CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

T. Biedl E. Kondratovsky M. Petrick O. Veksler
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Winter 2022

version 2022-02-07 15:28

Outline

1. Dictionaries for special keys
   - Lower bound
   - Interpolation Search
   - Tries
     - Standard Tries
     - Variations of Tries
     - Compressed Tries

Outline

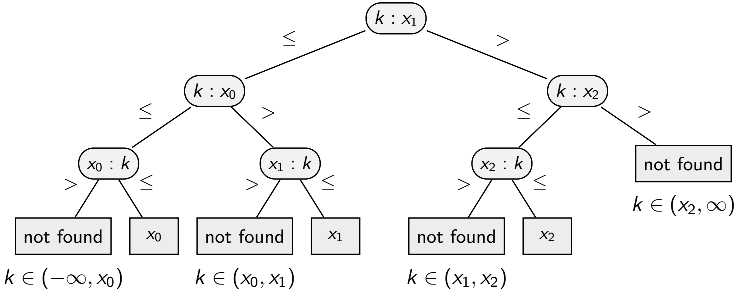
1. Dictionaries for special keys
   - Lower bound
   - Interpolation Search
   - Tries
     - Standard Tries
     - Variations of Tries
     - Compressed Tries

Lower bound for search

The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among n items. Is this the best possible?

Theorem: In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-n dictionary.

Proof: via decision tree



But can we beat the lower bound for special keys?

Outline

Binary Search

Recall the run-times in a *sorted array*:

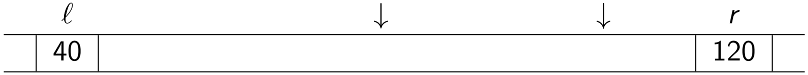
- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

binary-search(A, n, k)
A: Sorted array of size n, k: key

$\ell \leftarrow 0$, $r \leftarrow n-1$
while $(\ell \leq r)$
    $m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
    if $(A[m] == k)$ **then return** "found at $A[m]$"
    else if $(A[m] < k)$ **then** $\ell \leftarrow m+1$
    else $r \leftarrow m-1$
return "not found, but would be between $A[\ell-1]$ and $A[\ell]$"

Interpolation Search: Motivation

binary-search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$



Question: If keys are *numbers*, where would you expect key $k = 100$?

interpolation-search$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]}(r - \ell) \right\rfloor$

Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash during computation of m.

interpolation-search(A, n, k)
A: Sorted array of size n, k: key

$\ell \leftarrow 0$, $r \leftarrow n-1$
    while $(\ell \leq r)$
        if $(k < A[\ell]$ or $k > A[r])$ **return** "not found"
        if $(k = A[r])$ **then return** "found at $A[r]$"
        $m \leftarrow \ell + \lfloor \frac{k - A[\ell])}{A[r] - A[\ell]} \cdot (r - \ell) \rfloor$
        if $(A[m] == k)$ **then return** "found at $A[m]$"
        else if $(A[m] < k)$ **then** $\ell \leftarrow m + 1$
        else $r \leftarrow m - 1$
// We always return from somewhere within while-loop

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	449	450	600	800	1000	1200	1500

interpolation-search(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, found at $A[4]$

Works well if keys are *uniformly* distributed:

- Can show: Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

But: Worst case performance $\Theta(n)$

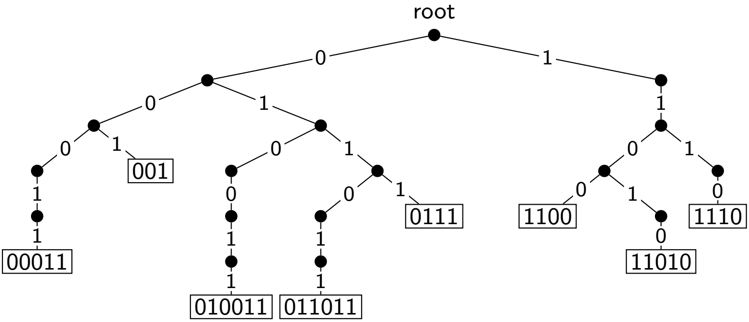
Outline

1. Dictionaries for special keys
   - Lower bound
   - Interpolation Search
   - Tries
     - Standard Tries
     - Variations of Tries
     - Compressed Tries

Tries: Introduction

Trie (also know as **radix tree**): A dictionary for bitstrings.
(Should know: string, word, $|w|$, alphabet, prefix, suffix, comparing words,....)

