# CS 240 – Data Structures and Data Management

# Module 8: Range-Searching in Dictionaries for Points

T. Biedl E. Kondratovsky M. Petrick O. Veksler
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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# Outline

1. Range-Searching in Dictionaries for Points
   - Range Searches
   - Multi-Dimensional Data
   - Quadtrees
   - kd-Trees
   - Range Trees
   - Conclusion

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   - Range Searches
   - Multi-Dimensional Data
   - Quadtrees
   - kd-Trees
   - Range Trees
   - Conclusion

# Range searches

- So far: *search*($k$) looks for *one* specific item.
- New operation **RangeSearch**: look for *all* items that fall within a given range.
  - Input: A **range**, i.e., an interval $I = (x, x')$
    It may be open or closed at the ends.
  - Want: Report all KVPs in the dictionary whose key $k$ satisfies $k \in I$

**Example:**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 10 | 11 | 17 | 19 | 33 | 45 | 51 | 55 | 59 |

*RangeSearch*( (18,45] ) should return $\{19, 33, 45\}$

- Let $s$ be the **output-size**, i.e., the number of items in the range.
- We need $\Omega(s)$ time simply to report the items.
- Note that sometimes $s = 0$ and sometimes $s = n$; we therefore keep it as a separate parameter when analyzing the run-time.

# Range searches in existing dictionary realizations

**Unsorted list/array/hash table**: Range search requires $\Omega(n)$ time: We have to check for each item explicitly whether it is in the range.

**Sorted array**: Range search in $A$ can be done in $O(\log n + s)$ time:

*RangeSearch*( (18,45] )

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 10 | 11 | 17 | 19 | 33 | 45 | 51 | 55 | 59 |
| | | | $\uparrow i$ | | | $\uparrow i'$ | | | |

- Using binary search, find $i$ such that $x$ is at (or would be at) $A[i]$.
- Using binary search, find $i'$ such that $x'$ is at (or would be at) $A[i']$
- Report all items $A[i+1...i'-1]$
- Report $A[i]$ and $A[i']$ if they are in range

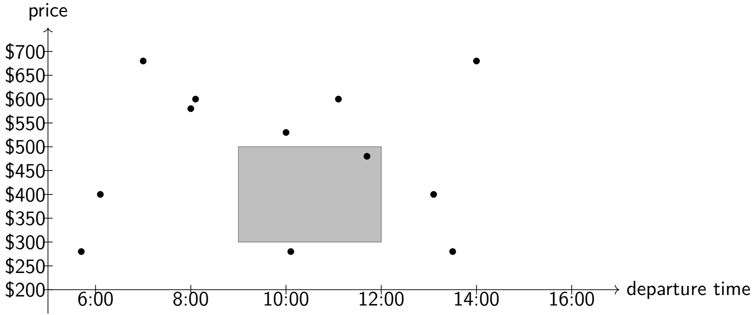
**BST**: Range searches can similarly be done in time $O(\text{height}+s)$ time. We will see this in detail later.

# Outline

# Multi-Dimensional Data

Range searches are of special interest for **multi-dimensional data**.

**Example**: flights that leave between 9am and noon, and cost \$300-\$500




- Each item has $d$ **aspects** (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
- Aspect values ($x_i$) are numbers
- Each item corresponds to a point in $d$-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane

# Multi-dimensional Range Search

(Orthogonal) $d$-**dimensional range search**: Given a **query rectangle** $A$, find all points that lie within $A$.

The time for range searches depends on how the points are stored.

- Could store a 1-dimensional dictionary (where the key is some combination of the aspects.)
  Problem: Range search on one aspect is not straightforward
- Could use one dictionary for each aspect
  Problem: inefficient, wastes space
- **Better idea**: Design new data structures specifically for points.
  - Quadtrees
  - kd-trees
  - range-trees
- **Assumption**: Point are in **general position**:
  No two $x$-coordinates or $y$-coordinates are the same.
  - Simplifies presentation; data structures can be generalized.

# Outline

# Quadtrees

We have $n$ points $S = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.

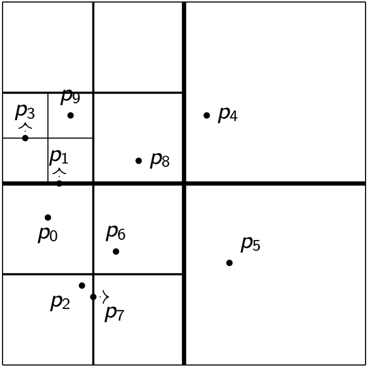
We need a **bounding box** $R$: a square containing all points.

