CS 240 – Data Structures and Data Management

Module 7E: Hashing - Enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

- Rehashing
- Multiplication method
- Randomly chosen hash-functions

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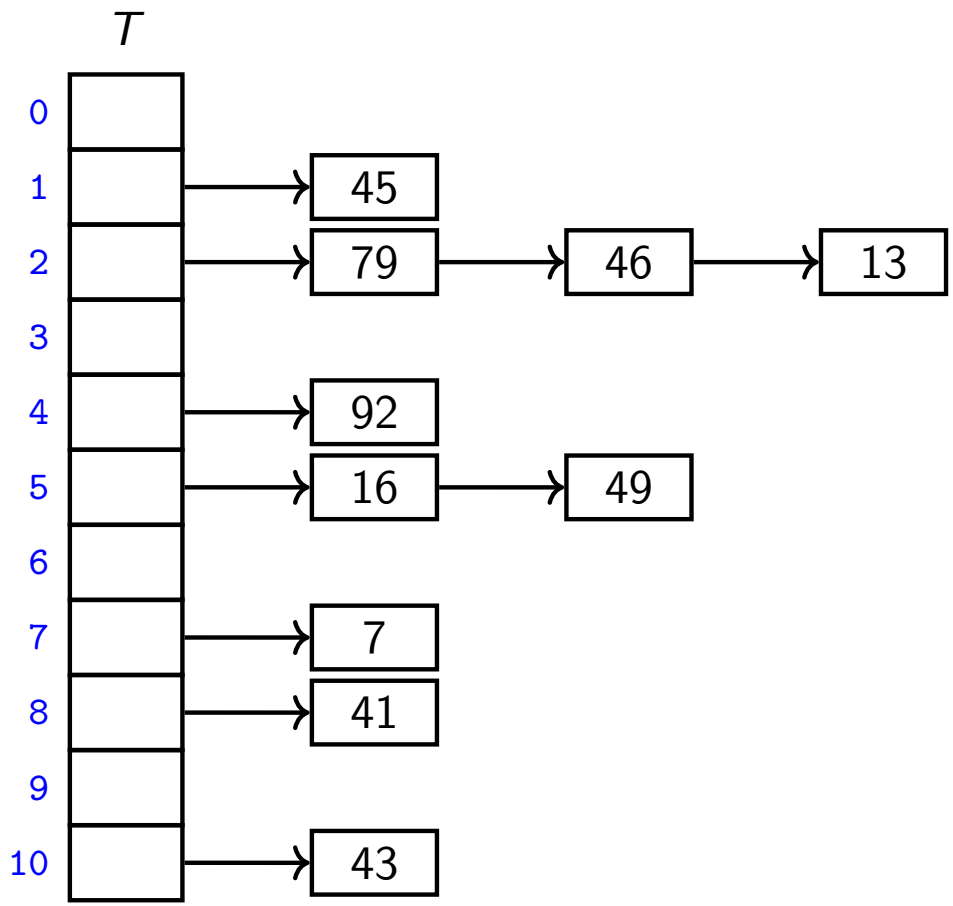
Rehashing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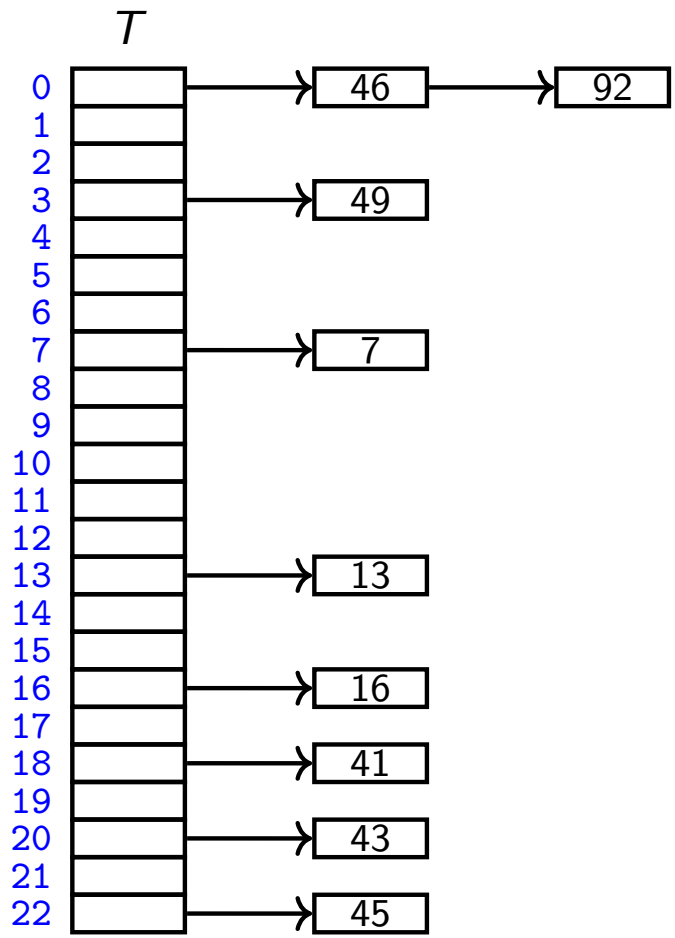
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Multiplication method

