

CS 240 – Data Structures and Data Management

Module 4: Dictionaries

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Based on lecture notes by many previous cs240 instructors

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Winter 2021

Outline

- 1 Dictionaries and Balanced Search Trees
 - ADT Dictionary
 - Review: Binary Search Trees
 - AVL Trees
 - Insertion in AVL Trees
 - Restoring the AVL Property: Rotations

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Dictionary ADT

Dictionary: An ADT consisting of a collection of items, each of which contains

- a *key*
- some *data* (the “value”)

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*(k) (also called *findElement*(k))
- *insert*(k, v) (also called *insertItem*(k, v))
- *delete*(k) (also called *removeElement*(k))
- optional: *closestKeyBefore*, *join*, *isEmpty*, *size*, *etc.*

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
- Keys can be compared in constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$ (except array occasionally needs to resize)

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$ (via binary search)

insert $\Theta(n)$

delete $\Theta(n)$

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Binary Search Trees (review)

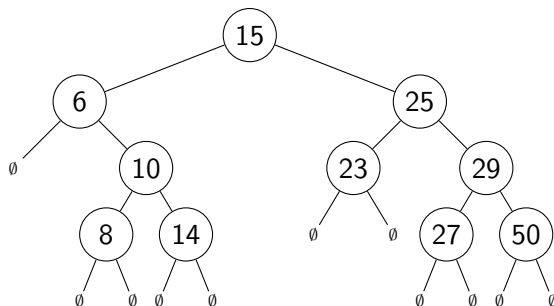
Structure Binary tree: all nodes have two (possibly empty) subtrees

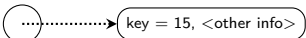
Every node stores a KVP

Empty subtrees usually not shown

Ordering Every key k in $T.left$ is less than the root key.

Every key k in $T.right$ is greater than the root key.



(In our examples we only show the keys, and we show them directly in the node. A more accurate picture would be )

BST as realization of ADT Dictionary

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

BST::insert(k, v) Search for k , then insert (k, v) as new node

Example:

Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.
- If x has one non-empty subtree, move child up
- Else, swap key at x with key at **successor** or **predecessor** node and then delete that node

Height of a BST

BST::search, *BST::insert*, *BST::delete* all have cost $\Theta(h)$, where h = height of the tree = max. path length from root to leaf

If n items are inserted one-at-a-time, how big is h ?

- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\Theta(\log n)$.
Any binary tree with n nodes has height $\geq \log(n + 1) - 1$
- Average-case: Can show $\Theta(\log n)$

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- **AVL Trees**
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- Restoring the AVL Property: Rotations

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an **AVL Tree** is a BST with an additional **height-balance property** at every node:

The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be -1 .)

Rephrase: If node v has left subtree L and right subtree R , then

balance(v) := $height(R) - height(L)$ must be in $\{-1, 0, 1\}$

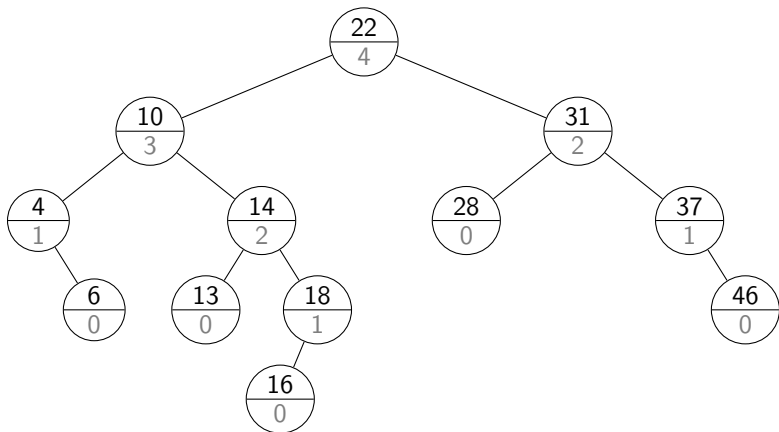
$balance(v) = -1$ means v is *left-heavy*