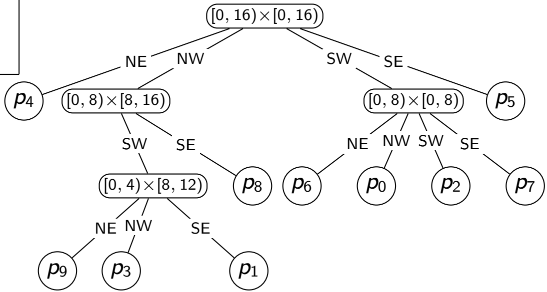
- Can find $R$ by computing minimum and maximum $x$ and $y$ values in $S$
- The width/height of $R$ should be a power of 2

**Structure** (and also how to *build* the quadtree that stores $S$):

- Root $r$ of the quadtree is associated with region $R$
- If $R$ contains 0 or 1 points, then root $r$ is a leaf that stores point.
- Else *split*: Partition $R$ into four equal subsquares (**quadrants**) $R_{NE}, R_{NW}, R_{SW}, R_{SE}$
- Partition $S$ into sets $S_{NE}, S_{NW}, S_{SW}, S_{SE}$ of points in these regions.
  - **Convention:** Points on split lines belong to right/top side
- Recursively build tree $T_i$ for points $S_i$ in region $R_i$ and make them children of the root.

# Quadtrees example

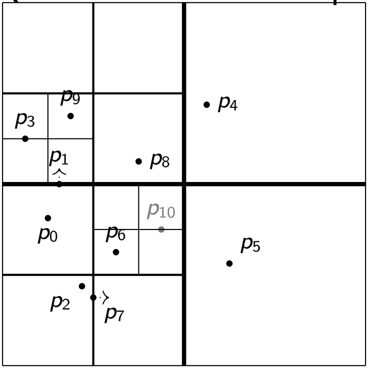


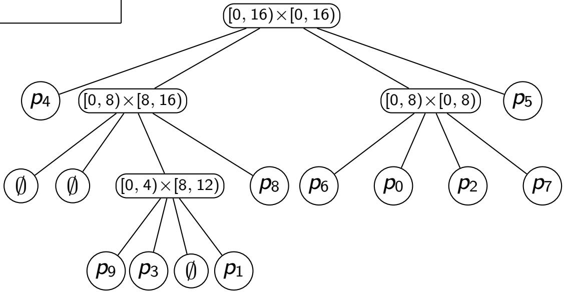
# Quadtree Dictionary Operations

- *search*: Analogous to binary search trees and tries
- *insert*:
  - Search for the point
  - Split the leaf while there are two points in one region
- *delete*:
  - Search for the point
  - Remove the point
  - If its parent has only one point left: delete parent (and recursively all ancestors that have only one point left)

# Quadtree Insert example




*insert*($p_{10}$)

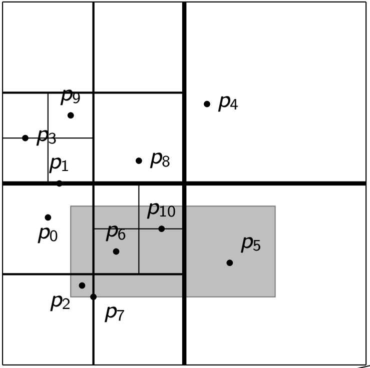


# Quadtree Range Search

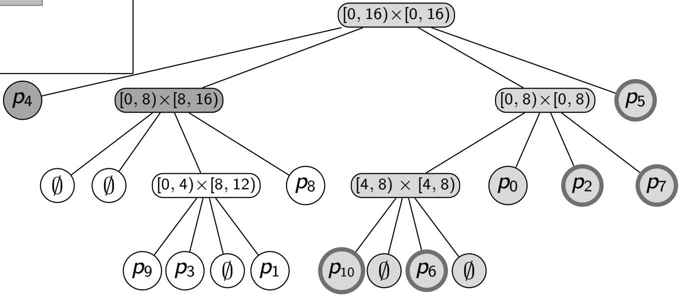
```
QTree::RangeSearch(r ← root, A)
r: The root of a quadtree, A: Query-rectangle
    R ← region associated with node r
    if (R ⊆ A) then                      // inside node
            report all points below r; return
    if (R ∩ A is empty) then             // outside node
            return

            // The node is a boundary node, recurse
    if (r is a leaf) then
        p ← point stored at r
        if p is in A return p
        else return
    for each child v of r do
        QTree::RangeSearch(v, A)
```

Note: We assume here that each node of the quadtree stores the associated square. Alternatively, these could be re-computed during the search (space-time tradeoff).