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key

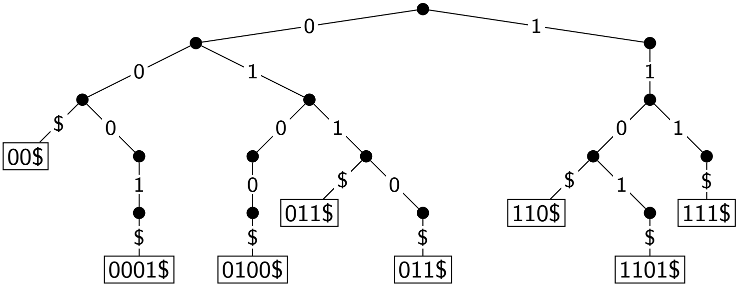


More on tries

Assumption: Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with 'end-of-word' character \$.

Example: A trie for $\{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}$



Then items (keys) are stored *only* in the leaf nodes

Tries: Search

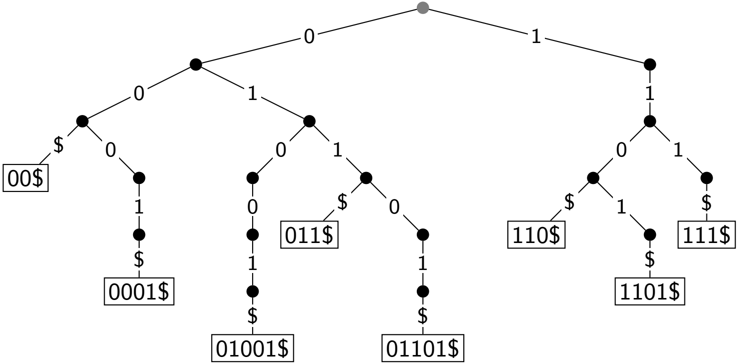
- start from the root and the most significant bit of x
- follow the link that corresponds to the current bit in x; return failure if the link is missing
- return success if we reach a leaf (it must store x)
- else recurse on the new node and the next bit of x

Trie::search$(v \leftarrow \text{root}, d \leftarrow 0, x)$
v: node of trie; d: level of v, x: word stored as array of chars

```
if v is a leaf
    return v
else
    let v' be child of v labelled with x[d]
    if there is no such child
        return "not found"
    else Trie::search(v', d + 1, x)
```

Tries: Search Example

Example: Trie::search(011$)

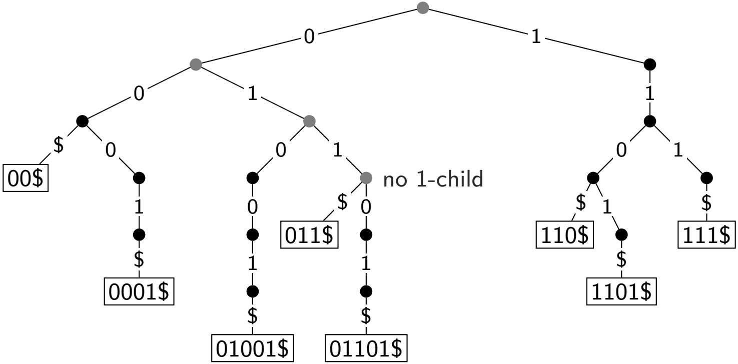


Tries: Insert & Delete

- *Trie::insert(x)*
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child. Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.
- *Trie::delete*(x)
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.
- **Time Complexity** of all operations: $\Theta(|x|)$
  $|x|$: length of binary string x, i.e., the number of bits in x