$balance(v) = +1$ means v is *right-heavy*

- Need to store at each node v the height of the subtree rooted at it
- Can show: It suffices to store $balance(v)$ instead
 - ▶ uses fewer bits, but code gets more complicated

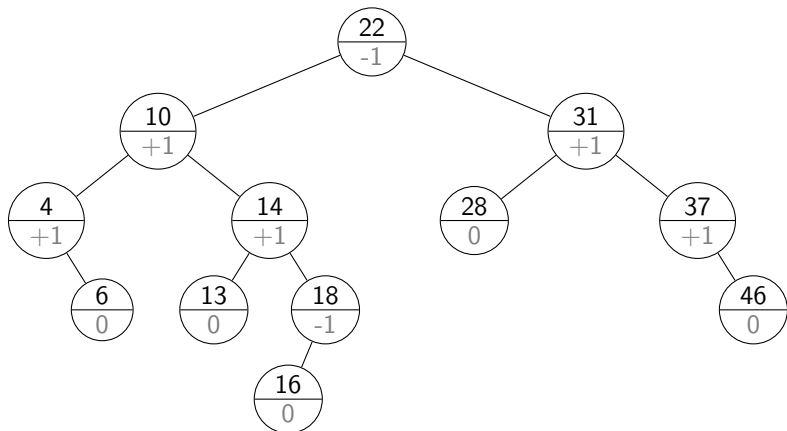
AVL tree example

(The lower numbers indicate the height of the subtree.)



AVL tree example

Alternative: store balance (instead of height) at each node.



Height of an AVL tree

Theorem: An AVL tree on n nodes has $\Theta(\log n)$ height.

\Rightarrow *search, insert, delete* all cost $\Theta(\log n)$ in the *worst case!*

Proof:

- Define $N(h)$ to be the *least* number of nodes in a height- h AVL tree.
- What is a recurrence relation for $N(h)$?
- What does this recurrence relation resolve to?

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AVL insertion

To perform *AVL::insert*(k, v):

- First, insert (k, v) with the usual BST insertion.
- We assume that this returns the new leaf z where the key was stored.
- Then, move up the tree from z , updating heights.
 - ▶ We assume for this that we have parent-links. This can be avoided if *BST::insert* returns the full path to z .
- If the height difference becomes ± 2 at node z , then z is **unbalanced**. Must re-structure the tree to rebalance.

AVL insertion

AVL::insert(k, v)

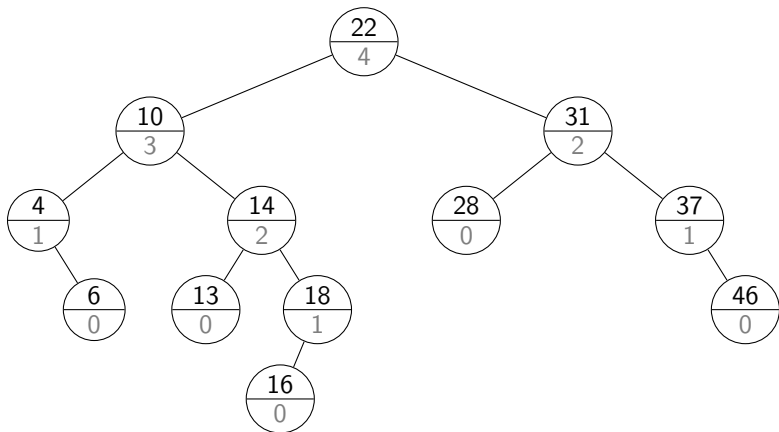
1. $z \leftarrow \text{BST::insert}(k, v)$ // leaf where k is now stored
2. **while** (z is not NIL)
3. **if** ($|z.\text{left}.\text{height} - z.\text{right}.\text{height}| > 1$) **then**
4. Let y be taller child of z
5. Let x be taller child of y
6. $z \leftarrow \text{restructure}(x, y, z)$ // see later
7. **break** // can argue that we are done
8. $\text{setHeightFromSubtrees}(z)$
9. $z \leftarrow z.\text{parent}$

setHeightFromSubtrees(u)

