text{ for } 0 \leq j \leq m-1$$

- If P does not occur in T, return FAIL
- Applications
  - information retrieval (text editors, search engines), bioinformatics, data mining

More Definitions [2]

antidisestablishmentarianism

- Substring $T[i \ldots j]$ $0 \leq i \leq j < n$ is a string consisting of characters $T[i], T[i+1], \ldots, T[j]$
  - length is $j - i + 1$
- Prefix of T is a substring $T[0 \ldots i]$ of T for some $0 \leq i \leq n-1$
- Suffix of T is a substring $T[i \ldots n-1]$ of T for some $0 \leq i \leq n-1$
- With this definition, prefix and suffix are never empty strings
  - sometimes want to allow empty string prefix and suffix

General Idea of Algorithms

guess at $i = 0$ | guess at $i = 1$ | ... | guess at $i = 6$ | guess at $i = 7$

abbbababbab
abba

a**bbba**babbab
 abba

abbbab**abba**b
abba

check at $j = 0$ check at $j = 1$

abbbaba**bbab**
abba

- Pattern matching algorithms consist of guesses and checks
  - a **guess** or **shift** is a position i such that P might start at $T[i]$
  - valid guesses (initially) are $0 \leq i \leq n - m$
  - a **check** of a guess is a single position j with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$
    - must perform m checks of a single correct guess
  - may make fewer checks of an incorrect guess

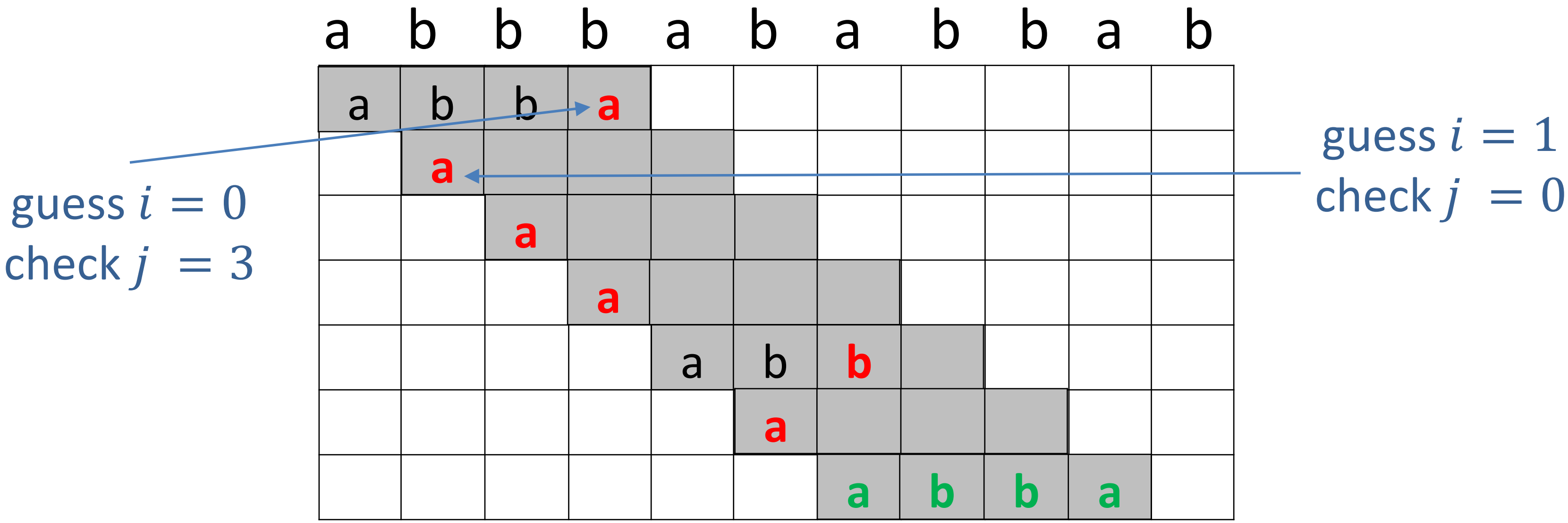
Diagrams for Matching

- Diagram single run of pattern matching algorithm by matrix of checks
    - each row represents a single guess

a	b	b	b	a	b	a	b	b	a	b
a	b	b	a							

Brute-Force Algorithm: Example

Example: T = abbbababbab, P = abba



- Worst possible input
  - $P = \underbrace{a \dots a}_{m-1 \text{ times}}b, T = \underbrace{aaaaaaaa \dots aaaaaaa}_{n \text{ times}}$
- Have to perform $(n - m + 1)m$ checks, which is $\Theta((n - m)m)$ runtime
  - this is $\Theta(nm)$ if $m \leq n/2$
  - worst running time if $m = n/2$
    - $\Theta(n^2)$

Brute-force Algorithm

- Checks every possible guess

```
Bruteforce::PatternMatching(T [0..n − 1], P[0..m − 1])
T : String of length n (text), P: String of length m (pattern)
      for i ← 0 to n − m do
            if strcmp(T [i ... i + m − 1], P) = 0
                  return "found at guess i"
      return FAIL
```

- Note: *strcmp* takes $\Theta(m)$ time

```
strcmp(T [i ... i + m − 1], P[0...m − 1])
for j ← 0 to m − 1 do
      if T [i + j] is before P[j] in Σ then return -1
      if T [i + j] is after P[j] in Σ then return 1
return 0
```

How to improve?

- Extra preprocessing on pattern P
  - **Karp-Rabin**
  - **KMP**
  - **Boyer-Moore**
  - Eliminate guesses based on completed matches and mismatches
- Do extra preprocessing on the text T
  - **Suffix-trees**
  - **Suffix-arrays**
  - Create a data structure to find matches easily

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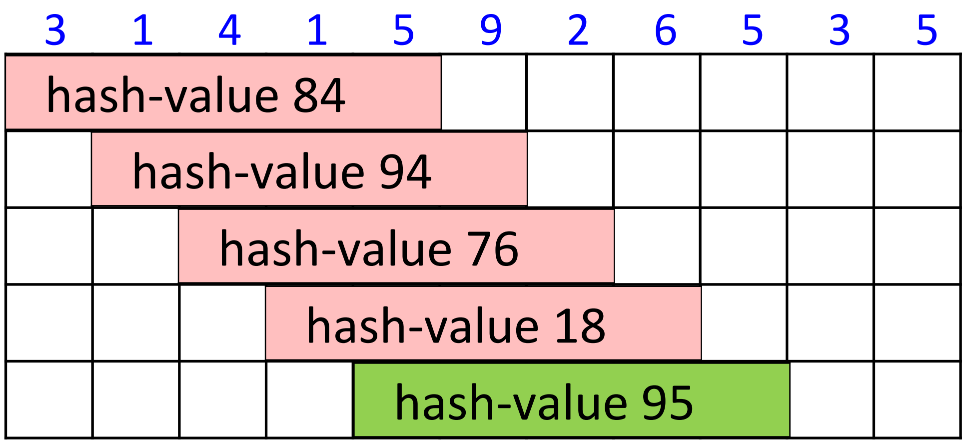
Karp-Rabin Fingerprint Algorithm: Idea

- Hash functions are useful not just for hash tables!
- **Idea:** use hashing to eliminate guesses faster
  - compute hash function for each guess, compare with pattern hash
    - if values are unequal, then current guess cannot match the pattern
    - if values are equal, verify that pattern actually matches text
      - equal hash value does not guarantee equal keys
      - although if hash function is good, most likely keys are equal
      - $O(m)$ time to verify, but happens rarely, and most likely only for true match
  - Example: P = 5 9 2 6 5, T = 3 1 4 1 5 9 2 6 5 3 5
      - standard hash function: flattening + modular (radix $R = 10$):

$$h(59265) = (5 \cdot 10^4 + 9 \cdot 10^3 + 2 \cdot 10^2 + 6 \cdot 10^1 + 5) \; mod \; 97 = 59265 \; mod \; 97 = 95$$



$h(31415) = 84$

$h(14159) = 94$

$h(41592) = 76$

$h(15926) = 18$

$h(59265) = 95$

Karp-Rabin Fingerprint Algorithm – First Attempt

Karp-Rabin-Simple::patternMatching(T, P)

$h_P \leftarrow h(P[0..m-1)])$

for $i \leftarrow 0$ to $n - m$

$h_T \leftarrow h(T\ [i...i+m-1])$

if $h_T = h_P$

if *strcmp*$(T\ [i\ ...\ i+m-1], P) = 0$

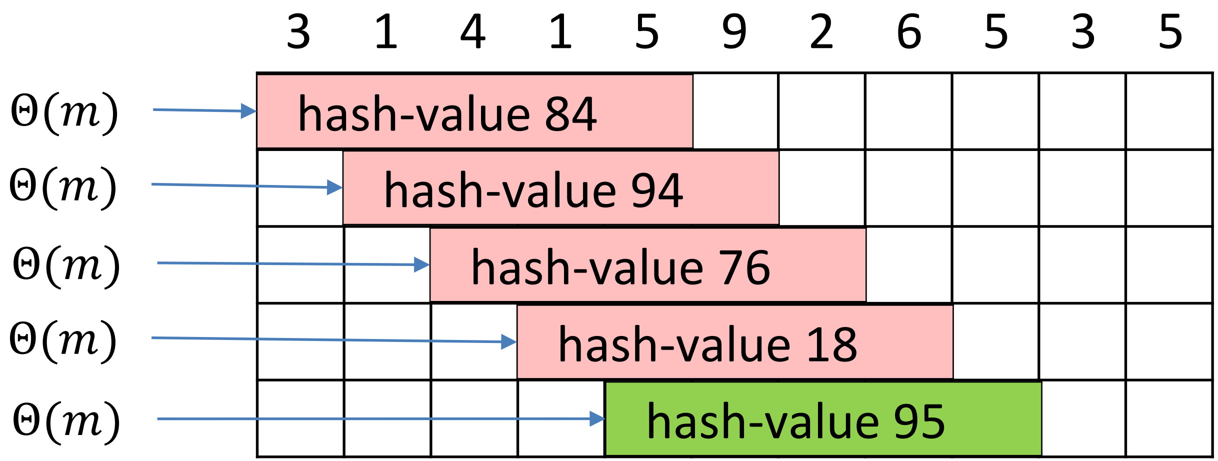
return "found at guess i"

return FAIL

$\Theta(m)$

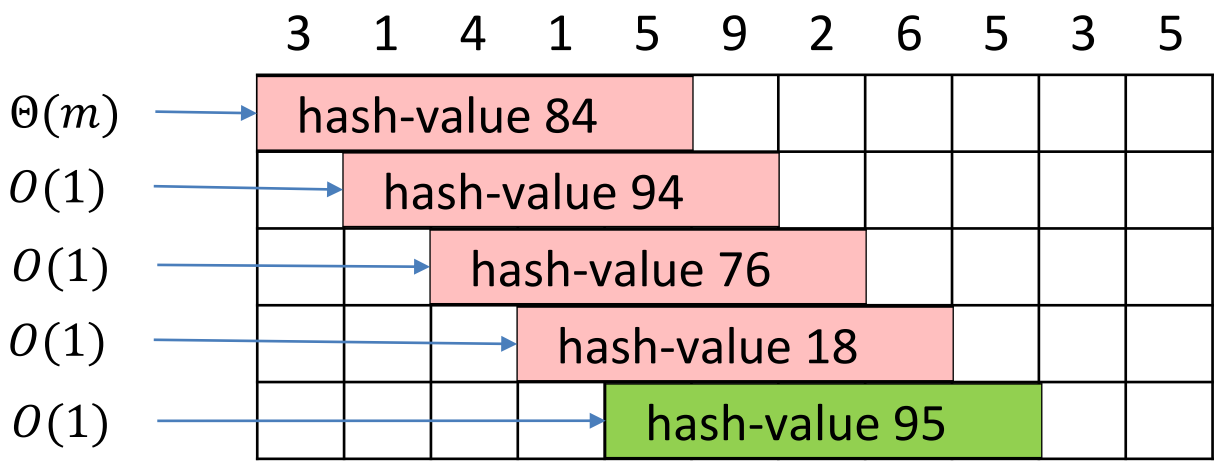
- Algorithm correctness: match is not missed
  - $h(T\ [i..i+m-1]) \neq h(P) \Rightarrow$ guess i is not P
- What about running time?