# Quadtree range search example

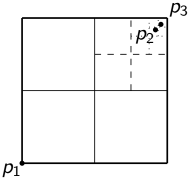



- Red: Search stopped due to $R \cap A = \emptyset$.
- Green: Search stopped due to $R \subseteq A$.
- Blue: Must continue search in children / evaluate.



# Quadtree Analysis

- Crucial for analysis: what is the height of a quadtree?
  - Can have very large height for bad distributions of points




  - **spread factor** of points $S$:

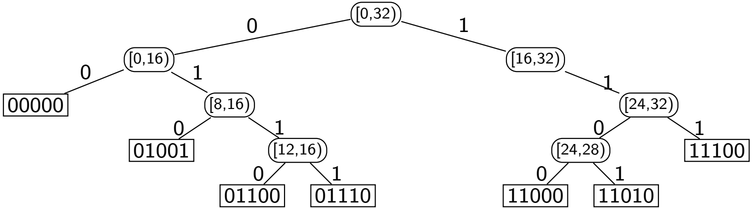
$$\beta(S) = \frac{\text{sidelength of } R}{\text{minimum distance between points in } S}$$

  - Can show: height $h$ of quadtree is in $\Theta(\log \beta(S))$
- Complexity to build initial tree: $\Theta(nh)$ worst-case
- Complexity of range search: $\Theta(nh)$ worst-case even if the answer is $\emptyset$
- But in practice much faster.

# Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| "Points:" | 0 | 9 | 12 | 14 | 24 | 26 | 28 |
| (in base-2) | 00000 | 01001 | 01100 | 01110 | 11000 | 11010 | 11100 |

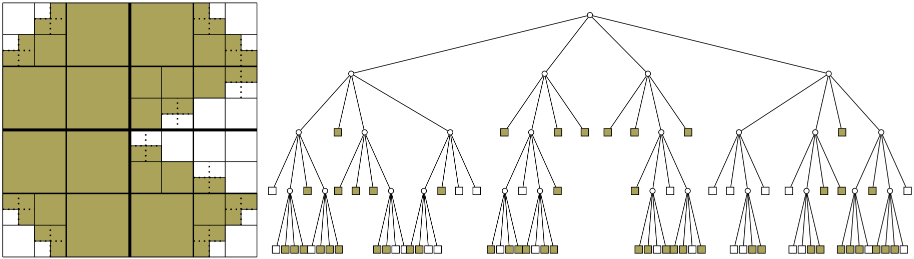



Same as a trie (with splitting stopped once key is unique)

- Quadtrees also easily generalize to higher dimensions (octrees, *etc.* ) but are rarely used beyond dimension 3.

# Quadtree summary

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (bit-shift!) if the width/height of $R$ is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation: We could stop splitting earlier and allow up to $S$ points in a leaf (for some fixed bound $S$).
- Variation: Use quad-tree to store pixelated images.