1. $u.\text{height} \leftarrow 1 + \max\{u.\text{left}.\text{height}, u.\text{right}.\text{height}\}$

AVL Insertion Example

Example:



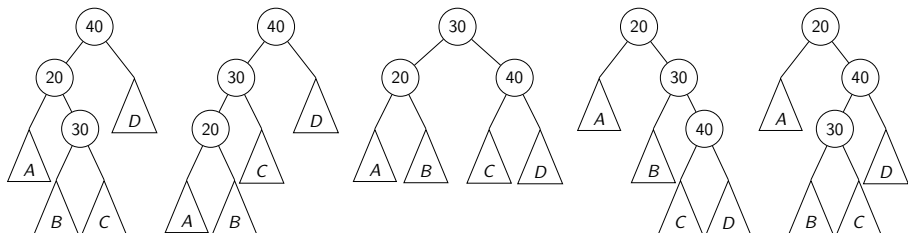
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How to “fix” an unbalanced AVL tree

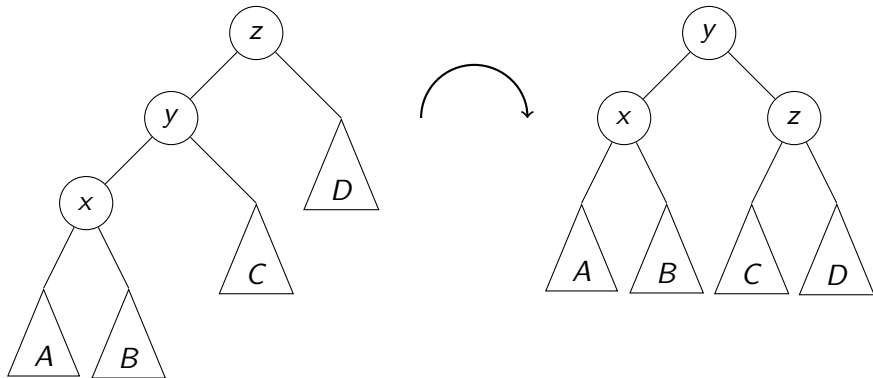
Note: there are many different BSTs with the same keys.



Goal: change the *structure* among three nodes without changing the *order* and such that the subtree becomes balanced.

Right Rotation

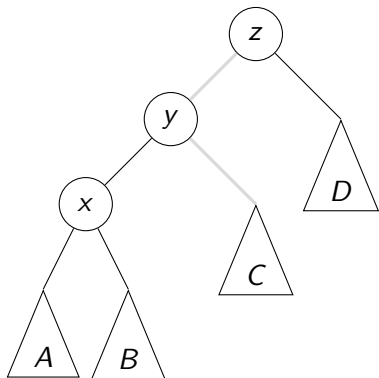
This is a **right rotation** on node z :



rotate-right(z)

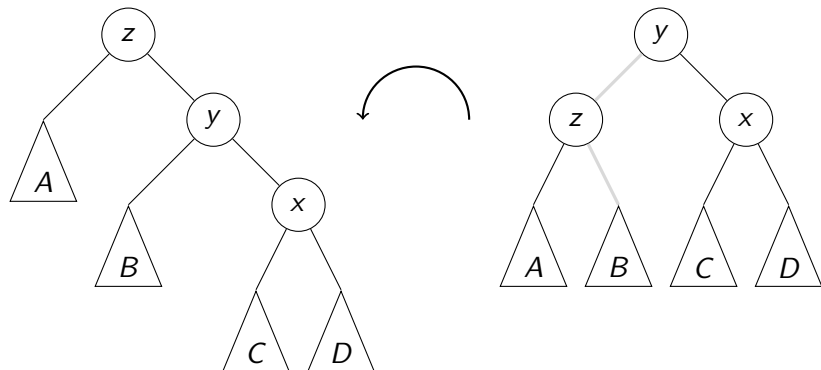
1. $y \leftarrow z.\text{left}$, $z.\text{left} \leftarrow y.\text{right}$, $y.\text{right} \leftarrow z$
2. *setHeightFromSubtrees*(z), *setHeightFromSubtrees*(y)
3. **return** y // returns new root of subtree

Why do we call this a rotation?