Karp-Rabin Fingerprint Algorithm: First Attempt



- For each shift, $\Theta(m)$ time to compute hash value
    - since $h(T[i \ldots i+m-1])$ depends on all m characters
    - worse than brute-force!
        - it is possible for brute force matching to use less than $\Theta(m)$ per shift, as it stops at the first mismatched character
- $n-m+1$ shifts in text to check
- Total time is $\Theta(mn)$ if pattern not in text
    - how can we improve this?

Karp-Rabin Fingerprint Algorithm: Idea



- Idea: compute next hash from previous one in $O(1)$ time
- $n - m + 1$ shifts in text to check
- $\Theta(m)$ to compute the first hash value
- $O(1)$ to compute all other hash values
- $\Theta(n + m)$ expected time
  - recall that we still need to check if the pattern actually matches text whenever hash value of text is equal to the hash value of pattern
  - if hash function is good, then whenever hash values are equal, pattern most likely matches the text

Karp-Rabin Fingerprint Algorithm – Fast Rehash

- For historical reasons, hashes are called **fingerprints**
- Insight: can update a fingerprint from previous fingerprint in constant time
  - $O(1)$ time to compute any hash, except first one
- **Example**

$$T = 4\ 1\ 5\ 9\ 2\ 6\ 5\ 3\ 5, \qquad P = 5\ \ 9\ \ 2\ \ 6\ \ 5$$

- Initialization of the algorithm
  1. compute first hash: $h(41592) = 41592\ mod\ 97 = 76$ [$\Theta(m)$ time]
  2. also compute $10000\ mod\ 97 = 9$
- Main loop: repeatedly compute next hash from the previous one
- Example: compute $\underline{15926}\ mod\ 97$ from $4\underline{1592}\ mod\ 97$
  - get rid of the old first digit and add new last digit

$$41592 \xrightarrow{-4 \cdot 10000} 1592 \xrightarrow{\times 10} 15920 \xrightarrow{+6} 15926$$

- Algebraically,

$$\big(41592 - (4 \cdot 10000)\big) \cdot 10 + 6 = 15926$$

Karp-Rabin Fingerprint Algorithm – Fast Rehash

- Insight: can update a fingerprint from previous fingerprint in constant time
- **Example**

T = 4 1 5 9 2 6 5 3 5, P = 5 9 2 6 5

- Initialization of the algorithm
  1. compute first hash: $h(41592) = 41592 \ mod \ 97 = 76$ [$\Theta(m)$ time]
  2. also compute $10000 \ mod \ 97 = 9$
- Main loop: repeatedly compute next hash from the previous one
- Example: compute $\underline{1592}6 \ mod \ 97$ from $4\underline{1592} \ mod \ 97$

$$\big(41592 - (4 \cdot 10000)\big) \cdot 10 + 6 = 15926$$

$$\Big(\big(41592 - (4 \cdot 10000)\big) \cdot 10 + 6\Big) \ mod \ 97 = 15926 \ mod \ 97$$

$$\Big(\big(\underbrace{41592 \ mod \ 97}_{\text{previous hash}} - (4 \cdot \underbrace{(10000 \ mod \ 97)}_{\text{precomputed}})\big) \cdot 10 + 6\Big) \ mod \ 97 = 15926 \ mod \ 97$$

$$\underbrace{\Big(\big(76 - (4 \cdot 9)\big) \cdot 10 + 6\Big) mod \ 97}_{\text{constant number of operations, independent of } m} = 15926 \ mod \ 97$$

$$18 = 15926 \ mod \ 97$$

Karp-Rabin Fingerprint Algorithm – Conclusion

Karp-Rabin-RollingHash::PatternMatching(T, P)

$M \leftarrow$ suitable prime number
$h_P \leftarrow h(P[0 \ldots m-1)])$
$h_T \leftarrow h(T\,[0..m-1)])$
$s \leftarrow 10^{m-1} \, mod \; M$
for $i \leftarrow 0$ to $n - m$
  if $h_T = h_P$
    if *strcmp*$(T\,[i \ldots i+m-1],\; P) = 0$
      return "found at guess i"
  if $i < n - m$ // compute hash-value for next guess
    $h_T \leftarrow \big((h_T - T[i] \cdot s) \cdot 10 + T\,[i+m]\big) mod \; M$
return FAIL

- Choose "table size" M at random to be prime in $\{2, \ldots, mn^2\}$
- Expected running time is $O(m+n)$
- $\Theta(mn)$ worst-case, but this extremely is unlikely
- Improvement: reset M if no match at $h_T = h_P$

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Knuth-Morris-Pratt (KMP) Overview

- KMP starts out similar to Brute-Force pattern matching

$$P = \ ababaca$$

T	c	a	b	a	b	a	a	b	a	b
	a									
		a	b	a	b	a	**c**			

mismatch at the first pattern letter, move to the next shift

letter matches the text, move on to the next check

mismatch at not the first pattern letter, do something different from Brute Force

Knuth-Morris-Pratt (KMP) Indexing

$P = cad$

	$j=0$	$j=1$	$j=2$		
T	c	a	b	a	b
$i=0$	c	a	b		

	$j=0$ $i=0$	$j=1$ $i=1$	$j=2$ $i=2$		
T	c	a	b	a	b
	c	a	b		

- Brute-force indexing
  - maintain variables i and j
  - j is the position in the pattern
  - i is equal to the current shift
  - check is performed by determining if $T[i+j] = P[j]$
- KMP indexing
  - maintain variables i and j
  - j is the position in the pattern
  - i is the position in the text where we do the next check
  - check is performed by determining if $T[i] = P[j]$
  - current shift is $i - j$

Knuth-Morris-Pratt (KMP) Derivation

$P = \ ababaca$

$j=0$
$i=0$

T

c	a	b	a	b	a	a	b	a	b
a									

- KMP starts similar to brute force pattern matching
  - maintain variables i and j
    - j is the position in the pattern
    - i is the position in the text where we do the check
    - check is performed by determining if $T[i] = P[j]$
      - current shift is $i - j$
- Begin matching with $i = 0, \ j = 0$
- If $T[i] \neq P[j]$ and $j = 0$, shift pattern by 1, the same action as in brute-force
  - $i = i + 1$
  - j is unchanged
    - shift was $i - j$ and it changes to $i + 1 - j$
      - it increases by 1 as needed

Knuth-Morris-Pratt Motivation

$P = \ ababaca$

	$j{=}0$ $i{=}0$	$j{=}0$ $i{=}1$	$j{=}1$ $i{=}2$	$j{=}2$ $i{=}3$	$j{=}3$ $i{=}4$	$j{=}4$ $i{=}5$	$j{=}5$ $i{=}6$			
T	c	a	b	a	b	a	a	b	a	b
	a									
		a	b	a	b	a	**c**			

- When $T[i] = P[j]$, the action is to check the next letter, as in brute-force
  - $i = i + 1$
  - $j = j + 1$
  - shift was $i - j$ and it stays unchanged

- Failure at text position $i = 6,$ pattern position $j = 5$
- When failure is at pattern position $j > 0$, do something smarter than brute force

Knuth-Morris-Pratt Motivation

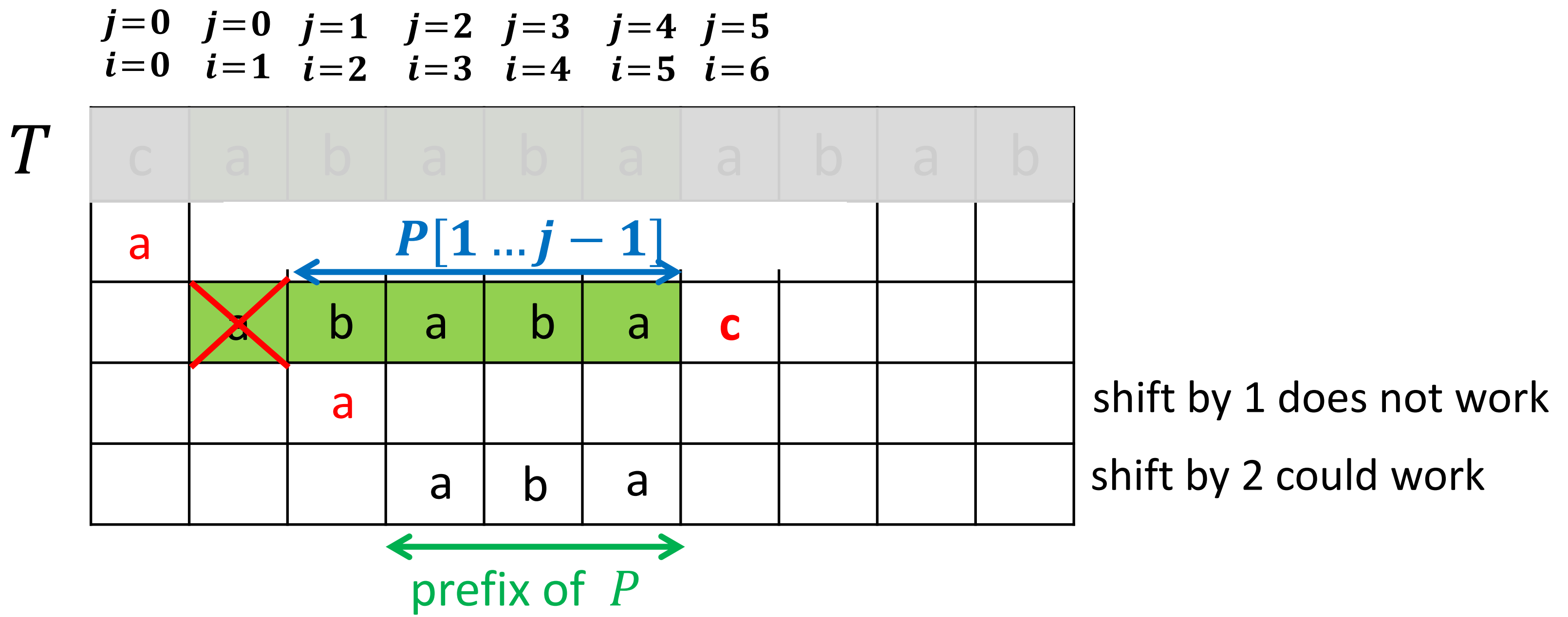
$P = \ ababaca$

	$j=0$ $i=0$	$j=0$ $i=1$	$j=1$ $i=2$	$j=2$ $i=3$	$j=3$ $i=4$	$j=4$ $i=5$	$j=5$ $i=6$				
T	c	a	b	a	b	a	a	b	a	b	
	a										
		a	b	a	b	a	c				
			a								shift by 1 does not work
				a	b	a					shift by 2 could work

