CS 240 – Data Structures and Data Management

Module 5: Other Dictionary Implementations - Enriched

T. Biedl É. Schost O. Veksler

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Winter 2021

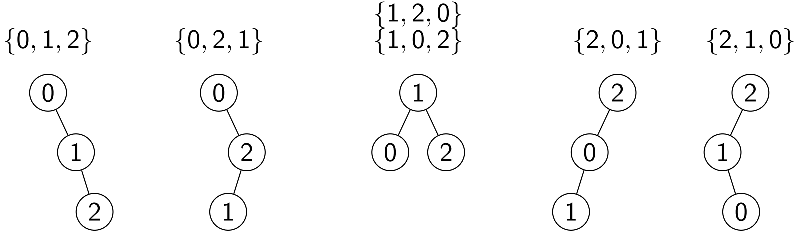
version 2021−02−10 12:28

Outline

- Expected height of a BST
- Treaps
- Optimal static binary search trees
- MTF-heuristic in a BST
- Splay Trees

Expected height of BSTs

Assume we *randomly* choose a permutation of $\{0, \ldots, n-1\}$ and build a binary search tree in this order:



Theorem: The expected height of the binary search tree is $O(\log n)$.
Proof:

Outline

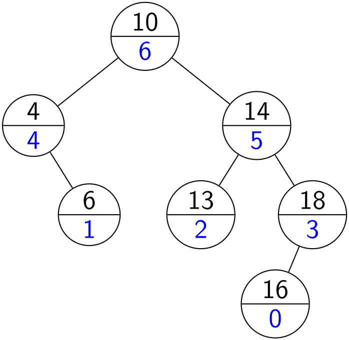
- Expected height of a BST
- **Treaps**
- Optimal static binary search trees
- MTF-heuristic in a BST
- Splay Trees

Treaps

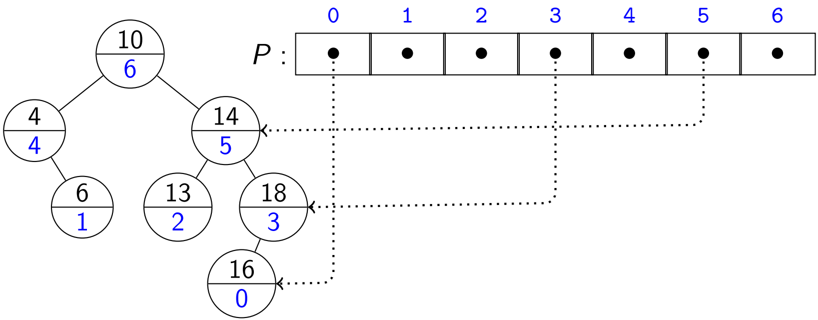
Goal: Build a binary search tree that acts as if it had been build in randomly picked insertion order.

Idea: Use binary search tree, but store a priority with each node.

- Priorities are a permutation of $\{0, \ldots, n-1\}$.
- Permutation has been picked *randomly*
- All permutations should be equally likely.
- Priorities are *decreasing* when going downwards (similar to a heap).

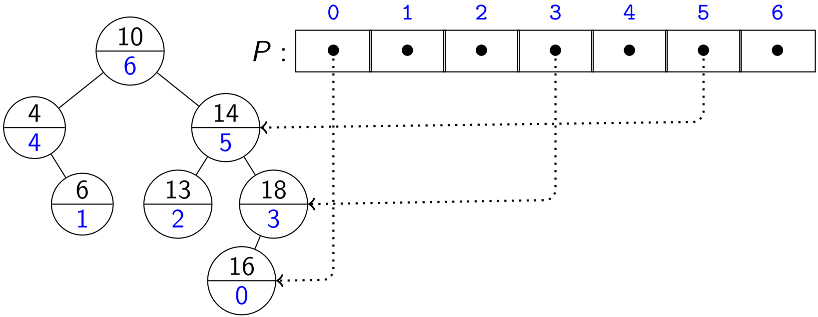


Treaps



- We will also need an array P where $P[i]$ stores node with priority i.
- We call this a **treap** (= tree + heap).

Treaps



- We will also need an array P where $P[i]$ stores node with priority i.
- We call this a **treap** (= tree + heap).

Theorem: The expected height of a treap is $O(\log n)$.
Proof: Root-item has priority $n-1$. This is picked randomly, so proof for expected height of BST applies.

Treap Insertion

Consider adding a new KVP. What priority should it get?

- We need a random permutation of $\{0, \ldots, n-1\}$
  - Currently we had a random permutation of $\{0, \ldots, n-2\}$.

Treap Insertion

Consider adding a new KVP. What priority should it get?