Left Rotation

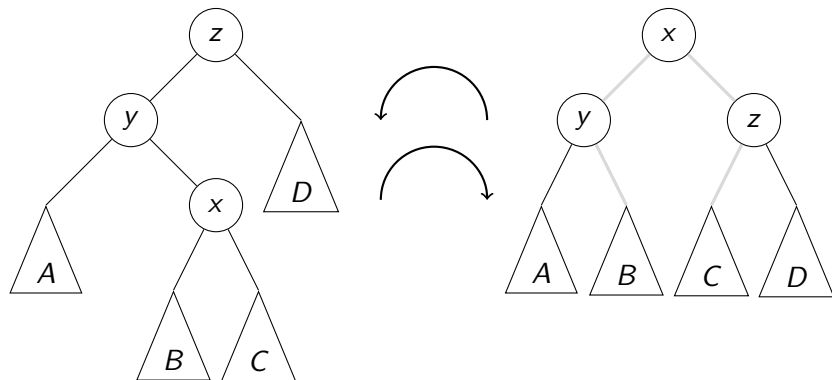
Symmetrically, this is a **left rotation** on node z :



Again, only two links need to be changed and two heights updated.
Useful to fix right-right imbalance.

Double Right Rotation

This is a **double right rotation** on node z :

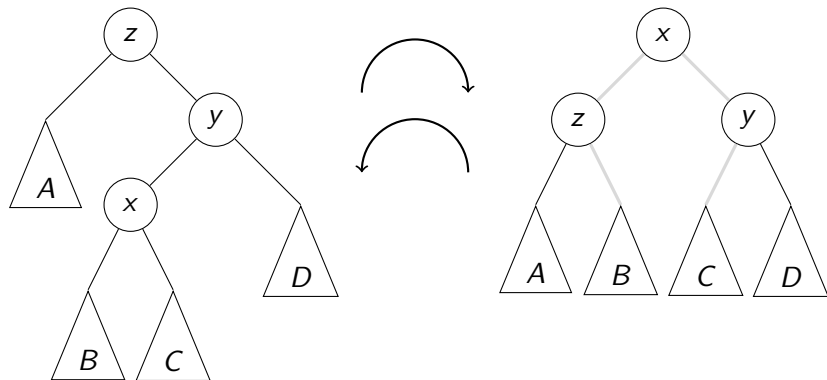


First, a left rotation at y .

Second, a right rotation at z .

Double Left Rotation

Symmetrically, there is a **double left rotation** on node z :



First, a right rotation at y .
Second, a left rotation at z .

Fixing a slightly-unbalanced AVL tree

restructure(x, y, z)

node x has parent y and grandparent z

1. **case**



: // Right rotation

return *rotate-right*(z)



: // Double-right rotation

$z.left \leftarrow$ *rotate-left*(y)

return *rotate-right*(z)



: // Double-left rotation

$z.right \leftarrow$ *rotate-right*(y)

return *rotate-left*(z)



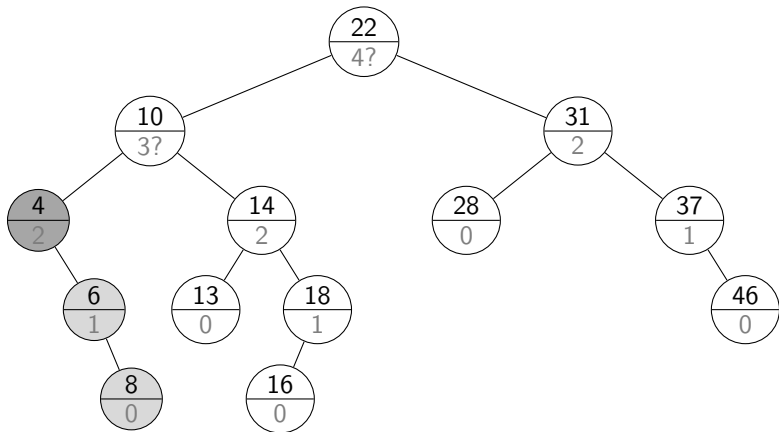
: // Left rotation

return *rotate-left*(z)

Rule: The middle key of x, y, z becomes the new root.