- When failure is at pattern position $j > 0$, do something smarter than brute force
- Prior to $j = 5$, pattern and text are equal
    - find how to shift pattern looking only at pattern
- If failure at $j = 5$, shift pattern by 2 **and** start matching with $j = 3$
    - equivalently: i stays the same, new $j = 3$
        - old shift was $i - 5$, the new shift is $i - 3$, so shift increased by 2
    - skipped one shift, and 3 character checks
    - can precompute the action of 'shift by 2 and skip 3 characters' before matching even begins, from the pattern, as we do not need text for this computation

Knuth-Morris-Pratt Motivation

$P = \ ababaca$



- If failure at $j = 5$: continue matching with the same i and new $j = 3$
    - precomputed from pattern before matching begins
- Brief rule for determining new j
    - find longest suffix of $P[1 \ldots j-1]$ which is also prefix of P
    - call a suffix valid if it is a prefix of P
    - new $j =$ the length of the longest valid suffix of $P[1 \ldots j-1]$

KMP Failure Array Computation: Slow

- **Rule**: if failure at pattern index $j > 0$, continue matching with the same i and new $j =$ the length of the longest valid suffix of $P[1 \dots j-1]$
- Computed previously for $j = 5$, but need to compute for all j
- Store this information in array $F[0 \dots m-1]$, also called failure-function
    - $F[j]$ is length of the longest valid suffix of $P[1 \dots j]$
    - if failure at pattern index $j > 0$, new $j = F[j-1]$

- We could have indexed failure array F differently
    - $F[j]$ is length of the longest valid suffix of $P[1 \dots j-1]$
    - if failure at pattern index $j > 0$, new $j = F[j]$
    - But then we have to remember, when computing $F[j]$ that
        - $F[j]$ is length of the longest valid suffix of $P[1 \dots j-1]$
        - inconvenient to remember

KMP Failure Array Computation: Slow

- **Rule**: if failure at pattern index $j > 0$, continue matching with the same i and new $j =$ the length of the longest valid suffix of $P[1 \dots j-1]$
- Store the length of the longest valid suffix of $P[1 \dots j]$ in $F[j]$
- If failure at pattern index $j > 0$, new $j = F[j-1]$
- $P = ababaca$

F	0	1	2	3	4	5	6
	0	0	1				

- $j = 0$
  - $P[1 \dots 0] =$ "", $P = ababaca$, longest valid suffix is ""
  - note that $F[0] = 0$ for any pattern
- $j = 1$
  - $P[1 \dots 1] = b$, $P = ababaca$, longest valid suffix is ""
- $j = 2$
  - $P[1 \dots 2] = ba$, $P = ababaca$, longest valid suffix is a

KMP Failure Array Computation: Slow

- Store the length of the longest valid suffix of $P[1 \ldots j]$ in $F[j]$

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

- $j = 3$
  - $P[1 \ldots 3] = bab$, $P = ababaca$, longest valid suffix is ab
- $j = 4$
  - $P[1 \ldots 4] = baba$, $P = ababaca$, longest valid suffix is aba
- $j = 5$
  - $P[1 \ldots 5] = babac$, $P = ababaca$, longest valid suffix is ""
- $j = 6$
  - $P[1 \ldots 6] = babaca$, $P = ababaca$, longest valid suffix is a
- Failure array is precomputed before matching starts
  - straightforward computation is $O(m^3)$ time

```
for j = 0 to m − 1        // go over all positions in the failure array
    for i = 1 to j        // go over all suffixes of P[1 ... j]
        for k = 1 to i    // compare next suffix to prefix of P
```

String matching with KMP: Example

- $T = cabababcabababaca, P = abababaca$

F

0	1	2	3	4	5	6
0	0	1	2	3	0	1

$\boldsymbol{i=0}$
$\boldsymbol{j=0}$

T:	c	a	b	a	b	a	b	c	a	b	a	b	a	c	a
P:															

rule 1

if $T[i] = P[j]$
- $i = i + 1$
- $j = j + 1$

rule 2

if $T[i] \neq P[j]$ and $j > 0$
- i unchanged
- $j = F[j-1]$

rule 3

if $T[i] \neq P[j]$ and $j = 0$
- $i = i + 1$
- j is unchanged

String matching with KMP: Example

- $T = cabababcabababaca$, $P = abababaca$

F							
	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

	$j=0$ $i=0$	$j=0$ $i=1$	$j=1$ $i=2$	$j=2$ $i=3$	$j=3$ $i=4$	$j=4$ $i=5$	$j=3$ $\cancel{j=5}$ $i=6$	$j=0$ $\cancel{j=2}$ $\cancel{j=4}$ $i=7$	$j=0$ $i=8$	$j=1$ $i=9$	$j=2$ $i=10$	$j=3$ $i=11$	$j=4$ $i=12$	$j=5$ $i=13$	$j=6$ $i=14$	
T:	c	a	b	a	b	a	b	c	a	b	a	b	a	c	a	
P:	a															
		a	b	a	b	a	c									new $j = 3$
				(a)	(b)	(a)	b	a								new $j = 2$
						(a)	(b)	a								new $j = 0$
								a								
									a	b	a	b	a	c	a	match!

if $T[i] = P[j]$
- $i = i + 1$
- $j = j + 1$

if $T[i] \neq P[j]$ and $j > 0$
- i unchanged
- $j = F[j-1]$

if $T[i] \neq P[j]$ and $j = 0$
- $i = i + 1$
- j is unchanged

Knuth-Morris-Pratt Algorithm

```
KMP(T, P)
    F ← failureArray(P)
    i ← 0 // current character of T
    j ← 0 // current character of P
    while i < n do
        if P[j] = T[i]
            if j = m − 1
                return "found at shift i − m + 1"
               // shift is equal to i − j
            else // rule 1
                i ← i + 1
                j ← j + 1
        else // P[j] ≠ T[i]
            if j > 0
                j ← F[j − 1] // rule 2
            else // rule 3
                i ← i + 1
    return FAIL
```

KMP: Running Time, informally

F							
	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

j decreases

i increases

								$j=0$							
							$j=3$	$\cancel{j=2}$							
	$j=0$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$\cancel{j=5}$	$\cancel{j=4}$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$	$i=11$	$i=12$	$i=13$	$i=14$
T:	c	a	b	a	b	a	b	c	a	b	a	b	a	c	a
P:															

if $T[i] = P[j]$

- $i = i + 1$
- $j = j + 1$

if $T[i] \neq P[j]$ and $j > 0$

- i unchanged
- $j = F[j-1]$

if $T[i] \neq P[j]$ and $j = 0$

- $i = i + 1$
- j is unchanged

- For now, ignore the cost of computing failure array
- Total time = 'horizontal iterations' + 'vertical iterations'
- i can increase at most n times $\rightarrow$ j can increase at most n times
- Total number of decreases of $j \leq$ total number of increases of j $\leq n$
- $O(n)$ total iterations, more formal analysis later

Fast Computation of F

- Failure array F
  - $F[0] = 0$, no need to compute
  - for $j > 0$, $F[j] =$ length of the longest suffix of $P[1...j]$ which is also prefix of P
    - i.e. $F[j] =$ longest valid suffix of $P[1 \dots j]$
- Crucial fact: after processing T, final value of j is longest valid suffix of T

$P = ababaca$

	$j=0$ $i=0$	$j=0$ $i=1$	$j=1$ $i=2$	$j=2$ $i=3$	$j=3$ $i=4$
T:	c	a	b	a	
P:	a				
		a	b	a	

- Use the crucial fact for computation of F
  - match $T = P[1 \dots 1]$ with P, and set $F[1] =$ final j
  - match $T = P[1 \dots 2]$ with P, and set $F[2] =$ final j
  - …
  - match $T = P[1 \dots m-1]$ with P, and set $F[m-1] =$ final j
  - but first, let us rename variable j as l (only for failure array computation)
    - since j is already used when we take $T = P[1 \dots j]$

Fast Computation of F

- $P = ababaca$
- Useful fact
  - after processing T, final value of l is longest valid suffix of T
- Failure array F
  - for $j > 0$, $F[j]$ = length of the longest valid suffix of $P[1...j]$
- Big idea

	$l=0$, $i=0$	$l=0$, $i=1$	$l=1$, $i=2$	$l=2$, $i=3$	$l=3$, $i=4$
T:	c	a	b	a	
P:	a				
		a	b	a	

$$T = P[1 \ldots 1] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[1] = l$$

$$T = P[1 \ldots 2] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[2] = l$$

$$\vdots$$

$$T = P[1 \ldots m-1] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[m-1] = l$$

'chicken and egg' problem with big idea: need F to put text through KMP

Fast Computation of F: Big Idea Saved

- $j = 1$

  $$T = P[1 \ldots 1] \longrightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[1] = l$$

  - start with $l = 0$
  - text has one letter, can reach at most $l = 1$
  - need at most $F[0]$, and already have it

- $j = 2$

  $$T = P[1 \ldots 2] \longrightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[2] = l$$

  - start with $l = 0$
  - text has two letters, can reach at most $l = 2$
  - need at most $F[0], F[1]$, and already have it

$\vdots$

- $j = m - 1$

  $$T = P[1 \ldots m - 1] \longrightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[m - 1] = l$$

  - start with $l = 0$
  - text has $m - 1$ letters, can reach at most $l = m - 1$
  - need at most $F[0], F[1], \ldots, F[m - 2]$, and already have it