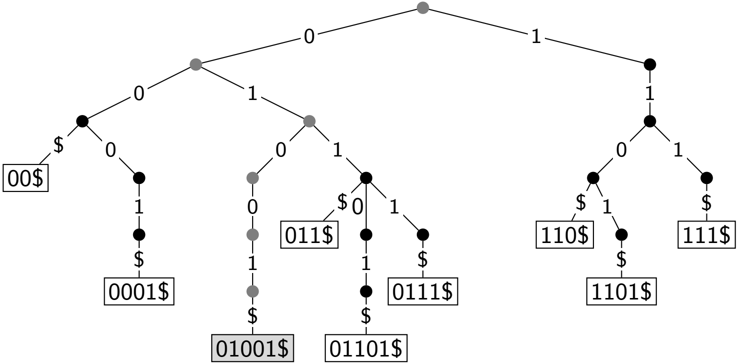
Tries: Insert Example

Example: *Trie::insert*(0111$)



Tries: Delete Example

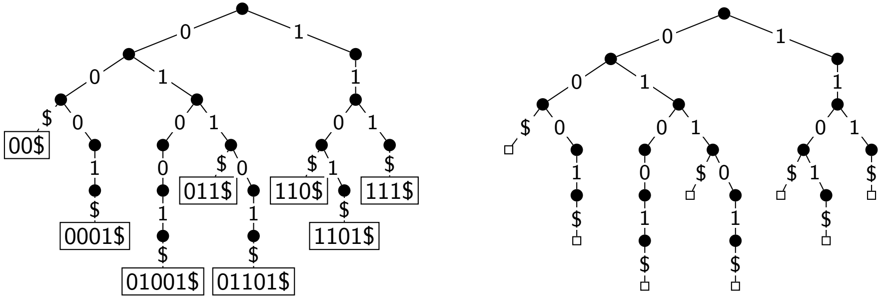
Example: *Trie::delete*(01001$)



Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

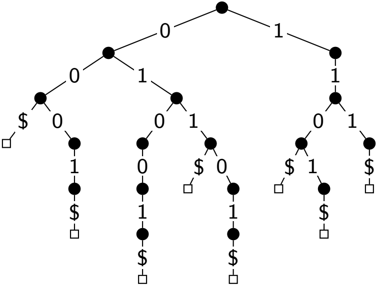
- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.

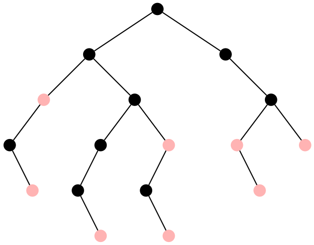


Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys.
  Use a *flag* to indicate such nodes.
- No need for end-of-word character \$
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.

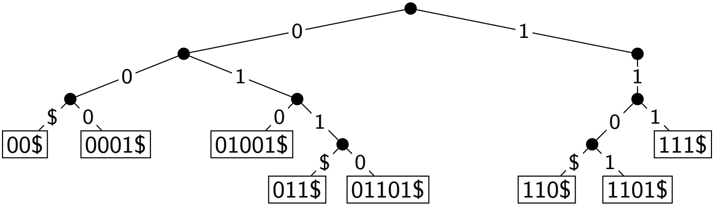




Variations 3 of Tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)

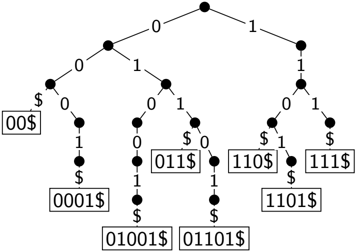


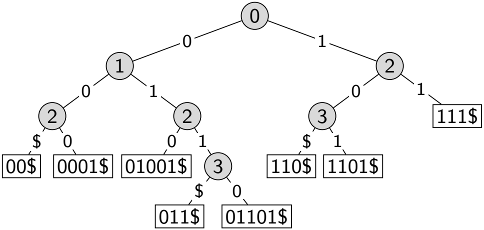
This is in practice the most efficient version of tries, but the operations get a bit more complicated.