- We need a random permutation of $\{0, \ldots, n-1\}$
  - Currently we had a random permutation of $\{0, \ldots, n-2\}$.
- Recall *shuffle* from long ago:

```
shuffle(A)
A: array of size n stores ⟨0, ...n−1⟩
1.    for i ← 1 to n − 1 do
2.        swap( A[i], A[random(i + 1)] )
```

- In ith round,
  - have random permutation of $\{0, \ldots, i-1\}$
  - build random permutation of $\{0, \ldots, i\}$ in $O(1)$ time
  - key insight: swap with randomly chosen item

Treap Insertion

Consider adding a new KVP. What priority should it get?

- We need a random permutation of $\{0, \ldots, n-1\}$
  - Currently we had a random permutation of $\{0, \ldots, n-2\}$.
- Recall *shuffle* from long ago:

```
shuffle(A)
A: array of size n stores ⟨0, ...n−1⟩
1.      for i ← 1 to n − 1 do
2.          swap( A[i], A[random(i + 1)] )
```

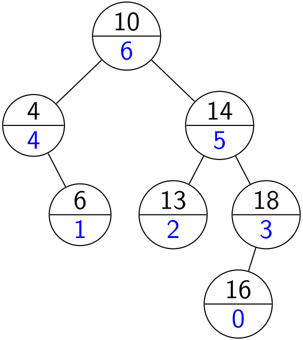
- In ith round,
  - have random permutation of $\{0, \ldots, i-1\}$
  - build random permutation of $\{0, \ldots, i\}$ in $O(1)$ time
  - key insight: swap with randomly chosen item

We can do the same by *randomly* picking priority p for new item.

- The item that had priority p previously now has priority $n-1$.
- If this violates the heap-property, then rotate to fix it.

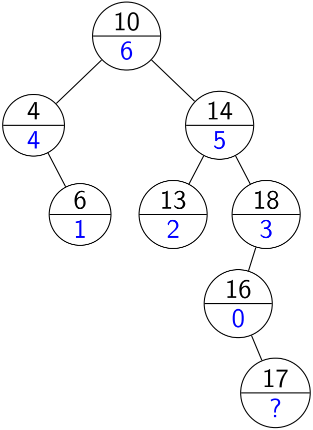
Treap Insertions Example

Example: *treap::insert*(17)



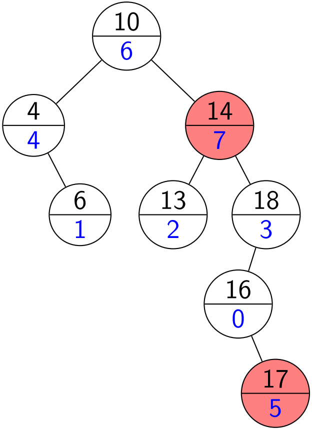
Treap Insertions Example

Example: *treap::insert*(17)



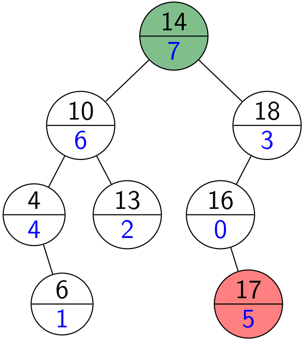
Treap Insertions Example

Example: *treap::insert*(17)
Randomly pick priority $5 \in \{0, \ldots, 7\}$



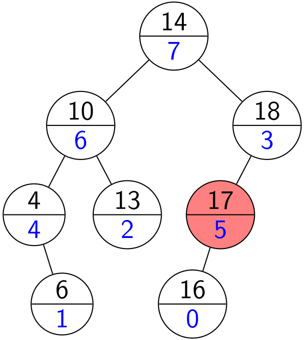
Treap Insertions Example

Example: *treap::insert*(17)
Randomly pick priority $5 \in \{0, \ldots, 7\}$



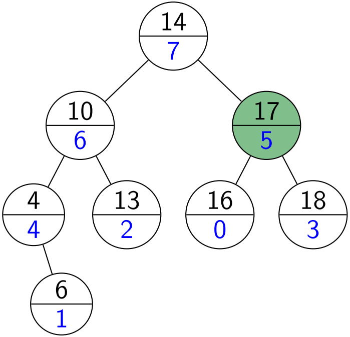
Treap Insertions Example

Example: *treap::insert*(17)
Randomly pick priority $5 \in \{0, \ldots, 7\}$



Treap Insertions Example

Example: *treap::insert*(17)
Randomly pick priority $5 \in \{0, \ldots, 7\}$



Treap Insertion Code

We assume that the treap stores array where $P[i]$ = node with priority i.

```
treap::insert(k, v)
1.    n ← P.size              // current size
2.    z ← BST::insert(k, v); n++
3.    p ← random(n)
4.    if p < n − 1 do
5.         z′ ← P[p], z′.priority ← n − 1, P[n − 1] ← z′
6.         fixUpWithRotations(z′)
7.    z.priority ← p; P[p] ← z
8.    fixUpWithRotations(z)
```

```
treap::fixUpWithRotations(z)
1.    while (y ← z.parent is not NIL and z.priority > y.priority) do
2.         if z is the left child of y do rotate-right(y)
3.         else rotate-left(y)
```