Fast Computation of F: Big Idea Made Bigger

$$T = P[1 \dots 1] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[1] = l$$

$$T = P[1 \dots 2] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[2] = l$$

do not start from scratch, start from where $P[1 \dots 1]$ finished

$$T = P[1 \dots 3] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[3] = l$$

do not start from scratch, start from where $P[1 \dots 2]$ finished

$$\vdots$$

$$T = P[1 \dots m-1] \rightarrow \boxed{\text{KMP}} \xrightarrow{\text{final } l} F[m-1] = l$$

do not start from scratch, start from where $P[1 \dots m-2]$ finished

- Cost of passing $P[1 \dots 1], P[1 \dots 2], \dots, P[1 \dots m-1]$ through KMP is equal to the cost of passing just $P[1 \dots m-1]$ through KMP
- In essence, we are just matching pattern with itself:
  - $T = P[1 \dots m-1], P = P$

Fast Computation of F

- Process $T = P[1 \dots j]$, $F[j] =$ final l
- $P = ababaca$
- Initialize $F[0] = 0$

	0	1	2	3	4	5	6
F	0						

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0					

- Process $T = P[1 \dots j]$, $F[j]$ = final l
- $P = ababaca$
- $j = 1, T = P[1 \dots j] = b$

	$l=0$, $i=0$	$l=0$, $i=1$									
T:	b										
P:	$\boldsymbol{a}$										

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l - 1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0	1				

- Process $T = P[1 \ldots j]$, $F[j] =$ final l
- $P = ababaca$
- $j = 2, T = P[1 \ldots j] = ba$

	$l=0$, $i=0$	$l=0$, $i=1$	$l=1$, $i=2$								
T:	b	a									
P:	$\boldsymbol{a}$										
		a									

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l - 1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0	1	2			

- Process $T = P[1 \ldots j]$, $F[j] =$ final l
- $P = ababaca$
- $j = 3, T = P[1 \ldots j] = bab$

	$l=0$ $i=0$	$l=0$ $i=1$	$l=1$ $i=2$	$l=2$ $i=3$							
T:	b	a	b								
P:	$\boldsymbol{a}$										
		a	b								

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l-1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0	1	2	3		

- Process $T = P[1 \dots j]$, $F[j]$ = final l
- $P = ababaca$
- $j = 4, T = P[1 \dots j] = baba$

	$l=0$ $i=0$	$l=0$ $i=1$	$l=1$ $i=2$	$l=2$ $i=3$	$l=3$ $i=4$						
T:	b	a	b	a							
P:	$\boldsymbol{a}$										
		a	b	a							

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l-1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F						
0	1	2	3	4	5	6
0	0	1	2	3	0	

- Process $T = P[1 \dots j]$, $F[j] =$ final l
- $P = ababaca$
- $j = 5, T = P[1 \dots j] = babac$

	$l=0$ $i=0$	$l=0$ $i=1$	$l=1$ $i=2$	$l=2$ $i=3$	$l=0$ ~~$l=1$~~ ~~$l=3$~~ $i=4$	$l=0$ $i=5$						
T:	b	a	b	a	c							
P:	$\boldsymbol{a}$											
		a	b	a	$\boldsymbol{b}$							new $l = 1$
				(a)	$\boldsymbol{b}$							new $l = 0$
					$\boldsymbol{a}$							

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l - 1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

- Process $T = P[1 \dots j]$, $F[j]$ = final l
- $P = ababaca$
- $j = 6, T = P[1 \dots j] = babaca$

	$l=0$, $i=0$	$l=0$, $i=1$	$l=1$, $i=2$	$l=2$, $i=3$	$l=0$, $\cancel{l=1}$, $\cancel{l=3}$, $i=4$	$l=0$, $i=5$	$l=1$, $i=6$					
T:	b	a	b	a	c	a						
P:	$\boldsymbol{a}$											
		a	b	a	$\boldsymbol{b}$							new $l = 1$
				(a)	$\boldsymbol{b}$							new $l = 0$
					$\boldsymbol{a}$							
						a						

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l-1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

- $P = ababaca$
- Matching $T = P[1 \ldots m-1]$ with pattern P, updating $F[i] = l$ after each text letter i is processed

	$l{=}0$ $i{=}0$	$l{=}0$ $i{=}1$	$l{=}1$ $i{=}2$	$l{=}2$ $i{=}3$	$l{=}0$ ~~$l{=}1$~~ ~~$l{=}3$~~ $i{=}4$	$l{=}0$ $i{=}5$	$l{=}1$ $i{=}6$					
T:	b	a	b	a	c	a						
P:	a											
		a	b	a	b							new $l = 1$
				(a)	b							new $l = 0$
					a							
						a						

if $T[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $T[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l-1]$

if $T[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

Fast Computation of F

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

- $P = ababaca$
- Matching $T = P[1 \ldots m-1]$ with pattern P, updating $F[i] = l$ after each text letter i is processed

	$l=0$ $i=1$	$l=0$ $i=2$	$l=1$ $i=3$	$l=2$ $i=4$	$l=0$ ~~$l=1$~~ ~~$l=3$~~ $i=5$	$l=0$ $i=6$	$l=1$ $i=7$					
P:	b	a	b	a	c	a						
P:	***a***											
		a	b	a	***b***							new $l = 1$
				(a)	***b***							new $l = 0$
					a							
						a						

if $\boldsymbol{P}[i] = P[l]$
- $i = i + 1$
- $l = l + 1$

if $\boldsymbol{P}[i] \neq P[l]$ and $l > 0$
- i unchanged
- $l = F[l-1]$

if $\boldsymbol{P}[i] \neq P[l]$ and $l = 0$
- $i = i + 1$
- l is unchanged

KMP: Computing Failure Array

- Pseudocode is almost identical to KMP(T, P)
  - main difference: $F[j]$ gets both used and updated
  - same code as in the example on previous slides, but we renamed i into j

```
failureArray(P)
P: String of length m (pattern)
    F[0] ← 0
    j ← 1 // matching P[1 ... j]
    l ← 0
    while j < m do
        if P[j] = P[l]   // rule 1
            l ← l + 1
            F[j] ← l
            j ← j + 1
        else if l > 0    // rule 2
            l ← F[l − 1]
        else             // rule 3
            F[j] ← 0     // l = 0
            j ← j + 1
```

KMP: FailureArray Runtime

- To bound the number of loop iterations, find an expression that increases by at least 1 at each iteration
- Have red, green and blue cases
  - red + blue: j increases by 1
  - green: l decreases by at least 1 $\rightarrow -l$ increases by at least 1
- So let us try $j - l$
  - green + blue: increases by at least 1
  - red: $j - l \rightarrow (j+1) - (l+1) = j - l$, no increase
- Next try $2j - l$
  - red: $2j - l \rightarrow 2(j+1) - (l+1) \rightarrow 2j - l + 1$
  - green: $2j - l$ increases by at least 1
  - blue: $2j - l \rightarrow 2(j+1) - l \rightarrow 2j - l + 2$
- At initialization, $2j - l = 2 \geq 0$
- At the end, $2j - l \leq 2m$
  - $j = m, l \geq 0$

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

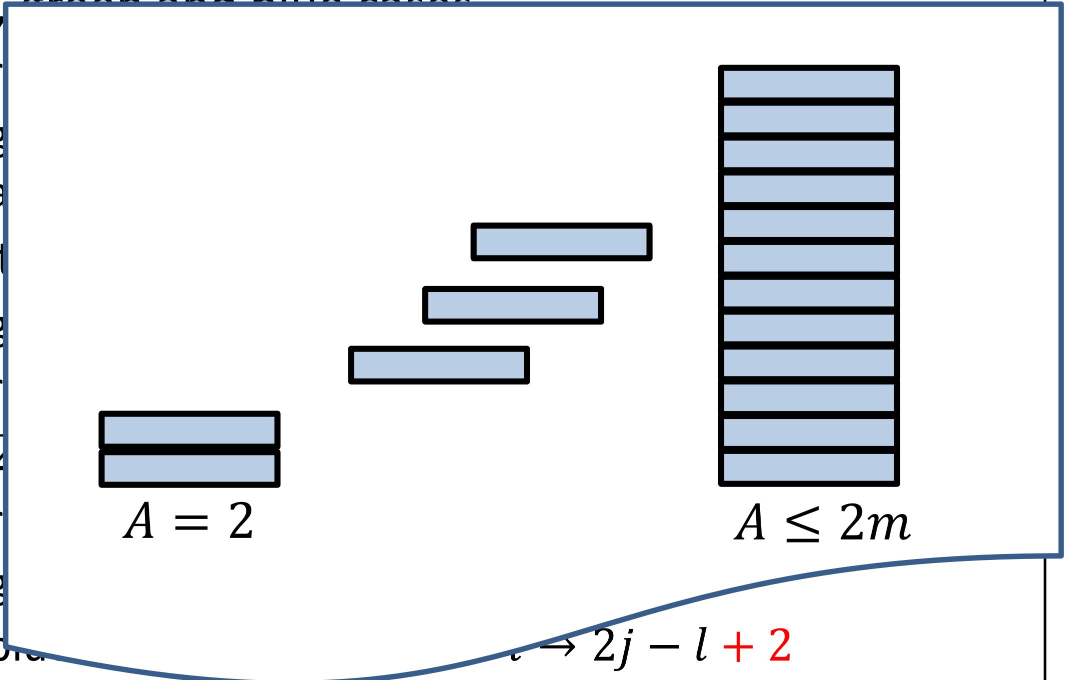
failureArray(P)
P: String of length m

$F[0] \leftarrow 0$
$j \leftarrow 1$
$l \leftarrow 0$
while $j < m$ **do**
  if $P[j] = P[l]$
    $l \leftarrow l + 1$
    $F[j] \leftarrow l$
    $j \leftarrow j + 1$
  else if $l > 0$
    $l \leftarrow F[l-1]$
  else
    $F[j] \leftarrow 0$
    $j \leftarrow j + 1$

KMP: FailureArray Runtime

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

- To bound the number of loop iterations, find an expression that increases by at least 1 at each iteration
- Have red,
  - r
  - g
    a
- So let us t
  - g
  - r
- Next try 2
  - r
  - g
  - b $\quad l \rightarrow 2j - l + 2$



- At initialization, $2j - l = 2 \geq 0$
- At the end, $2j - l \leq 2m$
  - $j = m, l \geq 0$

failureArray(P)
P: String of length m
$F[0] \leftarrow 0$
$j \leftarrow 1$
$l \leftarrow 0$
while $j < m$ **do**
  if $P[j] = P[l]$
    $l \leftarrow l + 1$
    $F[j] \leftarrow l$
    $j \leftarrow j + 1$
  else if $l > 0$
    $l \leftarrow F[l-1]$
  else
    $F[j] \leftarrow 0$
    $j \leftarrow j + 1$