- Pick $A \in (0,1)$ (preferably an irrational, e.g. $A = \frac{\sqrt{5}-1}{2} = \phi$)
- $h(k) = \Big\lfloor M \cdot \big(\underbrace{\underbrace{A \cdot k}_{\text{multiply}} - \underbrace{\lfloor A \cdot k \rfloor}_{\text{integral part}}}_{\text{fractional part, in } [0,1)} \big) \Big\rfloor$ (integer in $[0, M)$)
- Example:

$A = 0.a_1 a_2 a_3 \ldots$
$k = b_1 b_2 \ldots b_6$
(both in base 2)

Should use at least $\log |U| + \log |M|$ bits of A.

$A \cdot k$		(leading bits)						(bits of fractional part)					
	$=$	0	0	0	..	0	0						
	$+a_1 \cdot$	0	b_1	b_2	b_3	..	b_5	b_6					
	$+a_2 \cdot$	0	0	b_1	b_2	b_3	..	b_5	b_6				
	$+a_3 \cdot$	0	0	0	b_1	b_2	b_3	..	b_5	b_6			
	$\vdots$				$\ddots$	$\ddots$		$\ddots$	$\ddots$	$\ddots$	$\ddots$		
	$+a_5 \cdot$	0	0	0	0	0	b_1	b_2	b_3	..	b_5	b_6	
	$+a_6 \cdot$	0	0	0	0	0	0	b_1	b_2	b_3	..	b_5	b_6
	$+a_7 \cdot$	0	0	0	0	0	0	0	b_1	b_2	b_3	..	b_5 b_6
	$+a_8 \cdot$	0	0	0	0	0	0	0	0	b_1	b_2	b_3	.. b_5 b_6
	$\vdots$							$\leftarrow h(k) \rightarrow$			$\ddots$	$\ddots$	$\ddots$ $\ddots$ $\ddots$

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Randomly chosen hash-functions

- There are $|U|^M$ many possible hash-functions
- Ideally we would choose randomly among all of them.
- But then we cannot compute hash-value quickly!

- **Idea:** Fix a family $\mathcal{H}$ of hash-functions that are easy to compute. Then choose uniformly among them.
- Example:
  - $U = \mathbb{Z}_5$, $M = 2$
  - $h_b(k) = ((k + b) \bmod 5) \bmod 2$
  - $\mathcal{H} = \{h_b : b \in \mathbb{Z}_5\}$
  - Choose $b \in \mathbb{Z}_5$ randomly to get hash-function

$\mathcal{H}$	keys				
	0	1	2	3	4
h_0	0	1	0	1	0
h_1	1	0	1	0	0
h_2	0	1	0	0	1
h_3	1	0	0	1	0
h_4	0	0	1	0	1

- But how do we measure whether these are "good"?

Universal hash-functions

- For analysis, we needed *uniform hash-values*:,

$$P(h(k) = i) = \frac{1}{M}$$

- But this is *not* good enough.

$\mathcal{H}$	keys				
	0	1	2	4	5
h_0	0	0	0	0	0
h_1	1	1	1	1	1

- $P(h(k) = i) = \frac{1}{2}$ for $i = 0, 1$ and any k
- But these hash-functions are terrible!
- Problem: hash-values not independent

- Also want: Small probability of collisions (**universal hashing**):

$$P\Big(h(k) = h(k')\Big) = \frac{1}{M} \quad \text{for any two keys } k \neq k'$$

- This is enough for analyzing hashing with chaining as before.

Carter-Wegman hash-function

$$h_{a,b}(k) = \Big(\underbrace{a \cdot k + b \bmod p}_{f_{a,b}(k)} \Big) \bmod M$$

(where $k \in \mathbb{Z}_p$, p prime, $a, b \in \mathbb{Z}_p$ chosen randomly, $a \neq 0$), $M < p$

Example: $(p = 5, M = 2)$:

	keys				
	0	1	2	3	4
$f_{1,0}$	0	1	2	3	4
$f_{2,0}$	0	2	4	1	3
$f_{1,2}$	2	3	4	0	1
$f_{2,1}$	1	3	0	2	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

	keys				
	0	1	2	3	4
$h_{1,0}$	0	1	0	1	0
$h_{2,0}$	0	0	0	1	1
$h_{1,2}$	0	1	0	0	1
$h_{2,1}$	1	1	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Claim: $f_{a,b}$ is a permutation of $\mathbb{Z}_p$.