CS 240 – Data Structures and Data Management

Module 9e: String Matching - Enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

1. Pattern Matching - details
   - KMP failure function – fast computation
   - KMP failure function – improvement
   - Good suffix array

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KMP failure function – fast computation

$F[j]$ is the length of the longest prefix of P that is a suffix of $P[1..j]$.

- How can we compute this faster?
- Recall property of KMP-automaton of P:
  - If we are in state ℓ, then we have just seen $P[0..\ell-1]$

  $\Leftrightarrow$ $P[0..\ell-1]$ is a suffix of what we have just parsed.

  - Also, KMP is always in the rightmost state where this holds.

  $\Leftrightarrow$ $P[0..\ell-1]$ is the *longest* suffix of what we have just parsed.

  $\Leftrightarrow$ ℓ is the length of the longest prefix of P
  that is a suffix of what we have just parsed.

Combine this with the definition of $F[j]$ to get:

$F[j] = \ell \Leftrightarrow$
we reach state ℓ when parsing $P[1..j]$ on the KMP-automaton for P

KMP failure function – fast computation

$F[j]$ = **the state we reach when parsing** $P[1..j]$

This immediately gives algorithm: For $j = 1, 2, \ldots,$

- parse $P[1..j]$ on the KMP-automaton for P
- Set $F[j] = \ell$ if we reach state ℓ

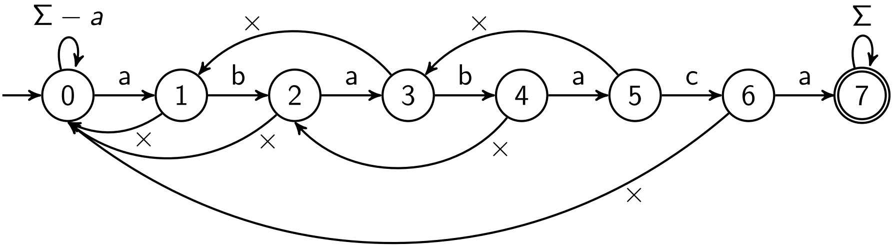
Observe: We don't need to re-start the parsing from scratch!

- Assume we have computed $F[j]$ already.
- To compute $F[j+1]$, parse $P[j+1]$ and note reached state.
- So can compute $F[0..m-1]$ with *one* parse of $P[1..m-1]$

But isn't this circular?

- We need failure-arcs for parsing, but we compute them only now!
- But: To compute $F[j]$, parse $P[1..j-1]$ first ($j-1$ characters)
  $\Rightarrow$ reach state $\leq j$
  $\Rightarrow$ don't need $F[j]$ (= arc from state $j+1$) to parse $P[j]$

KMP failure function – fast computation



Parse $P[1..m{-}1]$ = `babaca` while adding failure-arcs:

j	1	1	1	2	2	3	3	4	4	5	5	5	5	5	5	6	6
$P[i]$		b		a		b		a		c		c		c		a	
$P[j]$		a		a		b		a		b		b		a		a	
ℓ	0	$\xrightarrow{b}$	0	$\xrightarrow{a}$	1	$\xrightarrow{b}$	2	$\xrightarrow{a}$	3	$\xrightarrow{\times}$	1	$\xrightarrow{\times}$	0	$\xrightarrow{c}$	0	$\xrightarrow{a}$	1
$F[j]$			0		1		2		3						0		1

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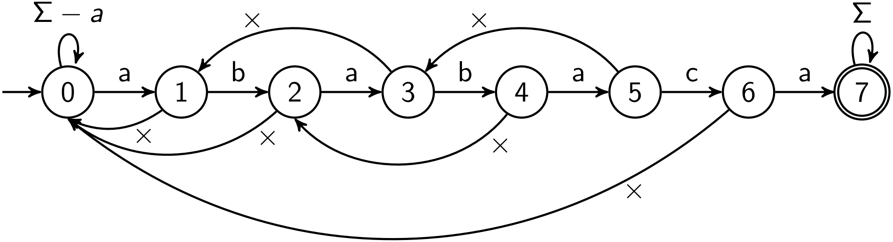
KMP failure function – improvement

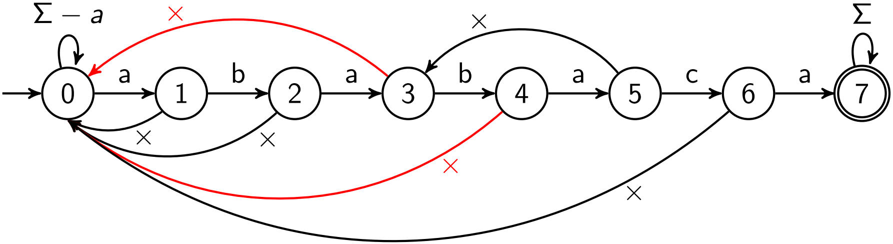
We can define an even better failure-function:



Consider failure-arc from state 4:

- This will be used if $T[i] \neq a = P[4]$ and leads to state 2.
- The next check will again compare $T[i]$ to $a = P[2]$.
- This *must* fail, and the failure-arc will lead to state 0.
- We might as well have gone to state 0 directly.