KMP: FailureArray Runtime

F	0	1	2	3	4	5	6
	0	0	1	2	3	0	1

- To bound the number of loop iterations, find an expression that increases by at least 1 at each iteration
- Have red, green and blue cases
  - red + blue: j increases by 1
  - green: l decreases by at least 1 $\rightarrow -l$ increases by at least 1
- So let us try $j - l$
  - green + blue: increases by at least 1
  - red: $j - l \rightarrow (j + 1) - (l + 1) = j - l$, no increase
- Next try $2j - l$
  - red: $2j - l \rightarrow 2(j + 1) - (l + 1) \rightarrow 2j - l + 1$
  - green: $2j - l$ increases by at least 1
  - blue: $2j - l \rightarrow 2(j + 1) - l \rightarrow 2j - l + 2$
- At initialization, $2j - l = 2 \geq 0$
- At the end, $2j - l \leq 2m$
  - $j = m, l \geq 0$
- No more than $2m$ loop iterations, and at least m iterations
- Time is $\Theta(m)$

failureArray(P)
P: String of length m

$F[0] \leftarrow 0$
$j \leftarrow 1$
$l \leftarrow 0$
while $j < m$ **do**
  if $P[j] = P[l]$
    $l \leftarrow l + 1$
    $F[j] \leftarrow l$
    $j \leftarrow j + 1$
  else if $l > 0$
    $l \leftarrow F[l - 1]$
  else
    $F[j] \leftarrow 0$
    $j \leftarrow j + 1$

KMP: Main Function Runtime

```
KMP(T, P)
    F ← failureArray (P)
    i ← 0
    j ← 0
    while i < n do
        if P[j] = T[i]
            if j = m − 1
                return "found at guess i − m + 1"
            else
                i ← i + 1
                j ← j + 1
        else // P[j] ≠ T [i]
            if j > 0
                j ← F[j − 1]
            else
                i ← i + 1
    return FAIL
```

- KMP main function
  - failureArray can be computed in $\Theta(m)$ time
  - Same analysis as for failure array gives $\Theta(n)$
  - Running time KMP altogether: $\Theta(n + m)$
    - which is the same as $\Theta(n)$ as $m \leq n$

Outline

- String Matching
  - Introduction
  - Karp-Rabin Algorithm
  - Knuth-Morris-Pratt algorithm
  - Boyer-Moore Algorithm
  - Suffix Trees
  - Suffix Arrays
  - Conclusion

Boyer-Moore Algorithm Motivation

- Fastest pattern matching in practice on English Text
- Important components
    - Reverse-order searching
        - compare P with a guess moving *backwards*
- When a mismatch occurs choose the better option among the two below
    1. Bad character heuristic
        - eliminate shifts based on mismatched character of T
    2. Good suffix heuristic
        - eliminate shifts based on the matched part (i.e.) suffix of P

Reverse Searching vs. Forward Searching

T = whereiswaldo, P = aldo

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

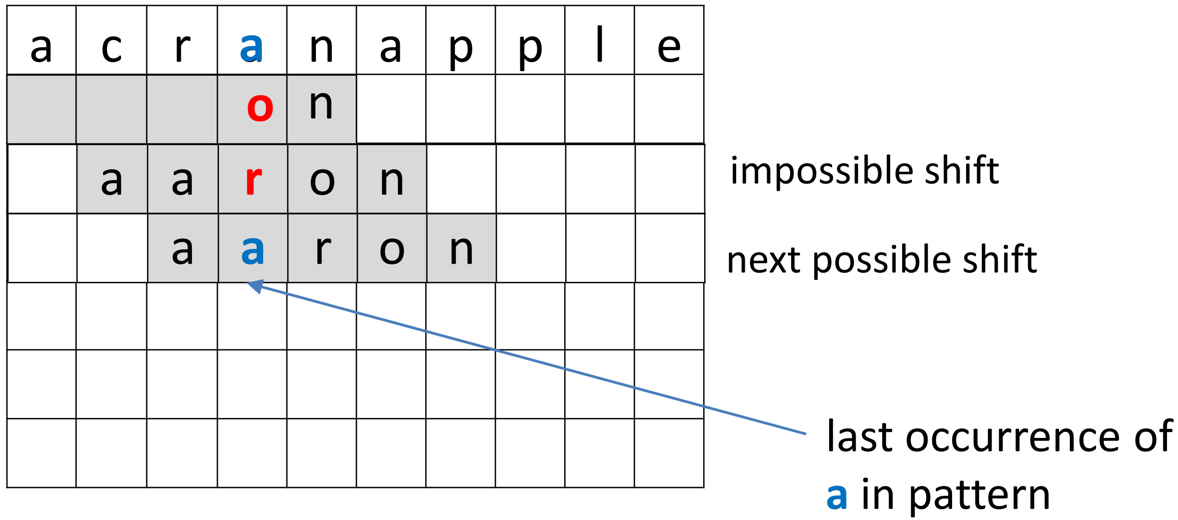
- **r** does not occur in P = aldo
- shift pattern past **r**
- **w** does not occur in P = aldo
- shift pattern past **w**
- bad character heuristic can rule out many shifts with reverse searching

w	h	e	r	e	i	s	w	a	l	d	o
a											

- **w** does not occur in P = aldo
- move pattern past **w**
- the first shift moves pattern past **w**
- no shifts are ruled out
- bad character heuristic does not rule out any shifts with forward searching when the first character of the pattern is mismatched

What if Mismatched Text Character Occurs in P?

T= acranapple, P = aaron



- Mismatched character in the text is **a**
- Find **last** occurrence of **a** in P
- Shift the pattern to the right until **last** **a** in P aligns with **a** in text
  - all smaller shifts are impossible since they do not match **a**
- Precompute last occurrence of any letter before matching starts

Bad Character Heuristic: Side Note

T= acranapple, P = aaron

a	c	r	a	n	a	p	p	l	e	
			o	n						
		a	a	r	o	n				next possible shift
			a	a	r	o	n			also a valid shift

- If we shifted until the **first** a in P aligns with a in text
  - this would give a possible shift, but misses a previous possible shift, possibly leading to a missed pattern

Bad Character Heuristic: Full Version

- Extends to the case when mismatched text character does occur in P

T= acranapple, P = aaron

a	c	r	a	n	a	p	p	l	e
			o	n					
			[a]						

- Mismatched character in the text is a
- Shift the pattern to the right so that the last a in P aligns with a in text
- Continue matching the pattern (in reverse)

Bad Character Heuristic: Full Version

- Extends to the case when mismatched text character does occur in P

T= acranapple, P = aaron

a	c	r	**a**	n	a	p	p	l	e
			o	n					
			[a]			**n**			

- Mismatched character in the text is **a**
- Shift the pattern to the right so that the last **a** in P aligns with **a** in text
- Continue matching the pattern (in reverse)

Bad Character Heuristic: Last Occurrence Array

- Compute the last occurrence array $L(c)$ of any character in the alphabet
  - $L(c) = -1$ if character c does not occur in P, otherwise
  - $L(c) =$ largest index j such that $P[j] = c$
- Example: P = aaron
  - initialization

$char$	a	n	o	r	all others
$L(c)$	-1	-1	-1	-1	-1

this means:

a	b	c	d	e	f	…	x	y	z
-1	-1	-1	-1	-1	-1		-1	-1	-1

in actual implementation:

0	1	2	3	4	5	…	24	25	26
-1	-1	-1	-1	-1	-1		-1	-1	-1

Bad Character Heuristic: Last Occurrence Array

- Compute the last occurrence array $L(c)$ of any character in the alphabet
  - $L(c) = -1$ if character c does not occur in P, otherwise
  - $L(c) =$ largest index j such that $P[j] = c$
- Example: P = aaron
  - computation

$char$	a	n	o	r	all others
$L(c)$	0	-1	-1	-1	-1

aaron

$i = 0$

L is valid for $P =$ a

Bad Character Heuristic: Last Occurrence Array

- Compute the last occurrence array $L(c)$ of any character in the alphabet
  - $L(c) = -1$ if character c does not occur in P, otherwise
  - $L(c) =$ largest index j such that $P[j] = c$
- Example: P = aaron
  - computation

aaron

$i = 1$

$char$	a	n	o	r	all others
$L(c)$	1	-1	-1	-1	-1

L is valid for $P =$ aa

Bad Character Heuristic: Last Occurrence Array

- Compute the last occurrence array $L(c)$ of any character in the alphabet
  - $L(c) = -1$ if character c does not occur in P, otherwise
  - $L(c) =$ largest index j such that $P[j] = c$
- Example: P = aaron
  - computation

aaron

$i = 2$

$char$	a	n	o	r	all others
$L(c)$	1	-1	-1	2	-1

L is valid for $P =$ aar

Bad Character Heuristic: Last Occurrence Array

- Compute the last occurrence array $L(c)$ of any character in the alphabet
  - $L(c) = -1$ if character c does not occur in P, otherwise
  - $L(c) =$ largest index j such that $P[j] = c$
- Example: P = aaron
  - computation

aaron

$i = 3$

$char$	a	n	o	r	all others
$L(c)$	1	-1	3	2	-1

L is valid for $P =$ aaro

Bad Character Heuristic: Last Occurrence Array

- Compute the last occurrence array $L(c)$ of any character in the alphabet
  - $L(c) = -1$ if character c does not occur in P, otherwise
  - $L(c) =$ largest index j such that $P[j] = c$
- Example: P = aaron
  - computation

$char$	a	n	o	r	all others
$L(c)$	1	4	3	2	-1

aaron

$i = 4$

L is valid for $P =$ aaron

- Total time is $O(m + |\Sigma|)$

Boyer-More Indexing

- Same as in KMP
    - maintain variables i and j
    - j is the position in the pattern
    - i is the position in the text where we do the next check
    - check is performed by determining if $T[i] = P[j]$
    - current shift is $i - j$

Bad Character Heuristic: Shifting Formula

$char$	a	n	o	r	all others
$L(c)$	1	4	3	2	-1

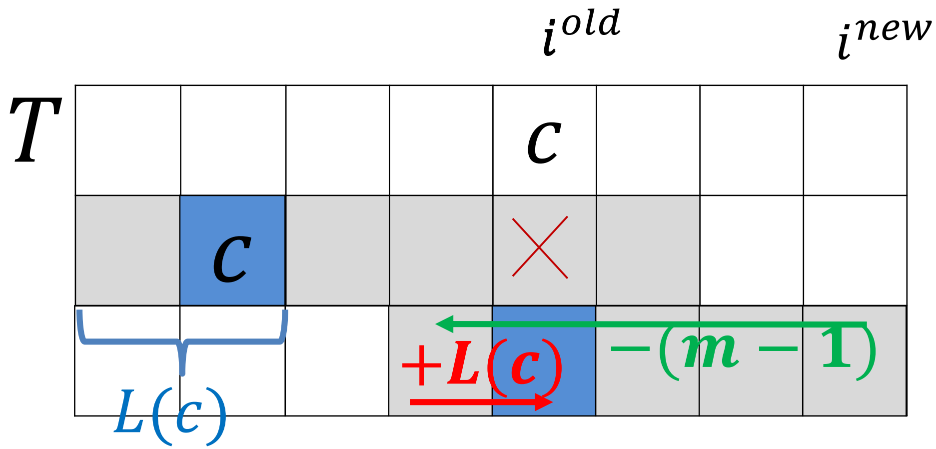
T= acranapple, P = aaron

			$j=3$ $i=3$			$j=4$ $i=6$			
a	c	r	a	n	a	p	p	l	e
			o	n					
			[a]			n			

- Let $L(c)$ be the last occurrence of character c in P
    - $L(\mathbf{a}) = 1$ in our example
- When mismatch occurs at text position i, pattern position j, update
    - $j \;= m - 1$
        - start matching at the end of the pattern
    - $i \;= i + m - 1 - L(c)$
    - for our example
        - $j \;= 5 - 1 = 4$
        - $i \;= 3 + 5 - 1 - 1 = 6$

