

# CS 240 – Data Structures and Data Management

## Module 1: Introduction and Asymptotic Analysis

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Based on lecture notes by many previous cs240 instructors

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# Outline

## 1 Introduction and Asymptotic Analysis

- CS240 Overview
- Algorithm Design
- Analysis of Algorithms I
- Asymptotic Notation
- Analysis of Algorithms II
- Example: Analysis of MergeSort
- Helpful Formulas

# Outline

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## Course Objectives: What is this course about?

- Much of Computer Science is *problem solving*: Write a program that converts the given input to the expected output.
- When first learning to program, we emphasize *correctness*: does your program output the expected results?
- Starting with this course, we will also be concerned with *efficiency*: is your program using the computer's resources (typically processor time) efficiently?
- We will study efficient methods of *storing*, *accessing*, and *organizing* large collections of data.

**Motivating examples:** Digital Music Collection, English Dictionary

Typical operations include: *inserting* new data items, *deleting* data items, *searching* for specific data items, *sorting*.

## Course Objectives: What is this course about?

- We will consider various **abstract data types** (ADTs) and how to realize them efficiently using appropriate **data structures**.
- There is a strong emphasis on mathematical analysis in the course.
- Algorithms are presented using pseudo-code and analyzed using order notation (big-Oh, etc.).

# Course Topics

- big-Oh analysis
- priority queues and heaps
- sorting, selection
- binary search trees, AVL trees
- skip lists
- hashing
- quadtrees, kd-trees
- range search
- tries
- string matching
- data compression

# CS Background

Topics covered in previous courses with relevant sections in [Sedgewick]:

- arrays, linked lists (Sec. 3.2–3.4)
- strings (Sec. 3.6)
- stacks, queues (Sec. 4.2–4.6)
- abstract data types (Sec. 4-intro, 4.1, 4.8–4.9)
- recursive algorithms (5.1)
- binary trees (5.4–5.7)
- sorting (6.1–6.4)
- binary search (12.4)
- binary search trees (12.5)
- probability and expectations (Goodrich & Tamassia, Section 1.3.4)

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# Problems (terminology)

First, we must introduce terminology so that we can precisely characterize what we mean by efficiency.

**Problem:** Given a problem instance, carry out a particular computational task.

**Problem Instance:** *Input* for the specified problem.

**Problem Solution:** *Output* (correct answer) for the specified problem instance.

**Size of a problem instance:**  $Size(I)$  is a positive integer which is a measure of the size of the instance  $I$ .

**Example:** Sorting problem

# Algorithms and Programs

**Algorithm:** An algorithm is a *step-by-step process* (e.g., described in pseudo-code) for carrying out a series of computations, given an arbitrary problem instance  $I$ .

**Solving a problem:** An Algorithm  $A$  *solves* a problem  $\Pi$  if, for every instance  $I$  of  $\Pi$ ,  $A$  finds (computes) a valid solution for the instance  $I$  in finite time.

**Program:** A program is an *implementation* of an algorithm using a specified computer language.

In this course, our emphasis is on algorithms (as opposed to programs or programming).

# Algorithms and Programs

**Pseudocode:** a method of communicating an algorithm to another person.

In contrast, a program is a method of communicating an algorithm to a computer.

## Pseudocode

- omits obvious details, e.g. variable declarations,
- has limited if any error detection,
- sometimes uses English descriptions,
- sometimes uses mathematical notation.

# Algorithms and Programs

For a problem  $\Pi$ , we can have several algorithms.

For an algorithm  $\mathcal{A}$  solving  $\Pi$ , we can have several programs (implementations).

Algorithms in practice: Given a problem  $\Pi$

- 1 Design an algorithm  $\mathcal{A}$  that solves  $\Pi$ . → **Algorithm Design**
- 2 Assess *correctness* and *efficiency* of  $\mathcal{A}$ . → **Algorithm Analysis**
- 3 If acceptable (correct and efficient), implement  $\mathcal{A}$ .

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# Efficiency of Algorithms/Programs

- How do we decide which algorithm or program is the most efficient solution to a given problem?
- In this course, we are primarily concerned with the *amount of time* a program takes to run. → **Running Time**
- We also may be interested in the *amount of additional memory* the program requires. → **Auxiliary space**
- The amount of time and/or memory required by a program will depend on *Size(I)*, the size of the given problem instance *I*.

# Running Time of Algorithms/Programs

First option: *experimental studies*

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition.
- Use a method like `clock()` (from `time.h`) to get an accurate measure of the actual running time.
- Plot/compare the results.

# Running Time of Algorithms/Programs

## Shortcomings of experimental studies

- Implementation may be complicated/costly.
- Timings are affected by many factors: *hardware* (processor, memory), *software environment* (OS, compiler, programming language), and *human factors* (programmer).
- We cannot test all inputs; what are good *sample inputs*?
- We cannot easily compare two algorithms/programs.

## We want a framework that:

- Does not require implementing the algorithm.
- Is independent of the hardware/software environment.
- Takes into account all input instances.

We need some *simplifications*.



# Overview of Algorithm Analysis

We will develop several aspects of algorithm analysis in the next slides.  
To overcome dependency on hardware/software:

- Algorithms are presented in structured high-level *pseudo-code* which is language-independent.
- Analysis of algorithms is based on an *idealized computer model*.
- Instead of time, count the number of *primitive operations*
- The efficiency of an algorithm (with respect to time) is measured in terms of its *growth rate* (this is called the *complexity* of the algorithm).