Treaps summary

- Randomized binary search tree, so expected height is $O(\log n)$
- Achieves $O(\log n)$ expected time for *search* and *insert*
- *delete* can be handled similar (but even more exchanges)

Treaps summary

- Randomized binary search tree, so expected height is $O(\log n)$
- Achieves $O(\log n)$ expected time for *search* and *insert*
- *delete* can be handled similar (but even more exchanges)

- Large space overhead (parent-pointers, priorities, P)
- Not particularly efficient in practice
  (except when priorities have meaning $\rightsquigarrow$ later)
- There are ways to avoid some of the space overhead, but in general randomized binary search trees are rarely used.
- We will soon see a randomization that works better (but is not a binary search tree)

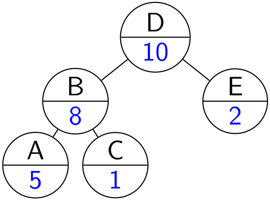
Outline

- Expected height of a BST
- Treaps
- Optimal static binary search trees
- MTF-heuristic in a BST
- Splay Trees

Optimal static binary search trees

- Can we find the optimal static order for a binary search tree?

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$



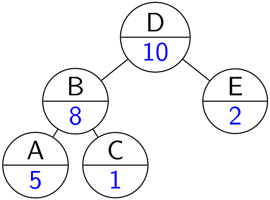
- Access-cost is now $\sum_k P(k) \cdot (1 + \text{depth of } k)$

  since we use $(1 + \text{depth of } k)$ comparisons to search for key k.

Optimal static binary search trees

- Can we find the optimal static order for a binary search tree?

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$



$1 \cdot \frac{10}{26} + 2 \cdot \frac{8}{26} + 2 \cdot \frac{2}{26} + 3 \cdot \frac{5}{26} + 3 \cdot \frac{1}{26} = \frac{48}{26}$

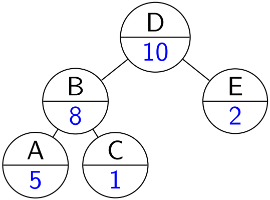
- Access-cost is now $\sum_k P(k) \cdot (1 + \text{depth of } k)$

  since we use $(1 + \text{depth of } k)$ comparisons to search for key k.

Optimal static binary search trees

- Can we find the optimal static order for a binary search tree?

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$



$1 \cdot \frac{10}{26} + 2 \cdot \frac{8}{26} + 2 \cdot \frac{2}{26} + 3 \cdot \frac{5}{26} + 3 \cdot \frac{1}{26} = \frac{48}{26}$

- Access-cost is now $\sum_k P(k) \cdot (1 + \text{depth of } k)$

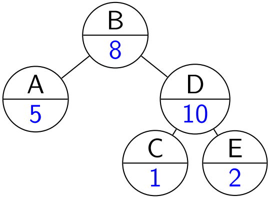
  since we use $(1 + \text{depth of } k)$ comparisons to search for key k.

- Natural greedy-algorithm:
  - Put item with highest access-probability at the root.
  - Split keys into left/right as dictated by the order-property.
  - Recurse in the subtree.

Optimal static binary search trees

The greedy-algorithm does *not* give the optimum!

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$

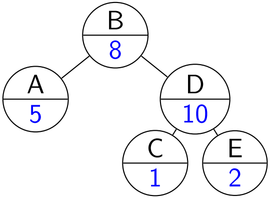


$1 \cdot \frac{8}{26} + 2 \cdot \frac{5}{26} + 2 \cdot \frac{10}{26} + 3 \cdot \frac{1}{26} + 3 \cdot \frac{2}{26} = \frac{47}{26}$

Optimal static binary search trees

The greedy-algorithm does *not* give the optimum!

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$



$$1 \cdot \frac{8}{26} + 2 \cdot \frac{5}{26} + 2 \cdot \frac{10}{26} + 3 \cdot \frac{1}{26} + 3 \cdot \frac{2}{26} = \frac{47}{26}$$

- To find the optimum, use "dynamic programming":
  - Effectively try *all* possible binary search trees
  - This would take exponential time if done in a straightfoward way.
  - Key idea: We can store and re-use solutions of subproblems to achieve polynomial run-time
- Many more details in cs341 (though not perhaps for this problem)

Outline

- Expected height of a BST
- Treaps
- Optimal static binary search trees
- MTF-heuristic in a BST
- Splay Trees

MTF-heuristic for binary search trees

What does 'move-to-front' mean in a binary search tree?