Bad Character Heuristic: Shifting Formula Explained

- Text character is c at the mismatch position i in the text
- $i = i + m - 1 - L(c)$



$$i^{new} - (m-1) + L(c) = i^{old}$$

$$i^{new} = i^{old} + m - 1 - L(c)$$

$$i = i + m - 1 - L(c)$$

Bad Character Heuristic: Important Use Condition

- Text character is c at the mismatch position i in the text
- $i \;\; = i + m - 1 - L(c)$
- Old shift: $i - j$
- New shift: $i + (m - 1) - L(c) - (m - 1) = i - L(c)$
- If $L(c) > j$, new shift < old shift, shifts P in the wrong direction, not useful
  - we already ruled that shift out, no point to come back to it
- Example: T= acranapple, P = reroa

$\boldsymbol{j=3}$
$\boldsymbol{i=8}$

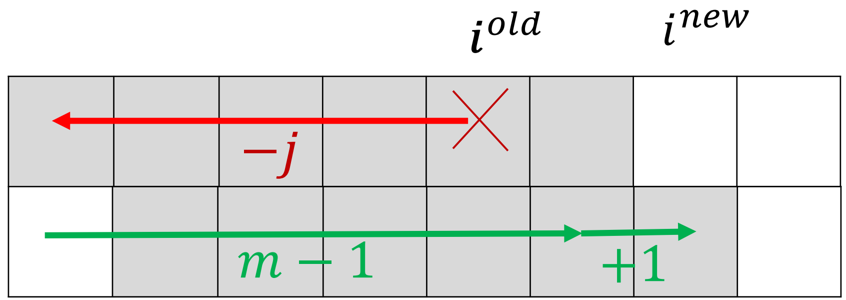
c	a	c	r	w	a	a	p	a	a	e
				a						
								o	a	
							o	a		

$L(\mathrm{a}) = 4$
$L(\mathrm{a}) > j = 3$
old shift: $i - j = 8 - 3 = 5$
$i \;\; = 8 + 5 \; - 1 - 4 = 8$
$j \;\; = 5 - 1 = 4$
new shift: $i - j = 8 - 4 = 4$

- **bad character heuristic makes sense to used only if $L(c) < j$**
  - $L(c) \neq j$ in case of a mismatch

Bad Character Heuristic: Brute-Force Step

- If $L(c) > j$
  - pattern would shift in wrong direction if used bad character heuristic
  - therefore, do brute-force step
  - $j = m - 1$
  - $i = i - j + m$



$$i^{old} \;\; -j \;\; +m-1 \;\; +1 \;\; = i^{new}$$

$$i^{new} = i^{old} - j + m$$

$$i = i - j + m$$

Bad Character Heuristic: Unified Formula

- If $L(c) < j$
  - $j = m - 1$
  - $i = i + m - 1 - L(c)$
- If $L(c) > j$
  - $j = m - 1$
  - $i = i - j + m$

- Unified formula for i that works in all cases

$$i = i + m - 1 - \min\{L(c), j - 1\}$$

Boyer-More Example

$char$	a	e	p	r	others
$L(c)$	1	3	2	4	-1

P = paper

					$j=4$ $i=4$			$j=4$ $i=7$		$j=4$ $i=9$				$j=3$ $i=13$	$j=4$ $i=14$	$j=4$ $i=15$			
T	f	e	e	d	a	l	l	p	o	o	r	p	a	r	r	o	t	s	
					r														$i=7$
					[a]			r											$i=9$
								[p]		r									$i=14$
														e	r				$i=15$
																r			$i=20$

not found!

- Unified formula for i that works in all cases

$$i = i + m - 1 - \min\{L(c), j - 1\}$$

Boyer-Moore Algorithm

BoyerMoore(T, P)

$L \leftarrow$ last occurrence array computed from P

$j \leftarrow m - 1$

$i \leftarrow m - 1$

while $i < n$ and $j \geq 0$ **do** //current guess begins at index $i - j$

if $T[i] = P[j]$ **then**

$i \leftarrow i - 1$

$j \leftarrow j - 1$

else

$i \leftarrow i + m - 1 - \min\{L(c), j - 1\}$

$j \leftarrow m - 1$

if $j = -1$ **return** "found at shift $i + 1$" // i moved one position to

// the left of the first char in T

else return FAIL

Good Suffix Heuristic

- Idea is similar to KMP, but applied to the suffix, since matching backwards

P = onobobo

$j=3$
$i=3$

T	o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
				b	o	b	o												
			o	n	o	b	o	b	o										

- Text has letters obo
- Do the smallest shift so that obo fits
- Can precompute this from the pattern itself, before matching starts
  - ‘if failure at $j = 3$, shift pattern by 2’
- Continue matching from the end of the new shift
- Will not study the precise way to do it

Boyer-Moore Algorithm with Good Suffix

```
BoyerMoore(T, P)
    L ← last occurrence array computed from P
    S ← good suffix array computed from P
    j ← m − 1
    i ← m − 1
    while i < n and j ≥ 0 do  //current guess begins at index i − j
        if  T[i] = P[j] then
            i ← i − 1
            j ← j − 1
        else
            i ← i + m − 1 − min{L(T[i]), S[j]}
            j ← m − 1
    if j = −1 return "found at shift i + 1"
    else  return FAIL
```

Boyer-Moore Summary

- Boyer-Moore performs very well, even when using only bad character heuristic
- Worst case run time is $O(nm)$ with bad character heuristic only, but in practice much faster
- On typical English text, Boyer-Moore looks only at $\approx$25% of text T
- With good suffix heuristic, can ensure $O(n + m + |\Sigma|)$ run time
    - no details

Outline

- String Matching
  - Introduction
  - Karp-Rabin Algorithm
  - Knuth-Morris-Pratt algorithm
  - Boyer-Moore Algorithm
  - Suffix Trees
  - Suffix Arrays
  - Conclusion

Suffix Tree: Trie of Suffixes

- What if we search for many patterns P within the same fixed text T?
- Idea: preprocess the text T rather than pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T
- Example: $P =$ ish

$$T = \text{establ}\underbrace{\overbrace{\text{ish}}^{\text{prefix}}\text{ment}}_{\text{suffix}}$$

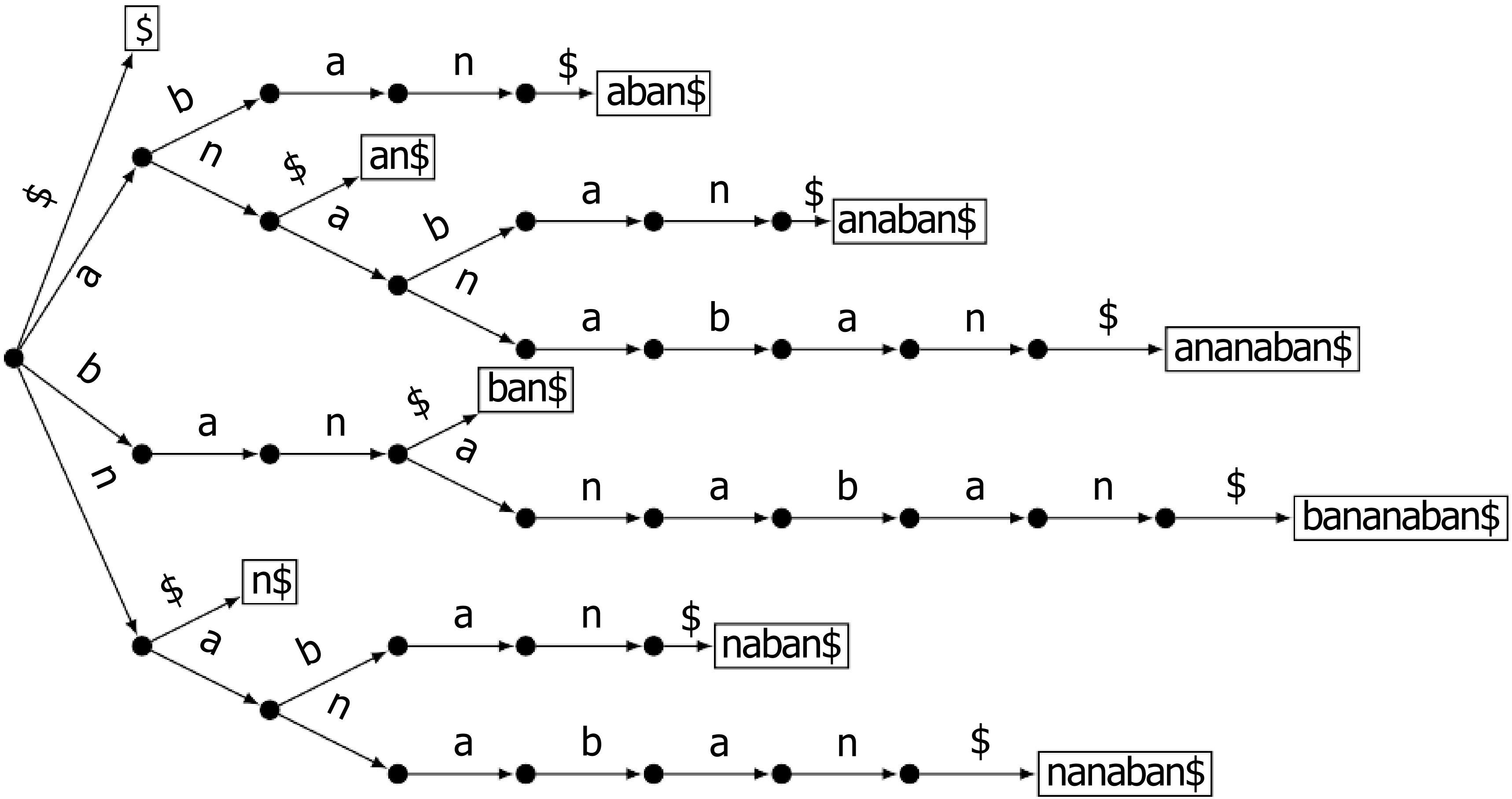
- Store all suffixes of T in a trie
- To save space
  - use compressed trie
  - store suffixes implicitly via indices into T
- This is called a **suffix tree**