Variation 4 of Tries

Compressed Trie: compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with n keys has at most $n-1$ internal nodes





Also known as **Patricia-Tries**:
Practical Algorithm to Retrieve Information Coded in Alphanumeric

Compressed Tries: Search

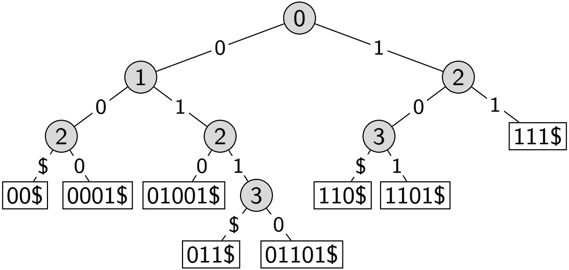
- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in x;
  return failure if the link is missing
- if we reach a leaf, expicitly check whether word stored at leaf is x
- else recurse on the new node and the next bit of x

CompressedTrie::search($v \leftarrow$ root, x)
v: node of trie; x: word

```
if v is a leaf
    return strcmp(x, v.key)
d ← index stored at v
if x has at most d bits
    return "not found"
v′ ← child of v labelled with x[d]
if there is no such child
    return "not found"
CompressedTrie::search(v′, x)
```

Compressed Tries: Search Example

Example: CompressedTrie::search(10$) unsuccessful



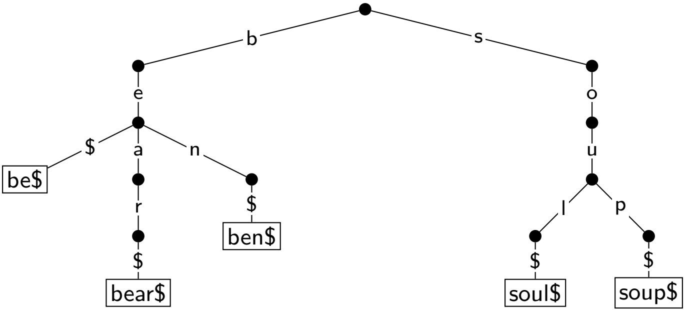
Compressed Tries: Insert & Delete

- *CompressedTrie::delete*(x):
  - Perform *search*(x)
  - Remove the node v that stored x
  - Compress along path to v whenever possible.
- *CompressedTrie::insert*(x):
  - Perform *search*(x)
  - Let v be the node where the search ended.
  - Conceptually simplest approach:
    - ★ Uncompress path from root to v.
    - ★ Insert x as in an uncompressed trie.
    - ★ Compress paths from root to v and from root to x.

    But it can also be done by only adding those nodes that are needed, see the textbook for details.
- All operations take $O(|x|)$ time.

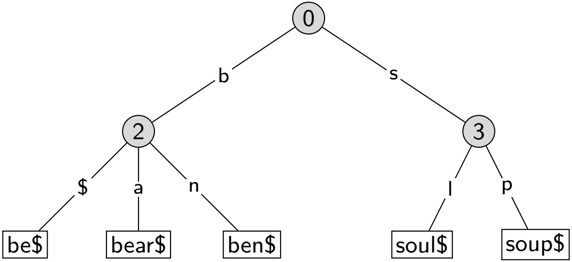
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings $\{$bear\$, ben\$, be\$, soul\$, soup\$$\}$



Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



Multiway Tries: Summary

- Operations *search*(x), *insert*(x) and *delete*(x) are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot \text{(time to find the appropriate child)})$

Each node now has up to $|\Sigma| + 1$ children. How should they be stored?

Solution 1: Array of size $|\Sigma| + 1$ for each node.
Complexity: $O(1)$ time to find child, $O(|\Sigma| n)$ space.

Solution 2: List of children for each node.
Complexity: $O(|\Sigma|)$ time to find child, $O(\#\text{children})$ space.

Solution 3: Dictionary (AVL-tree?) of children for each node.
Complexity: $O(\log(\#\text{children}))$ time, $O(\#\text{children})$ space.
Best in theory, but not worth it in practice unless $|\Sigma|$ is huge.

In practice, use *hashing* (keys are in (typically small) range Σ).