- Front = the place that is easiest to access
- In a binary search tree, that's the root.

⇒ After every access, bring item to the root of BST

MTF-heuristic for binary search trees

What does ‘move-to-front’ mean in a binary search tree?

- Front = the place that is easiest to access
- In a binary search tree, that’s the root.

$\Rightarrow$ After every access, bring item to the root of BST

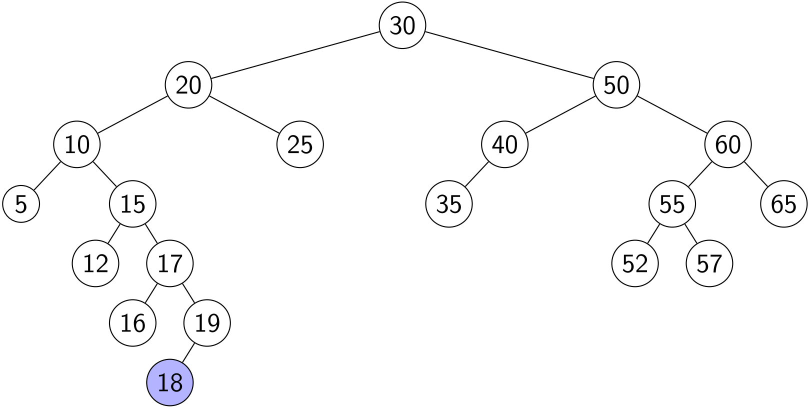
- But: order-property must be maintained!

$\Rightarrow$ Use *rotations*!

(This should remind you of treaps.)

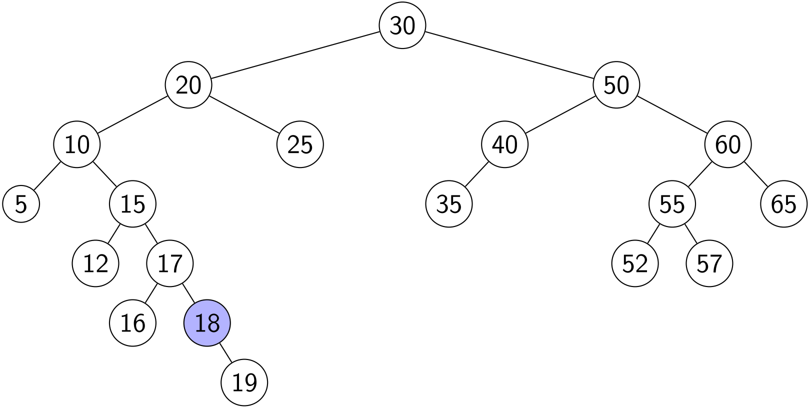
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



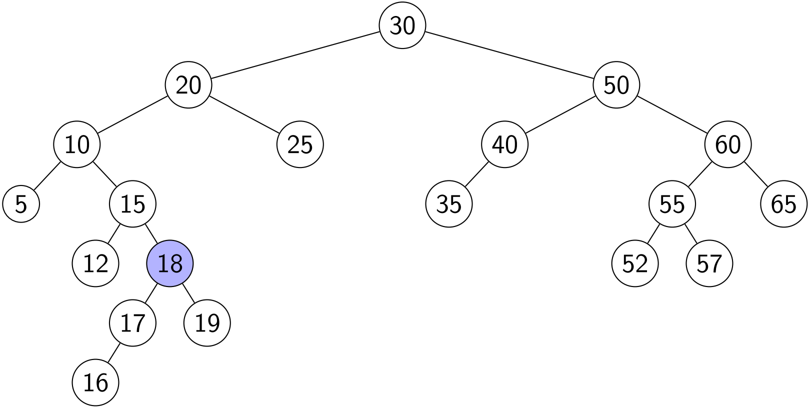
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



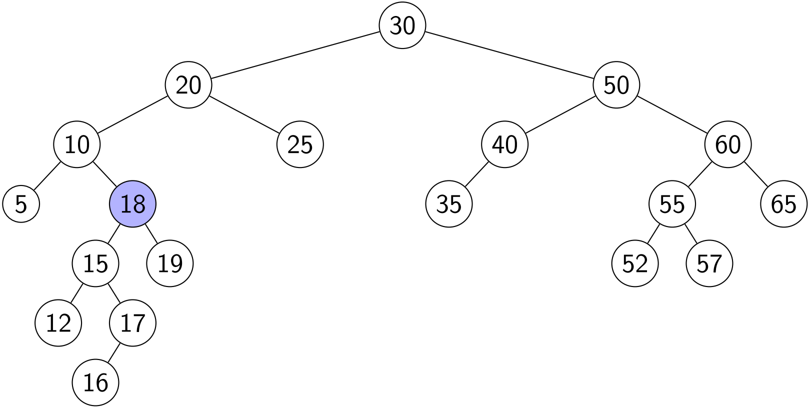
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