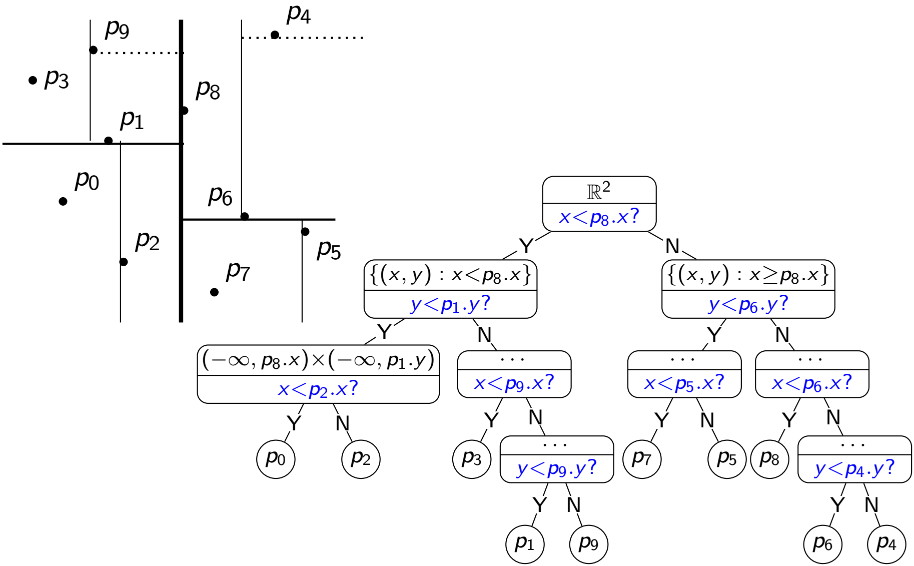
# Outline

# kd-trees

- We have $n$ points $S = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$
- Quadtrees split square into quadrants regardless of where points are
- (Point-based) kd-tree idea: Split the region such that (roughly) half the point are in each subtree
- Each node of the kd-tree keeps track of a **splitting line** in one dimension (2D: either vertical or horizontal)
- **Convention:** Points on split lines belong to right/top side
- Continue splitting, switching between vertical and horizontal lines, until every point is in a separate region

  (There are alternatives, e.g., split by the dimension that has better aspect ratios for the resulting regions. No details.)

# kd-tree example




For ease of drawing, we will usually not show the associated regions.

# Constructing kd-trees

Build kd-tree with initial split by $x$ on points $S$:

- If $|S| \leq 1$ create a leaf and return.
- Else $X :=$ *quick-select*$(S, \lfloor \frac{n}{2} \rfloor)$ (select by $x$-coordinate)
- Partition $S$ by $x$-coordinate into $S_{x<X}$ and $S_{x \geq X}$
  - $\lfloor \frac{n}{2} \rfloor$ points on one side and $\lceil \frac{n}{2} \rceil$ points on the other. (Recall: Points in general position.)
- Create left subtree recursively (splitting by $y$) for points $S_{x<X}$.
- Create right subtree recursively (splitting by $y$) for points $S_{x \geq X}$.

Building with initial $y$-split symmetric.

# Constructing kd-trees

**Run-time:**

- Find $X$ and partition $S$ in $\Theta(n)$ expected time using *randomized-quick-select*.
- Both subtrees have $\approx n/2$ points.

$$T^{\exp}(n) = 2T^{\exp}(n/2) + O(n) \quad \text{(sloppy recurrence)}$$

  This resolves to $\Theta(n \log n)$ expected time.
- This can be reduced to $\Theta(n \log n)$ *worst-case* time by pre-sorting (no details).

**Height:** $h(1) = 0$, $h(n) \leq h(\lceil n/2 \rceil) + 1$.

- This resolves to $O(\log n)$ (specifically $\lceil \log n \rceil$).

# kd-tree Dictionary Operations

- *search* (for single point): as in binary search tree using indicated coordinate
- *insert*: search, insert as new leaf.
- *delete*: search, remove leaf.

**Problem:** After insert or delete, the split might no longer be at exact median and the height is no longer guaranteed to be $\lceil \log_2 n \rceil$.

We can maintain $O(\log n)$ height by occasionally re-building entire subtrees. (No details.) But *rangeSearch* will be slower.

kd-trees do not handle insertion/deletion well.