KMP failure function – improvement



$$F^+[j] = \begin{cases} \text{length } \ell \text{ of the longest prefix of } P \text{ that is a suffix of } P[1..j] \\ \qquad\qquad\qquad\qquad \textbf{and where } \mathbf{P}[\ell] \neq \mathbf{P}[\mathbf{j}+\mathbf{1}]. \\ 0 \text{ if no such } \ell \text{ exists} \end{cases}$$

Easy to compute: $F^+[j] = \begin{cases} F[j] & \text{if } P[j+1] \neq P[F[j]] \text{ or } F[j]=0 \\ F^+[F[j]-1] & \text{otherwise} \end{cases}$

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Good suffix array - example

$P =$onobobo

n	o	o	o	b	o	o	o	o	o	b	n	b	b	o	b	o	…	…
			b	**o**	**b**	**o**												
				(o)	(b)	(o)	**b**	**o**										
								(o)		**o**								
										[b]	**o**							
											[n]	**(o)**	**b**	**o**	**b**	**o**		

- Do smallest shift so that **obo** fits in the new guess
- Do smallest shift so that matched suffix fits in the new guess
- No suffix matched $\rightsquigarrow$ shift over by one (or by last-char heuristic)
- What to do if the matched part does not repeat?

Good suffix array - if matched part doesn't repeat

$P =$nbonnnbo (different from before)

n	b	b	n	n	n	b	o	o	n	n	b	o	b	b	o	b	o	… .
		o	n	n	n	b	o											
					(n)	(b)	(o)	**n**	n	n	b	o						
										(n)	(b)	(o)			**n**	b	o	

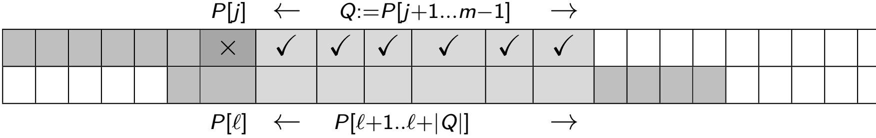
- Cannot match all of **nnnbo**
- But **nbo** fits a prefix of $P \rightsquigarrow$ shift to that guess
- Generally: Re-use longest suffix of matched part that fits a prefix of P
- If nothing fits: Shift guess all the way past previous guess.

$P =$nobnnnbo (different from before)

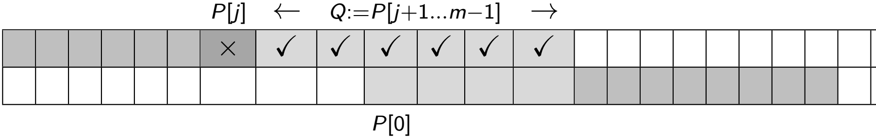
n	b	b	n	n	n	b	o	o	n	n	b	o	b	b	o	b	o	… …
		o	n	n	n	b	o											

Definition of good suffix array

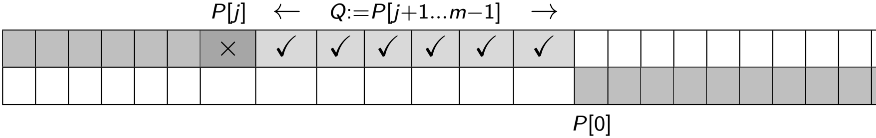
- Assume search failed at $P[j]$, but had matched $P[j+1..m-1] =: Q$
- Case 1: Q appears as substring of P elsewhere



- Case 2: A suffix of Q is a prefix of P.

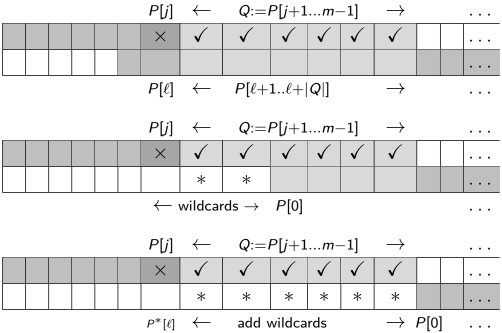


- Case 3: Neither (i.e., only empty suffix fits).



Definition of good suffix array

- Can unify all three cases into one!
- Let P^* be P with m *wildcards* attached in front.