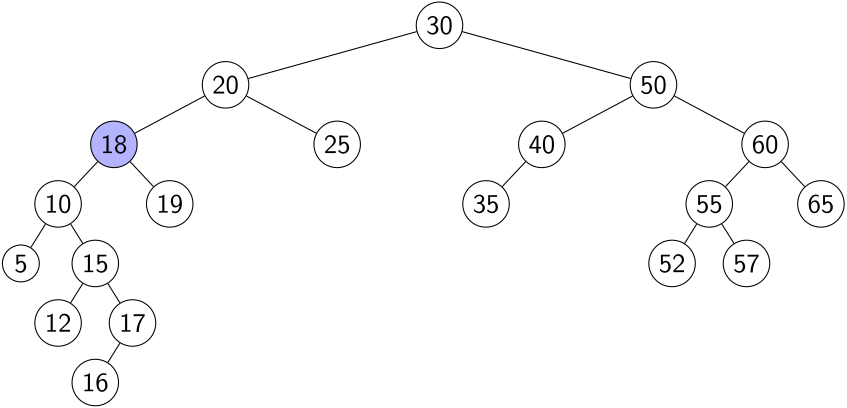
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



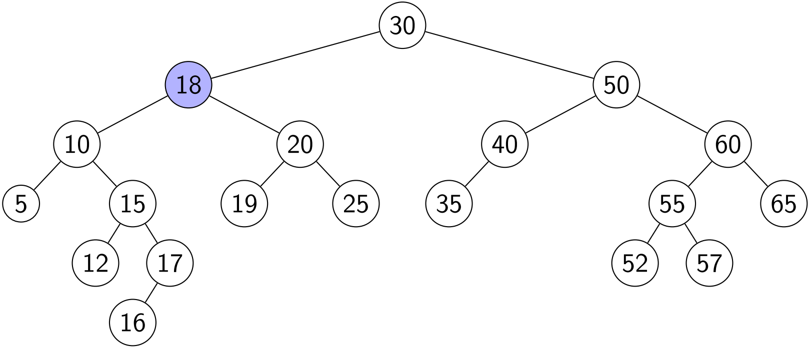
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



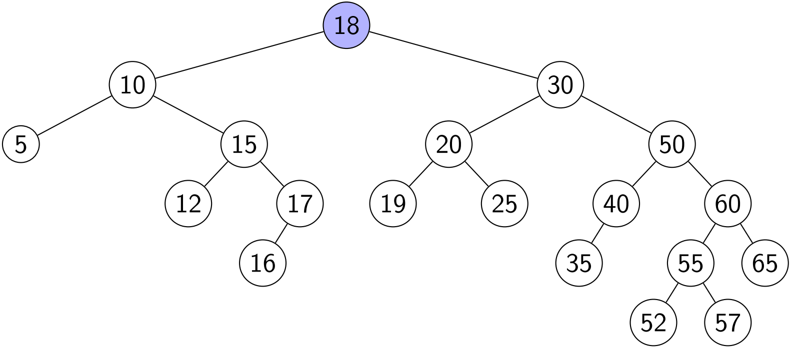
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



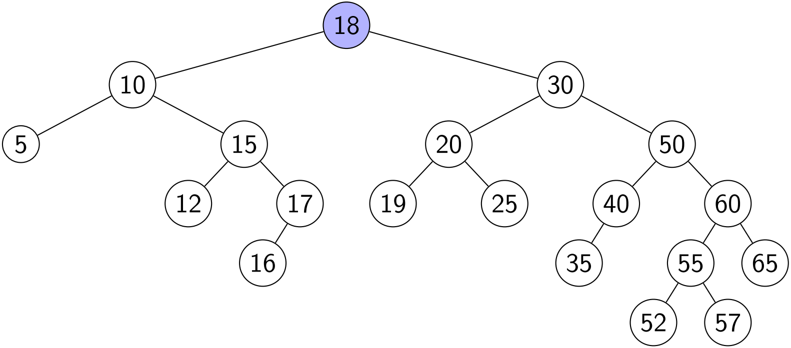
MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



This should work well, but we can do better by moving two level at a time.