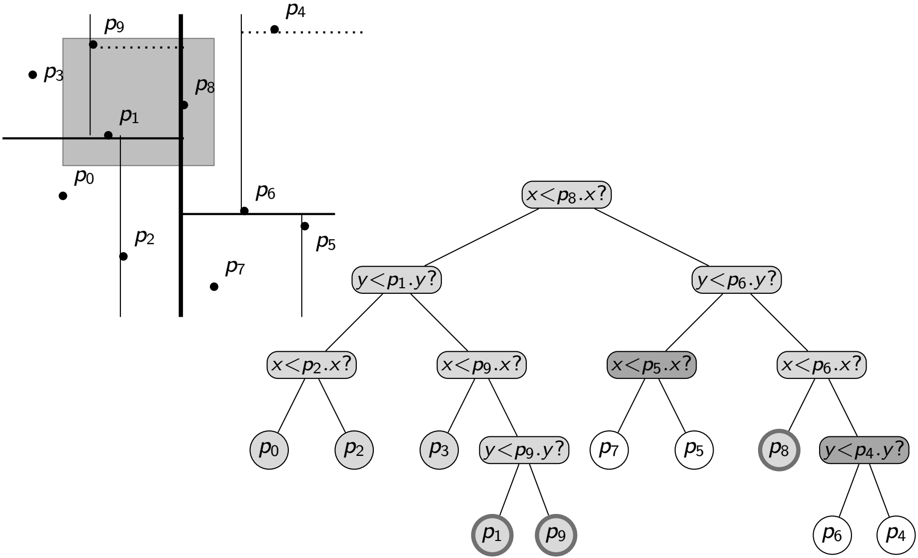
# kd-tree Range Search

- Range search is *exactly* as for quad-trees, except that there are only two children.

```
kdTree::RangeSearch(r ← root, A)
r: The root of a kd-tree, A: Query-rectangle
    R ← region associated with node r
    if (R ⊆ A) then report all points below r; return
    if (R ∩ A is empty) then return
    if (r is a leaf) then
        p ← point stored at r
        if p is in A return p
        else return
    for each child v of r do
        kdTree::RangeSearch(v, A)
```

- We assume again that each node stores its associated region.
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line.

# kd-tree: Range Search Example




Red: Search stopped due to $R \cap A = \emptyset$. Green: Search stopped due to $R \subseteq A$.

# kd-tree: Range Search Complexity

- The complexity is $O(s + Q(n))$ where
  - $s$ is the output-size
  - $Q(n)$ is the number of "boundary" nodes (blue):
    - *kdTree::RangeSearch* was called.
    - Neither $R \subseteq A$ nor $R \cap A = \emptyset$
- **Can show:** $Q(n)$ satisfies the following recurrence relation (no details):

$$Q(n) \leq 2Q(n/4) + O(1)$$

- This solves to $Q(n) \in O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(s + \sqrt{n})$

# kd-tree: Higher Dimensions

- kd-trees for $d$-dimensional space:
  - At the root the point set is partitioned based on the first coordinate
  - At the subtrees of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth $d$ we start all over again, partitioning on first coordinate
- **Storage**: $O(n)$
- **Height**: $O(\log n)$
- **Construction time**: $O(n \log n)$
- **Range search time**: $O(s + n^{1-1/d})$

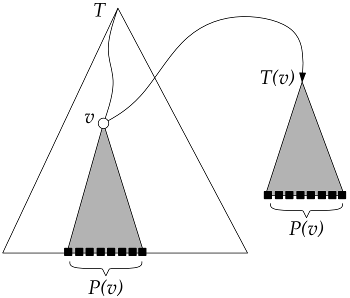
This assumes that $d$ is a constant.

# Outline

# Towards Range Trees

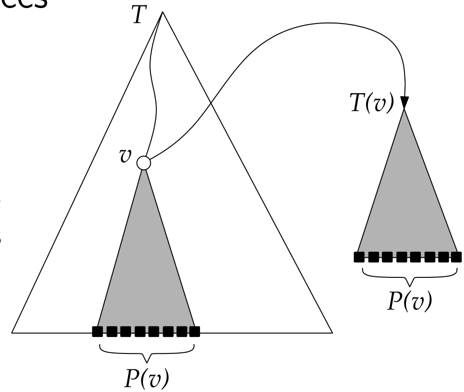
- Both Quadtrees and kd-trees are intuitive and simple.
- But: both may be very slow for range searches.
- Quadtrees are also potentially wasteful in space.

New idea: **Range trees**




- Somewhat wasteful in space, but much faster range search.
- **Tree of trees** (a *multi-level* data structure)