Trie of suffixes: Example

- T = bananaban

Suffixes = {bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n, Λ}

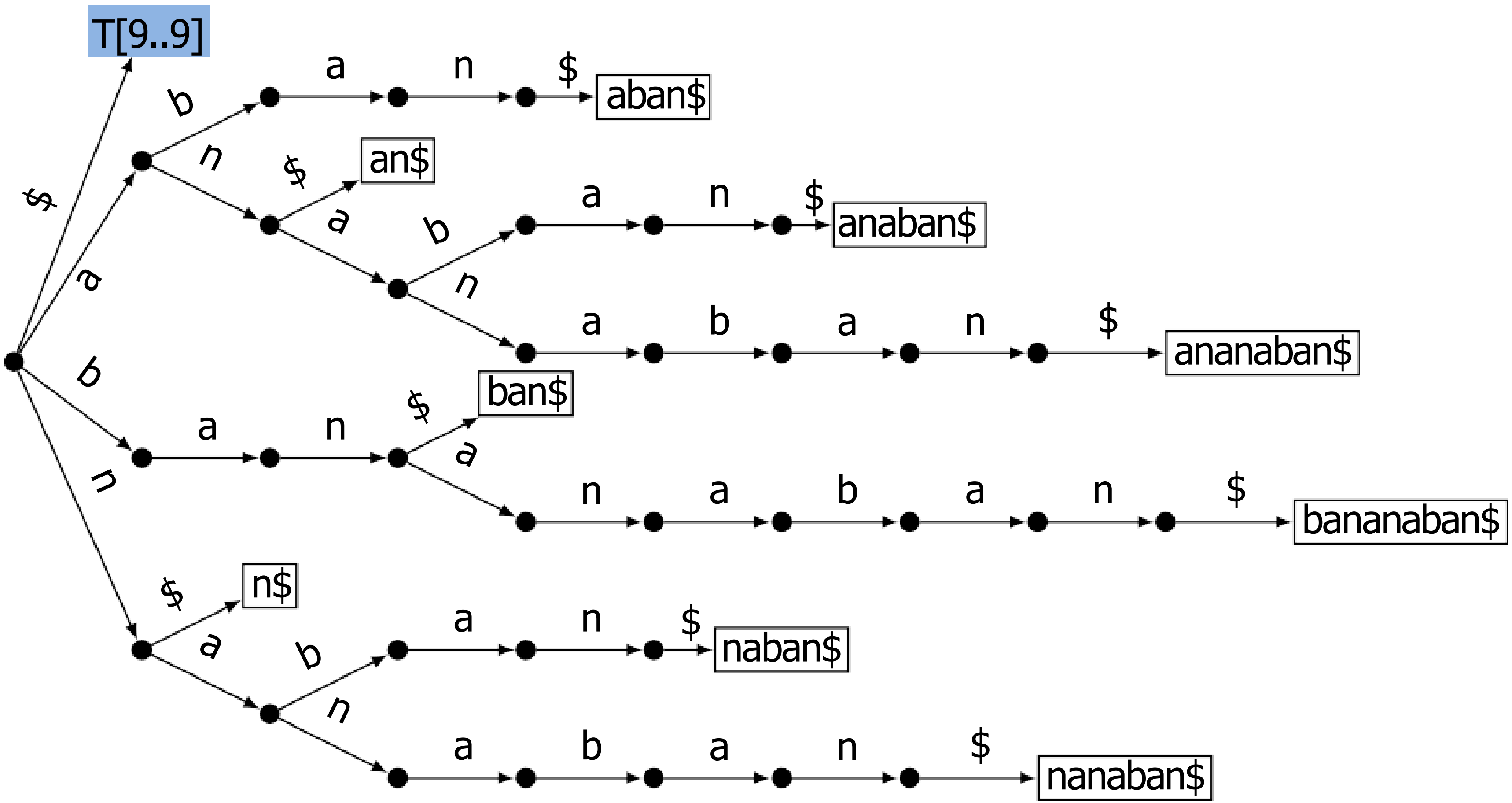
S = {bananaban\$, ananaban\$, nanaban\$, anaban\$,naban\$,…, ban\$, n\$, \$}



Trie of suffixes: Example

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$

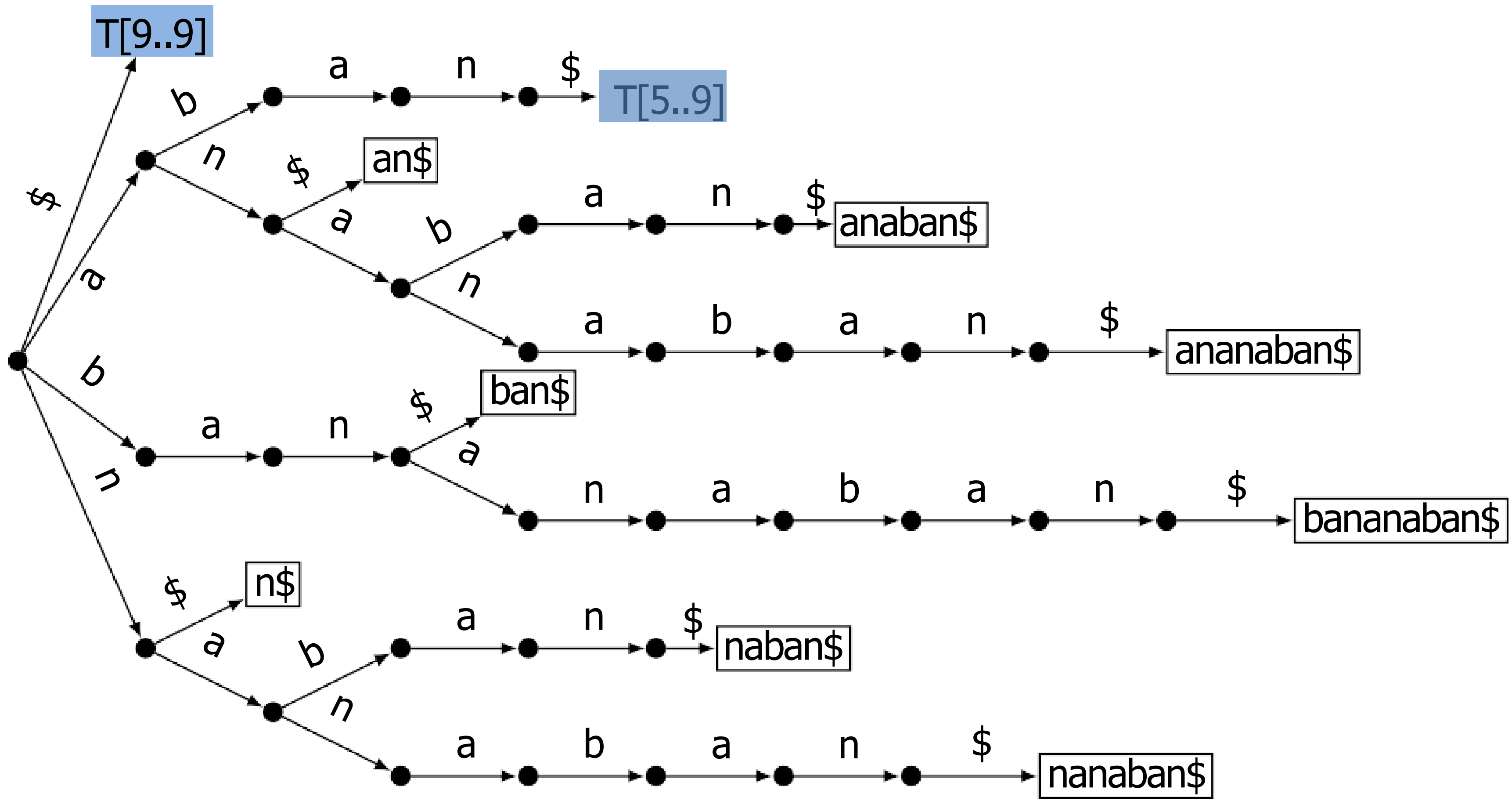
- Store suffixes via indices



Trie of suffixes: Example

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$

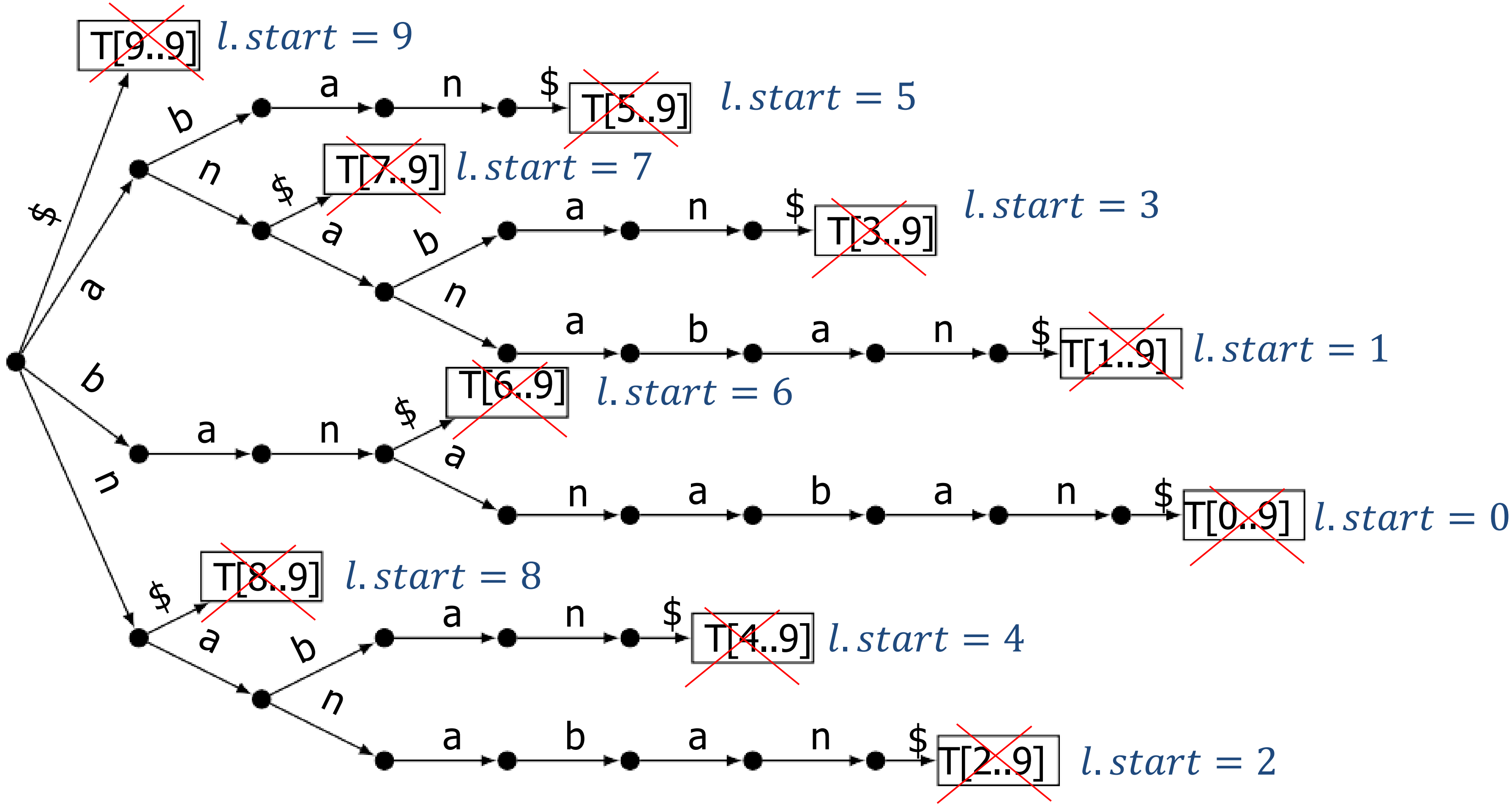
- Store suffixes via indices



Tries of suffixes

- In actual implementation, each leaf l stores the start of its suffix in variable $l.start$