In all cases:

- Q is a substring of P^*
- Align old $P[j]$ with new $P[\ell]$ (then $S[j] \leftarrow \ell$ fits update)
- So Q is prefix of $P^*[\ell+1...m-1]$
- Want $\ell \neq j$ so that we actually shift

$$S[j] = \max_{\ell \neq j} \quad P[j+1..m-1] \textbf{ is a prefix of } P^*[\ell+1..m-1]$$

Good Suffix Array Computation - human

$$
\begin{aligned}
S[j] &= \max_{\ell \neq j} \quad P[j{+}1..m{-}1] \text{ is a prefix of } P^*[\ell{+}1..m{-}1] \\
&= \max_{\ell} \quad P[j{+}1..m{-}1] \text{ is a prefix of } P^*[\ell{+}1..m{-}2]
\end{aligned}
$$

$P =$ boobobo		$P^*[-m..m-2]$														
		-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5		
j	$P[j{+}1..m{-}1]$	*	*	*	*	*	*	*	b	o	o	b	o	b	$\ell+1$	$S[j]$
5	o												o		4	3
4	bo											b	o		3	2
3	obo										o	b	o		2	1
2	bobo						b	o	b	o					-2	-3
1	obobo					o	b	o	b	o					-3	-4
0	oobobo				o	o	b	o	b	o					-4	-5

Easy to compute in polynomial time:

- Write down P^*, omitting rightmost character.
- For each j, write down $P[j{+}1..m{-}1]$
- Find rightmost match $\rightsquigarrow$ gives $\ell + 1$ $\rightsquigarrow$ gives $S[j]$

Good Suffix Array Computation - computer

Idea: After reformulations, this resembles the KMP failure function!

$$
\begin{aligned}
S[j] &= \max_{\ell \neq j}\{\ P[j{+}1..m{-}1] \text{ is a prefix of } P^*[\ell{+}1..m{-}1]\ \} \\
&= \max_{\ell}\{\ P[j{+}1..m{-}1] \text{ is a prefix of } P^*[\ell{+}1..m{-}2]\ \} \\
&= \max_{\ell}\{\ (P[j{+}1..m{-}1])^{\text{reverse}} \text{ is a suffix of } (P^*[\ell{+}1..m{-}2])^{\text{reverse}}\ \}
\end{aligned}
$$

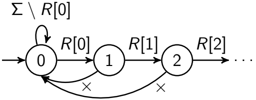
$\big($define R to be reverse of P^*: $R[j] = P^*[m{-}1{-}j]\big)$

$$
\begin{aligned}
&= \max_{\ell}\{\ R[0..m{-}j{-}2]\underbrace{R[0..m{-}j{-}2]}_{\text{prefix of } R} \text{ is a suffix of } R[1..m{-}\ell{-}2]R[1..\,m \\
&= m - 2 - \min_{k}\{\ \text{the } (m{-}j{-}1)^{\text{st}} \text{ prefix of } R \text{ is a suffix of } R[1..k]\ \}
\end{aligned}
$$

This should remind you of properties of a KMP-automaton.

Good Suffix Array Computation - computer

Recall KMP-automaton for R:



We reach state q if the q^{th} prefix of R was a suffix of what was parsed.

$$S[j] \;=\; m-2-\min_{k}\{ \text{ the } (m-j-1)^{\text{st}} \text{ prefix of } R \text{ is a suffix of } R[1..k] \;\}$$

$$\;=\; m-2-\min_{k}\{ \text{ state } m-j-1\underbrace{m-j-1}_{q} \text{ is reached when } \underbrace{\text{parsing}}_{\text{on KMP-automaton for } R}\; R[$$

$$S[{\scriptstyle m-q-1}] \;=\; m-2-\min_{k}\{ \text{ state } q \text{ reached when parsing } R[1..k] \;\}$$

Good Suffix Array Computation - computer

Final result:

$$S[m-q-1] \;=\; m-2-\min_k \left\{ \begin{array}{l} \text{parsing } R[1..k] \text{ on the KMP-automaton for} \\ R = P^{\text{reverse}} {*}{*}{*}\ldots{*} \text{ brings us to state } q \end{array} \right\}$$

So to compute $S[\cdot]$:

- Create KMP-automaton $\mathcal{K}_R$ for $R := P^{\text{reverse}} {*}{*}{*}\ldots{*}$
- Parse $R[1..2m-1]$ on $\mathcal{K}_R$
- Whenever we reach a state q
  - Check whether q was visited already
  - If not: $S[m-q-1] \leftarrow m-2-k$, where $R[k]$ is last parsed character.

Run-time: $O(m)$ since R has $O(m)$ characters.