Outline

- Expected height of a BST
- Treaps
- Optimal static binary search trees
- MTF-heuristic in a BST
- Splay Trees

Splay trees

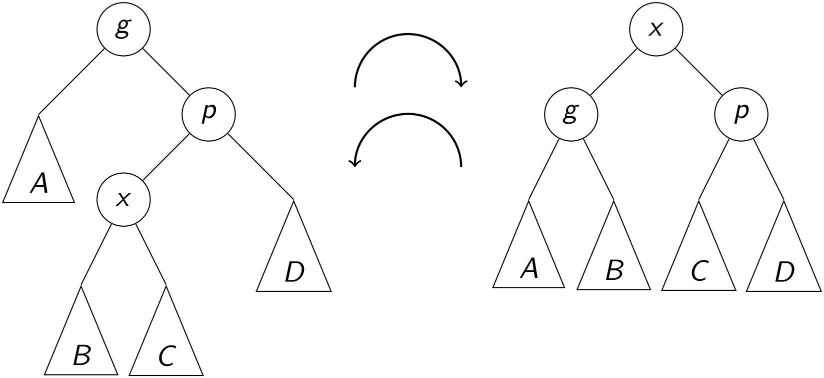
Splay tree overview:

- Binary search tree
- *No* extra information (such as height, balance, size) needed at nodes
- After search/insert, bring accessed node to the root with rotations
- Move node up two layers at a time (except when near root)
  - Use **zig-zig-rotation** or **zig-zag-rotation** to move up two levels.

Goal: This has amortized run-time $O(\log n)$.

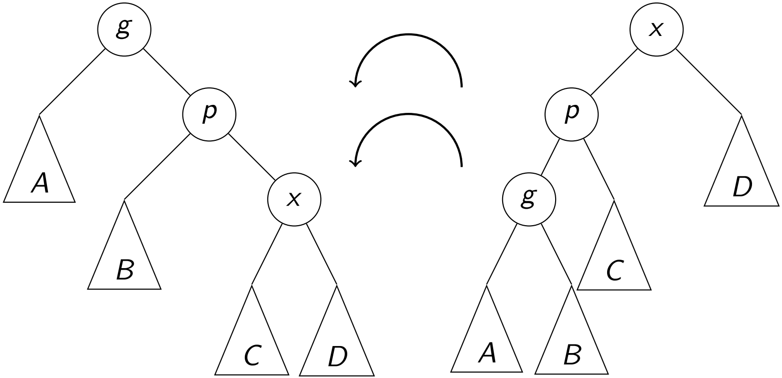
Zig-zag Rotation = Double Rotation

- Let x be the node that we want to move up.
- Let p and g be its parent and grandparent.
- If they are in zig-zag formation, apply a double-rotation.



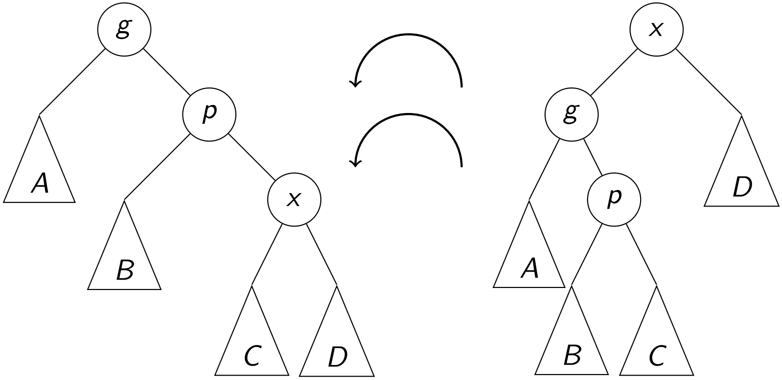
Zig-zig Rotation

- If they are in zig-zig formation, apply a new kind of rotation.



First, a left rotation at g. Second, a left rotation at p.

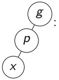
Compare to doing two single rotations

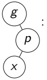


- Both operations bring x two levels higher.
- But using the zig-zig rotation allows to do amortized analysis.