# 2-dimensional Range Trees

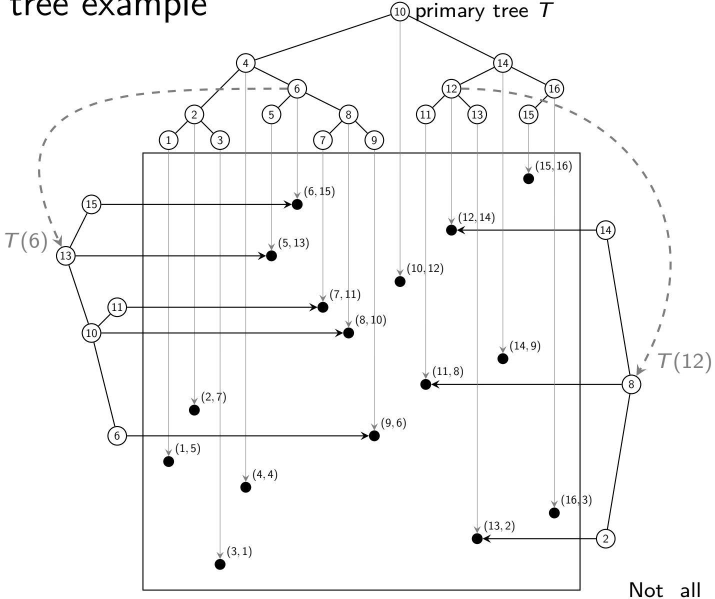



**Primary structure**:
Balanced binary search tree $T$ that stores $P$ and uses *x-coordinates* as keys.

Each node $v$ of $T$ stores an **associate structure** $T(v)$:

- Let $P(v)$ be all points in subtree of $v$ in $T$ (including point at $v$)
- $T(v)$ stores $P(v)$ in a balanced binary search tree, using the *y-coordinates* as key
- Note: $v$ is not necessarily the root of $T(v)$

# Range tree example




Not all associate trees are shown.

# Range Tree Space Analysis

- Primary tree uses $O(n)$ space.
- Associate tree $T(v)$ uses $O(|P(v)|)$ space (where $P(v)$ are the points at descendants of $v$ in $T$)
- **Key insight**: $w \in P(v)$ means that $v$ is an ancestor of $w$ in $T$
  - Every node $w$ has $O(\log n)$ ancestors in $T$ (Recall that we assume $T$ to be balanced.)
  - Every node $w$ belongs to $O(\log n)$ sets $P(v)$
  - So $\sum_v |P(v)| \leq \sum_w \#\{\text{ancestors of } w\} \in O(n \log n)$

**Therefore:** A range-tree with $n$ points uses $O(n \log n)$ space.

# Range Trees Operations

- *search*: search by $x$-coordinate in $T$
- *insert*: First, insert point by $x$-coordinate into $T$.
  Then, walk back up to the root and insert the point by $y$-coordinate in *all* associate trees $T(v)$ of nodes $v$ on path to the root.
- *delete*: analogous to insertion
- **Problem:** We want the binary search trees to be balanced.
  - This makes *insert*/*delete* very slow if we use AVL-trees.
    (A rotation at $v$ changes $P(v)$ and hence requires a re-build of $T(v)$.)
  - **Solution**: Completely rebuild highly unbalanced subtrees (no details)
- *range-search*: search by $x$-range in $T$.
  Among found points, search by $y$-range in some associated trees.
- Must understand first: How to do (1-dimensional) range search in binary search tree?

# BST Range Search recursive

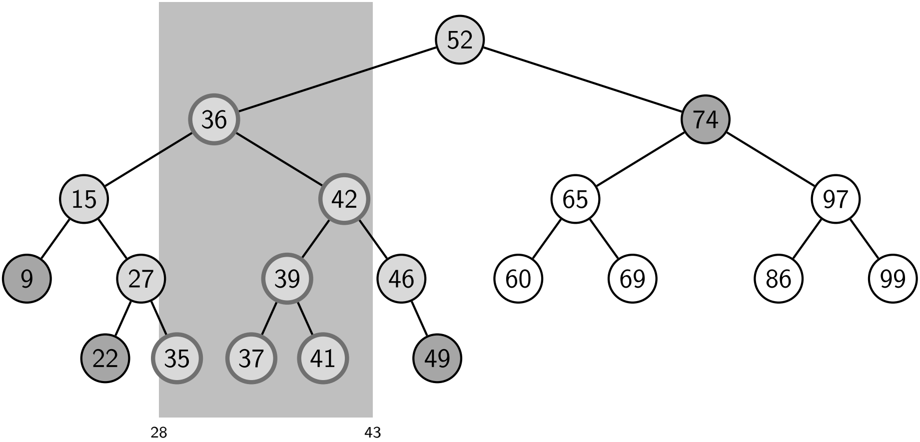
*BST::RangeSearch-recursive*$(r \leftarrow root, x_1, x_2)$
$r$: root of a binary search tree, $x_1, x_2$: search keys
Returns keys in subtree at $r$ that are in range $[x_1, x_2]$

