CS 240 – Data Structures and Data Management

Module 6E: Dictionaries for special keys - Enriched

T. Biedl É. Schost O. Veksler

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Winter 2021

version 2021–02–10 12:29

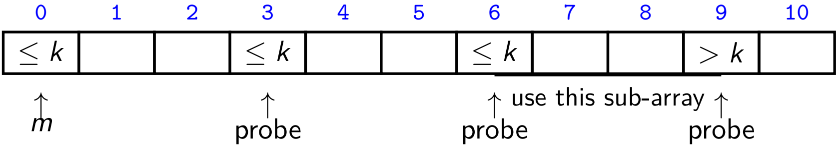
Improving Interpolation Search

- Had: Average-case run-time of *interpolation-search* is $O(\log \log n)$.
- This is very complicated to prove!

  (
  - Study error, i.e., distance between index of k and where we probed.
  - Argue that error is in $O(\sqrt{n})$ in first round.
  - Argue that error is in $O(\frac{1}{2^i} n)$ after i rounds.
  - Study the martingale formed by the errors in the rounds.
  - Argue that its expected length is $O(\log \log n)$.

)
- Instead: Define a variant of *interpolatation-search*
  - Better worst-case run-time.
  - Easier to analyze.
- Idea: *Force* the sub-array to have size $\sqrt{n}$
- To do so, search for suitable sub-array with probes.
- Crucial question: how many probes are needed?

Improving Interpolation Search



- First compare at m as before.
- If $A[m] \leq k$, probe rightward.
- Probes always go $\lfloor\sqrt{N}\rfloor$ indices rightward (where $N = r - \ell$ is the size of the sub-array where k could be)
- Continue probing until $> k$ or out-of-bounds
- Observe: $\#$ probes $\leq \frac{N}{\lfloor\sqrt{N}\rfloor} \leq \sqrt{N} + 1$.

Improving Interpolation Search

Interpolation-search-modified(A, n, k)
A: sorted array of size n, k: key

if $(k < A[0]$ or $k > A[n-1])$ **return** "not found"
if $(k = A[n-1])$ **return** "found at index $n-1$"
$\ell \leftarrow 0,\ r \leftarrow n-1$ // have $A[\ell] \leq k < A[r]$
while $(N \leftarrow (r-\ell) \geq 2)$
    $m \leftarrow \ell + \frac{k-A[\ell])}{A[r]-A[\ell]} \cdot (r-\ell)$
    if $(A[m] \leq k)$ // probe rightward
        $\ell \leftarrow m, m_r \leftarrow \min\{r, m + \lfloor\sqrt{N}\rfloor\}$
        while $(m_r < r$ and $A[m_r] < k)$
            $\ell \leftarrow m_r,\ m_r \leftarrow \min\{r, m + \lfloor\sqrt{N}\rfloor\}$
        $r \leftarrow m_r$ // found suitable sub-array
    else
        ⋮ // symmetrically probe leftward
if $(k = A[\ell])$ **return** "found at index ℓ"
else return "not found"

Analysis of *interpolation-search-improved*

- $T(n) \leq T(\sqrt{n}) + O(\#\text{probes})$
- $\#$ probes $\leq \sqrt{n} + 1$.
- The worst-case run-time satisfies

$$T^{\text{worst}}(n) \leq T^{\text{worst}}(\sqrt{n}) + c \cdot (\sqrt{n} + 1)$$

- Show: $T^{\text{worst}}(n) \leq \frac{5}{4} c \sqrt{n}$ for $n \geq 16$

- Therefore worst-case run-time is $O(\sqrt{n})$.

Analysis of *interpolation-search-improved*

- What is the number of probes on average?
- Rephrase: If numbers are chosen uniformly at random, what is the expected number of probes?
- **Can show:** Expected number of probes is in $O(1)$.
- The average-case run-time satisfies

$$T^{\text{avg}}(n) \leq T^{\text{avg}}(\sqrt{n}) + c$$

- Show: $T^{\text{avg}}(n) \leq c\lceil \log \log n \rceil$ for $n \geq 4$.

- Therefore the average-case run-time is $O(\log \log n)$.