Splay Tree Operations

SplayTree::insert(k, v)

$x \leftarrow$ *BST::insert*(k, v)
while (x is not the root)
    $p \leftarrow x.parent$
    if (x is the left child of p)
        if (p is the root)
            rotate-right(p)
        else $g \leftarrow p.parent$
        case


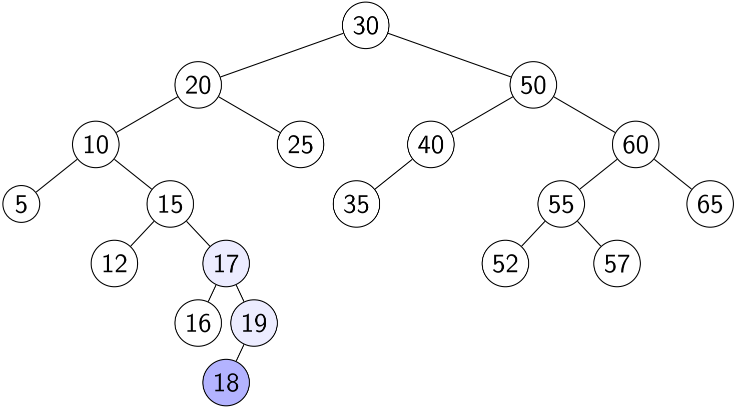
: ; // Zig-zig rotation
rotate-right(g)
rotate-right(p)


: // Zig-zag rotation
rotate-right(p)
rotate-left(g)
    else ... // symmetric case, x is right child

search is exactly the same, except use *BST::search*.

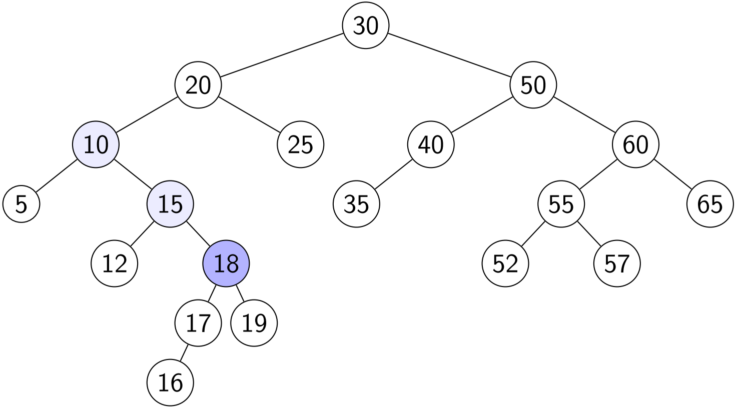
Splay Tree Insert

Example: *SplayTree::search*(18)



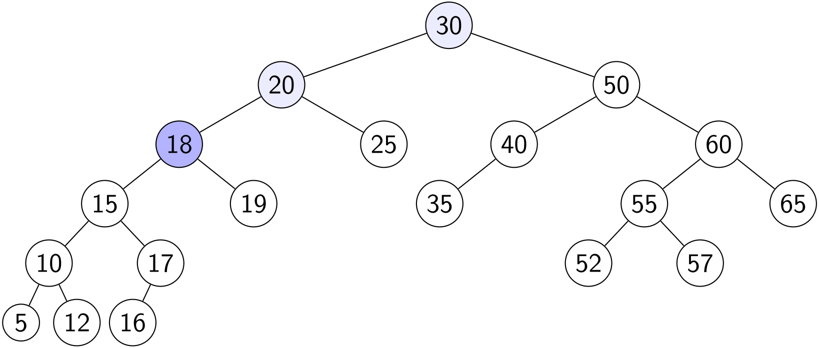
Splay Tree Insert

Example: *SplayTree::search*(18)



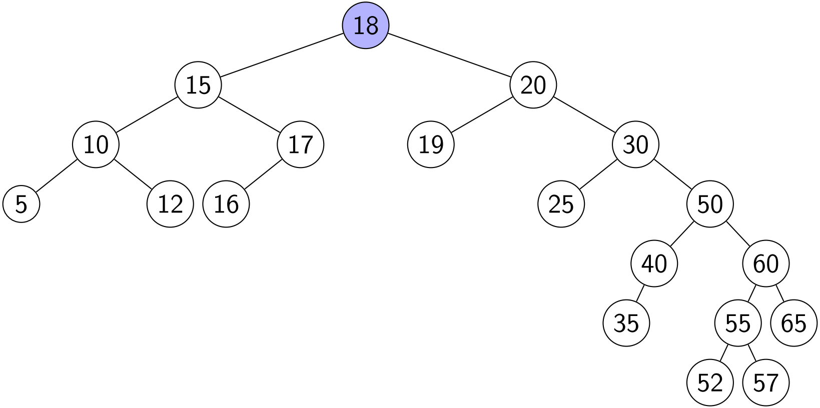
Splay Tree Insert

Example: *SplayTree::search*(18)



Splay Tree Insert

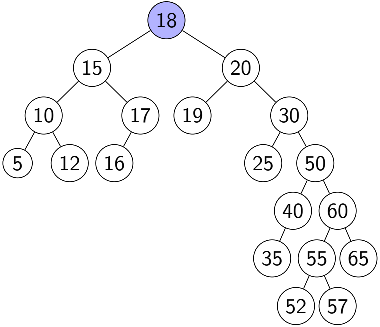
Example: *SplayTree::search*(18)