1. **if** $r = \texttt{NIL}$ **then return**
2. **if** $x_1 \leq r.key \leq x_2$ **then**
3. $\quad L \leftarrow$ *BST::RangeSearch-recursive*$(r.left, x_1, x_2)$
4. $\quad R \leftarrow$ *BST::RangeSearch-recursive*$(r.right, x_1, x_2)$
5. $\quad$ **return** $L \cup r.\{key\} \cup R$
6. **if** $r.key < x_1$ **then**
7. $\quad$ **return** *BST::RangeSearch-recursive*$(r.right, x_1, x_2)$
8. **if** $r.key > x_2$ **then**
9. $\quad$ **return** *BST::RangeSearch-recursive*$(r.left, x_1, x_2)$

Keys are reported in in-order, i. e., in sorted order.

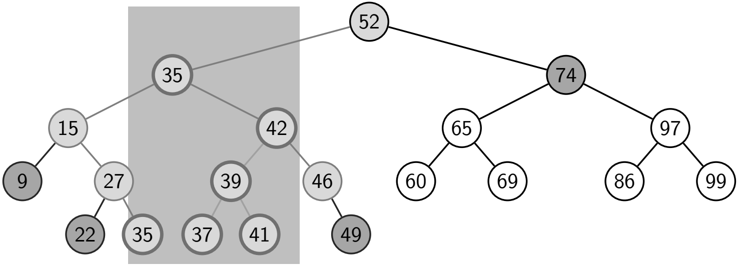
# BST Range Search example

*BST::RangeSearch-recursive*$(T, 28, 43)$



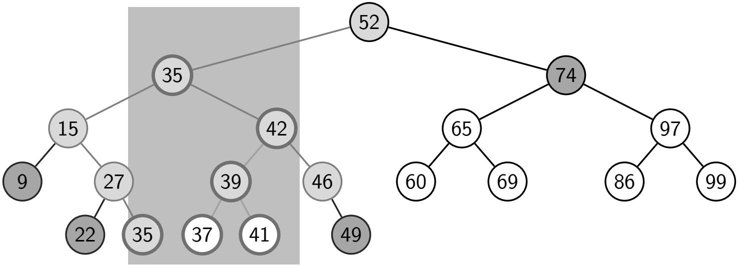

Note: Search from 39 was unnecessary: *all* its descendants are in range.

# BST Range Search re-phrased




- Search for left boundary $x_1$: this gives path $P_1$
- Search for right boundary $x_2$: this gives path $P_2$
- This partitions $T$ into three groups: outside, on, or between the paths.
- This classification will be crucial later!

# BST Range Search re-phrased

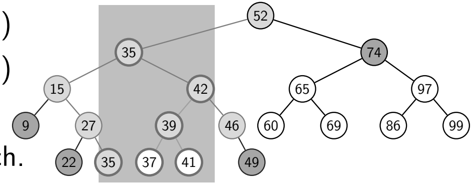



- boundary nodes: nodes in $P_1$ or $P_2$
  - For each boundary node, test whether it is in the range.
- outside nodes: nodes that are left of $P_1$ or right of $P_2$
  - These are *not* in the range, we do not visit them.
- inside nodes: nodes that are right of $P_1$ and left of $P_2$
  - We keep a list of the topmost inside nodes.
  - All descendants of such a node are *in* the range.
    For a 1d range search, report them.

# BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path $P_1$: $O(\log n)$
- Search for path $P_2$: $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.




- We spend $O(1)$ time per topmost inside node $v$.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- For 1d range search, also report the descendants of $v$.
  - We have $\sum_{v \text{ topmost inside}} \#\{\text{descendants of } v\} \leq s$ since subtrees of topmost inside nodes are disjoint. So this takes time $O(s)$ overall.