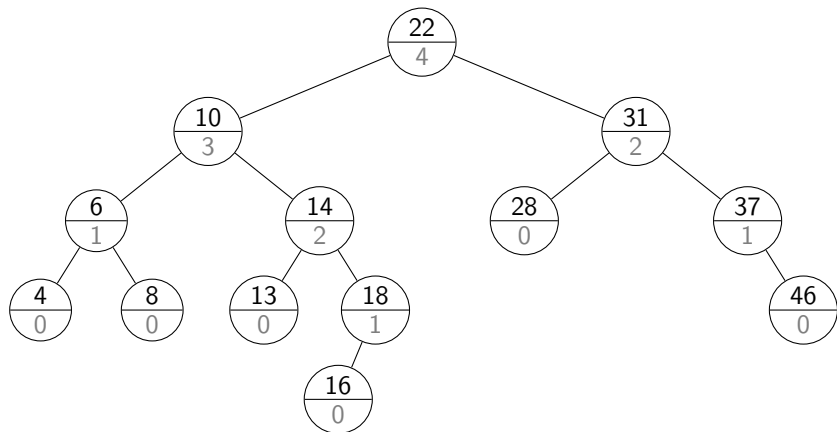
AVL Insertion Example revisited

Example:



AVL Insertion: Second example

Example: `AVL::insert(45)`



AVL Deletion

Remove the key k with *BST::delete*.

Find node where *structural* change happened.

(This is not necessarily near the node that had k .)

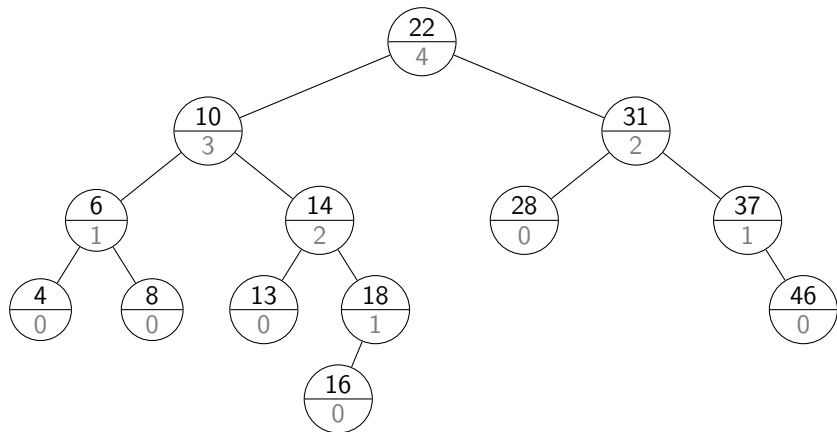
Go back up to root, update heights, and rotate if needed.

```
AVL::delete(k)
```

1. $z \leftarrow \text{BST::delete}(k)$
2. // Assume z is the parent of the BST node that was removed
3. **while** (z is not NIL)
4. **if** ($|z.\text{left}.\text{height} - z.\text{right}.\text{height}| > 1$) **then**
5. Let y be taller child of z
6. Let x be taller child of y (break ties to prefer single rotation)
7. $z \leftarrow \text{restructure}(x, y, z)$
8. // Always continue up the path and fix if needed.
9. $\text{setHeightFromSubtrees}(z)$
10. $z \leftarrow z.\text{parent}$

AVL Deletion Example

Example:



AVL Tree Operations Runtime

search: Just like in BSTs, costs $\Theta(\text{height})$

insert: *BST::insert*, then check & update along path to new leaf

- total cost $\Theta(\text{height})$
- *restructure* restores the height of the subtree to what it was,
- so *restructure* will be called *at most once*.

delete: *BST::delete*, then check & update along path to deleted node

- total cost $\Theta(\text{height})$
- *restructure* may be called $\Theta(\text{height})$ times.

Worst-case cost for all operations is $\Theta(\text{height}) = \Theta(\log n)$.

But in practice, the constant is quite large.