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$

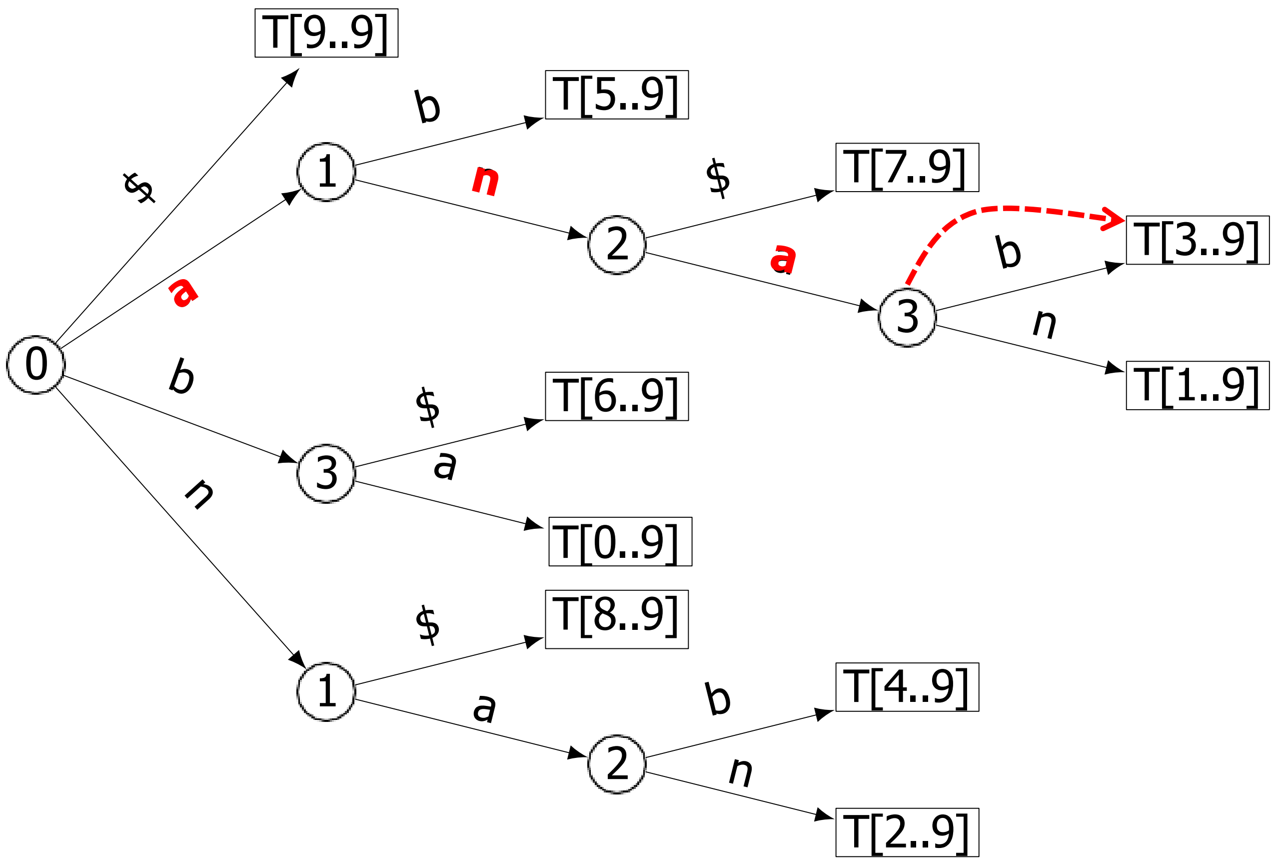


Suffix tree

T =

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$

- Suffix tree: compressed trie of suffixes
- If P occurs in the text, it is a prefix of one (or more) strings stored in the trie
- Have to modify search in a trie to allow search for a prefix



Building Suffix Tree

- Building
  - text T has n characters and $n+1$ suffixes
  - can build suffix tree by inserting each suffix of T into compressed trie
    - takes $\Theta(|\Sigma|n^2)$ time
  - there is a way to build a suffix tree of T in $\Theta(|\Sigma|n)$ time
    - beyond the course scope
- Pattern Matching
  - essentially search for P in compressed trie
    - some changes needed, since P may only be prefix of stored word
  - run-time is
    - $O(|\Sigma|m)$, assuming each node stores children in a linked list
    - $O(m)$, assuming each node stores children in an array
- Summary
  - theoretically good, but construction is slow or complicated and lots of space-overhead
  - rarely used in practice

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Suffix Arrays

- Relatively recent development (popularized in the 1990s)
- Sacrifice some performance for simplicity
  - slightly slower (by a log-factor) than suffix trees
  - much easier to build
  - much simpler pattern matching
  - very little space, only one array
- Idea
  - store suffixes implicitly, by storing start indices
  - store the sorting permutation of the suffixes in T

Suffix Array Example

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$

	0	1	2	3	4	5	6	7	8	9
Suffix Array =	9	5	7	3	1	6	0	8	4	2

i	**suffix** $T[i \ldots n]$
0	bananaban$
1	ananaban$
2	nanaban$
3	anaban$
4	naban$
5	aban$
6	ban$
7	an$
8	n$
9	$

sort lexicographically →

j	$A^s[j]$	
0	9	$
1	5	aban$
2	7	an$
3	3	anaban$
4	1	ananaban$
5	6	ban$
6	0	bananaban$
7	8	n$
8	4	naban$
9	2	nanaban$

Suffix Array Construction

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$

- Easy to construct using MSD-Radix-Sort (pad with any character to get the same length)

	round 1	round 2	...	round n
bananaban$	$********	$********		$********
ananaban$*	ananaban$	aban$****		aban$****
nanaban$**	anaban$***	ananaban$		an$*******
anaban$***	aban$*****	anaban$**		anaban$***
naban$****	an$*******	an$******		ananaban$*
aban$*****	bananaban$	bananaban$		ban$******
ban$******	ban$******	ban$******		bananaban$
an$*******	nanaban$**	nanaban$**		n$********
n$********	naban$****	naban$****		naban$****
$*********	n$********	n$********		nanaban$**

- Fast in practice, suffixes are unlikely to share many leading characters
- But worst case run-time is $\Theta(n^2)$
  - recursion depth is n, $\Theta(n)$ time at each recursion depth, example: $T = aa \ldots a$

Suffix Array Construction

- Idea: we do not need n rounds
  - $\Theta(\log n)$ rounds enough $\rightarrow \Theta(n \log n)$ run time
- Construction-algorithm
  - MSD-radix sort plus some bookkeeping
    - needs only one extra array
    - easy to implement
  - details are covered in an algorithms course

Pattern Matching in Suffix Arrays

- Suffix array stores suffixes (implicitly) in sorted order
- Idea: apply binary search

P = ban

0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$

T =

b a n

ban > a

	j	$A^s[j]$	
$l \rightarrow$	0	9	$
	1	5	aban$
	2	7	an$
	3	3	anaban$
$v \rightarrow$	4	1	ananaban$
	5	6	ban$
	6	0	bananaban$
	7	8	n$
	8	4	naban$
$r \rightarrow$	9	2	nanaban$

Pattern Matching in Suffix Arrays

- Suffix array stores suffixes (implicitly) in sorted order
- Idea: apply binary search

P = ban

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	**n**	$
									b	a	n

ban < **n**

	j	$A^s[j]$	
	0	9	$
	1	5	aban$
	2	7	an$
	3	3	anaban$
	4	1	ananaban$
$l \rightarrow$	5	6	ban$
	6	0	bananaban$
$v \rightarrow$	7	8	n$
	8	4	naban$
$r \rightarrow$	9	2	nanaban$

Pattern Matching in Suffix Arrays

- Suffix array stores suffixes (implicitly) in sorted order
- Idea: apply binary search

P = ban

	0	1	2	3	4	5	6	7	8	9
T =	b	a	n	a	n	a	b	a	n	$
							b	a	n	

found!

	j	$A^s[j]$	
	0	9	$
	1	5	aban$
	2	7	an$
	3	3	anaban$
	4	1	ananaban$
$v = l \rightarrow$	5	6	ban$
$r \rightarrow$	6	0	bananaban$
	7	8	n$
	8	4	naban$
	9	2	nanaban$

- $\Theta(\log n)$ comparisons
- Each comparison is *strcmp*$(P,\ T[A^s[v] \ldots A^s[v+m-1]])$
- $\Theta(m)$ per comparison $\Longrightarrow$ run-time is $\Theta(m \log n)$

Pattern Matching in Suffix Arrays

SuffixArray-Search$(T, P, A^s[0 \ldots n-1])$

A^s: suffix array of T, P: pattern

$l \leftarrow 0, r \leftarrow n-1$

while $l < r$

$v \leftarrow \left\lfloor \frac{l+r}{2} \right\rfloor$

$i \leftarrow A^s[v]$

// assume *strcmp* handles out of bounds suitably

$s \leftarrow$ *strcmp*$(P, T[i \ldots i+m-1])$

if $(s > 0)$ **do** $l \leftarrow v+1$

else $(s < 0)$ **do** $r \leftarrow v-1$

else return 'found at guess $T[i \ldots i+m-1]$'

if *strcmp*$(P, T[A^s[l], A^s[l]+m-1]]) = 0$

return 'found at guess $T\left[A^s[l], A^s[l]+m-1\right]$'

return FAIL

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String Matching Conclusion

	Brute Force	KR	BM	KMP	Suffix Trees	Suffix Array
preproc.	—	$O(m)$	$O(m + \|\Sigma\|)$	$O(m)$	$O(\|\Sigma\|n^2)$ $\rightarrow O(\|\Sigma\|n)$	$O(nlogn)$ $\rightarrow O(n)$
search time (preproc excluded)	$O(nm)$	$O(n + m)$ expected	$O(n + \|\Sigma\|)$ with good suffix often better	$O(n)$	$O(m)$	$O(mlogn)$ $\rightarrow O(m + logn)$
extra space	—	$O(1)$	$O(m + \|\Sigma\|)$	$O(m)$	$O(n)$	$O(n)$

- Algorithms stop once they found one occurrence
- Most of them can be adapted to find *all* occurrences within the same worst-case run-time