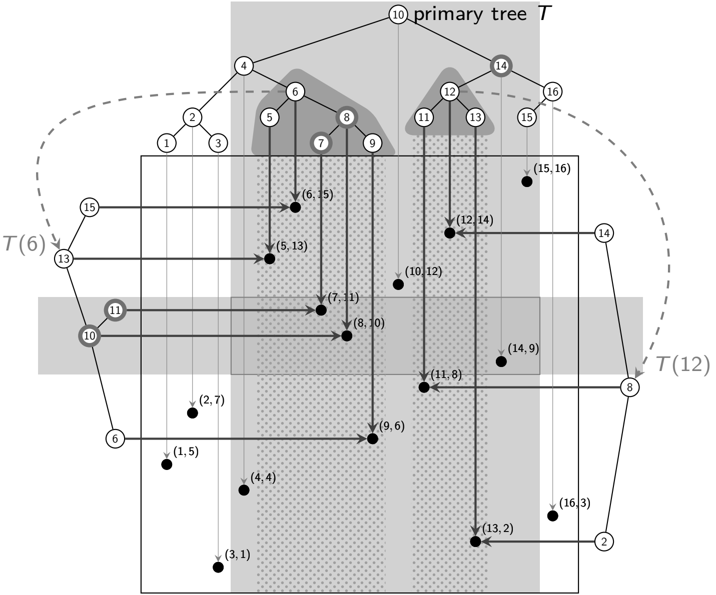
Run-time for 1d range search: $O(\log n + s)$. This is no faster overall, but topmost inside nodes will be important for 2d range search.

# Range Trees: Range Search

Range search for $A = [x_1, x_2] \times [y_1, y_2]$ is a two stage process:

- Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in primary tree $T$ (*BST::RangeSearch*$(T, x_1, x_2)$)
- Get boundary and topmost inside nodes as before.
- For every boundary node, test to see if the corresponding point is within the region $A$.
- For every topmost inside node $v$:
  - Let $P(v)$ be the points in the subtree of $v$ in $T$.
  - We know that all $x$-coordinates of points in $P(v)$ are within range.
  - Recall: $P(v)$ is stored in $T(v)$.
  - To find points in $P(v)$ where the $y$-cordinates are within range as well, perform a range search in $T(v)$: *BST::RangeSearch*$(T(v), y_1, y_2)$

# Range tree range search example



# Range Trees: Range Search Run-time

- $O(\log n)$ time to find boundary and topmost inside nodes in primary tree.
- There are $O(\log n)$ such nodes.
- $O(\log n + s_v)$ time for each topmost inside node $v$, where $s_v$ is the number of points in $T(v)$ that are reported
- Two topmost inside nodes have no common point in their trees
  $\Rightarrow$ every point is reported in at most one associate structure
  $\Rightarrow \sum_{v \text{ topmost inside}} s_v \leq s$

Time for range search in range-tree is proportional to

$$\sum_{v \text{ topmost inside}} (\log n + s_v) \in O(\log^2 n + s)$$

(There are ways to make this even faster. No details.)

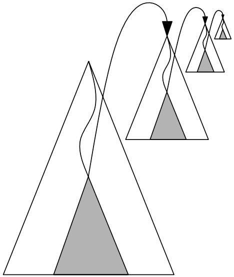
# Range Trees: Higher Dimensions

- Range trees can be generalized to $d$-dimensional space.

| | | |
|---|---|---|
| **Space** | $O(n\,(\log n)^{d-1})$ | **kd-trees**: $O(n)$ |
| **Construction time** | $O(n\,(\log n)^{d})$ | **kd-trees**: $O(n \log n)$ |
| **Range search time** | $O(s + (\log n)^{d})$ | **kd-trees**: $O(s + n^{1-1/d})$ |

(Note: $d$ is considered to be a constant.)

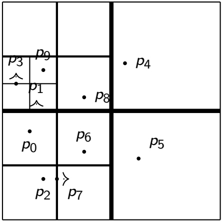
- Space/time trade-off compared to kd-trees.



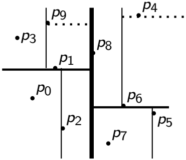
# Outline

# Range search data structures summary

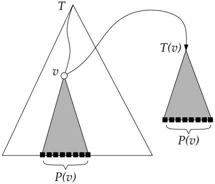
- Quadtrees
  - simple (also for dynamic set of points)
  - work well only if points evenly distributed
  - wastes space for higher dimensions




- kd-trees
  - linear space
  - range search time $O(\sqrt{n} + s)$
  - inserts/deletes destroy balance and range search time (no simple fix)




- range-trees
  - range search time $O(\log^2 n + s)$
  - wastes some space
  - inserts/deletes destroy balance (can fix this with occasional rebuilt)




**Convention:** Points on split lines belong to right/top side.