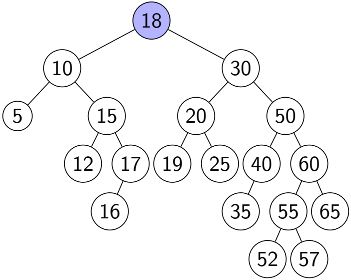
Zig-zig rotations vs. single rotations

Compare the resulting trees:

With zig-zig rotations:



With single rotations:

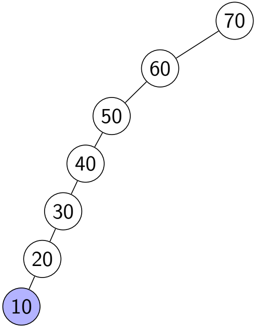


This is *not* more balanced, why do we apply zig-zig-rotations?

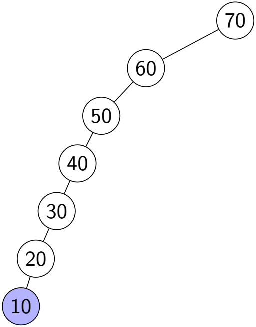
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



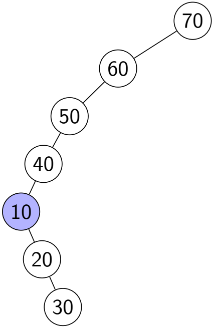
With single rotations:



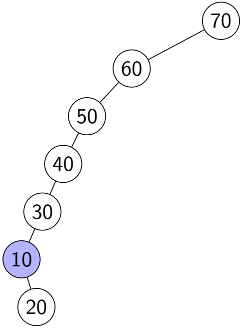
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



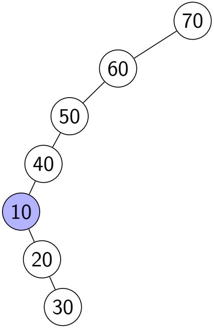
With single rotations:



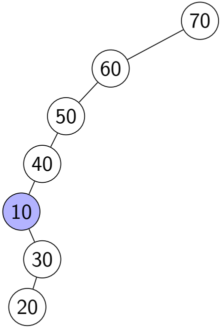
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



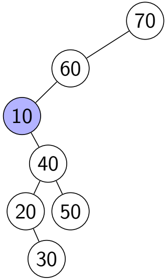
With single rotations:



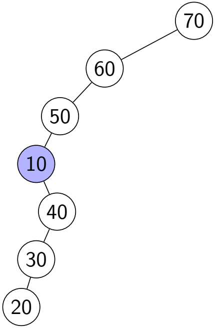
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



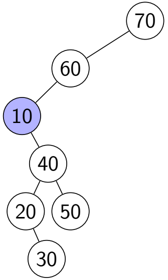
With single rotations:



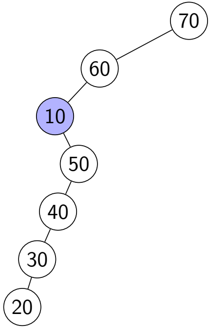
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



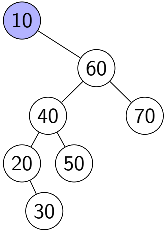
With single rotations:



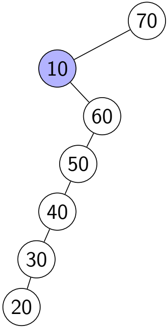
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



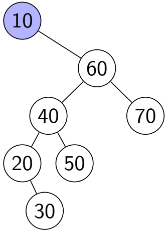
With single rotations:



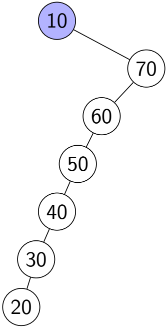
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



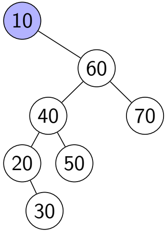
With single rotations:



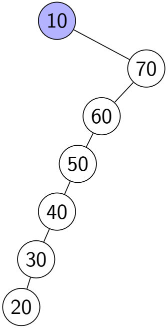
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



With single rotations:



Splay tree intuition:

- For any node on search-path, the depth (roughly) halves
- For all nodes, the depth increases by at most 2

Splay tree summary

Theorem: In a splay tree, all operations take $O(\log n)$ amortized time.
(The formal proof does not follow the intuition and uses a potential function.)

In summary:

- Needs *no* extra information (such as height or size) needed at nodes
- Our pseudo-code assumed parent-references; this can be avoided by temporarily storing search-path.
- According to experiments this is the most efficient binary search tree.