# Random Access Machine

## Random Access Machine (RAM) model:

- A set of memory cells, each of which stores one item (word) of data. Implicit assumption: memory cells are big enough to hold the items that we store.
- Any *access to a memory location* takes constant time.
- Any *primitive operation* takes constant time. Implicit assumption: primitive operations have fairly similar, though different, running time on different systems
- The *running time* of a program is proportional to the number of memory accesses plus the number of primitive operations.

This is an idealized model, so these assumptions may not be valid for a “real” computer.

# Running Time Simplifications

We will simplify our analysis by considering the behaviour of algorithms for large inputs sizes.

- **Example 1:** What is larger,  $100n$  or  $10n^2$ ?
- **Example 2:** What is larger,  $1000000n + 2000000000000000$  or  $0.01n^2$ ?
- To simplify comparisons, use **order notation**
- Informally: ignore constants and lower order terms

# Outline

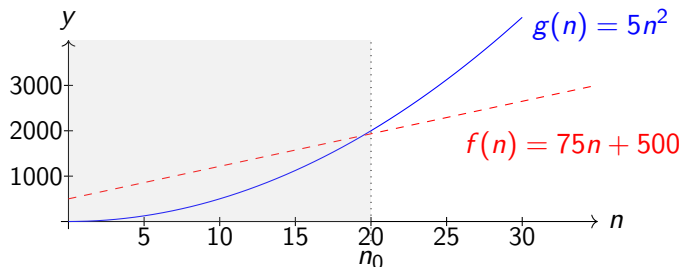
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# Order Notation

**O-notation:**  $f(n) \in O(g(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that  $|f(n)| \leq c |g(n)|$  for all  $n \geq n_0$ .

Example:  $f(n) = 75n + 500$  and  $g(n) = 5n^2$  (e.g.  $c = 1, n_0 = 20$ )



**Note:** The absolute value signs in the definition are irrelevant for analysis of run-time or space, but are useful in other applications of asymptotic notation.

## Example of Order Notation

In order to prove that  $2n^2 + 3n + 11 \in O(n^2)$  from first principles, we need to find  $c$  and  $n_0$  such that the following condition is satisfied:

$$0 \leq 2n^2 + 3n + 11 \leq c n^2 \text{ for all } n \geq n_0.$$

note that not all choices of  $c$  and  $n_0$  will work.

## Asymptotic Lower Bound

- We have  $2n^2 + 3n + 11 \in O(n^2)$ .
- But we also have  $2n^2 + 3n + 11 \in O(n^{10})$ .
- We want a *tight* asymptotic bound.

**$\Omega$ -notation:**  $f(n) \in \Omega(g(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that  $c|g(n)| \leq |f(n)|$  for all  $n \geq n_0$ .

**$\Theta$ -notation:**  $f(n) \in \Theta(g(n))$  if there exist constants  $c_1, c_2 > 0$  and  $n_0 \geq 0$  such that  $c_1|g(n)| \leq |f(n)| \leq c_2|g(n)|$  for all  $n \geq n_0$ .

$$f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))$$

## Example of Order Notation

Prove that  $f(n) = 2n^2 + 3n + 11 \in \Omega(n^2)$  from first principles.

Prove that  $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$  from first principles.

Prove that  $\log_b(n) \in \Theta(\log n)$  for all  $b > 1$  from first principles.



## Strictly smaller/larger asymptotic bounds

- We have  $f(n) = 2n^2 + 3n + 11 \in \Theta(n^2)$ .
- How to express that  $f(n)$  is *asymptotically strictly smaller* than  $n^3$ ?

***o*-notation:**  $f(n) \in o(g(n))$  if for all constants  $c > 0$ , there exists a constant  $n_0 \geq 0$  such that  $|f(n)| \leq c |g(n)|$  for all  $n \geq n_0$ .

***ω*-notation:**  $f(n) \in \omega(g(n))$  if  $g(n) \in o(f(n))$ .

- Main difference to  $O, \Omega$  is the quantifier for  $c$ .
- Rarely proved from first principles.

# Algebra of Order Notations

**Identity rule:**  $f(n) \in \Theta(f(n))$  **Transitivity:**

- If  $f(n) \in O(g(n))$  and  $g(n) \in O(h(n))$  then  $f(n) \in O(h(n))$ .
- If  $f(n) \in \Omega(g(n))$  and  $g(n) \in \Omega(h(n))$  then  $f(n) \in \Omega(h(n))$ .

**Maximum rules:** Suppose that  $f(n) > 0$  and  $g(n) > 0$  for all  $n \geq n_0$ .  
Then:

- $O(f(n) + g(n)) = O(\max\{f(n), g(n)\})$
- $\Omega(f(n) + g(n)) = \Omega(\max\{f(n), g(n)\})$

## Techniques for Order Notation

Suppose that  $f(n) > 0$  and  $g(n) > 0$  for all  $n \geq n_0$ . Suppose that

$$L = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \quad (\text{in particular, the limit exists}).$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \omega(g(n)) & \text{if } L = \infty. \end{cases}$$

The required limit can often be computed using *l'Hôpital's rule*. Note that this result gives *sufficient* (but not necessary) conditions for the stated conclusions to hold.

## Example 1

Let  $f(n)$  be a polynomial of degree  $d \geq 0$ :

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \dots + c_1 n + c_0$$

for some  $c_d > 0$ .

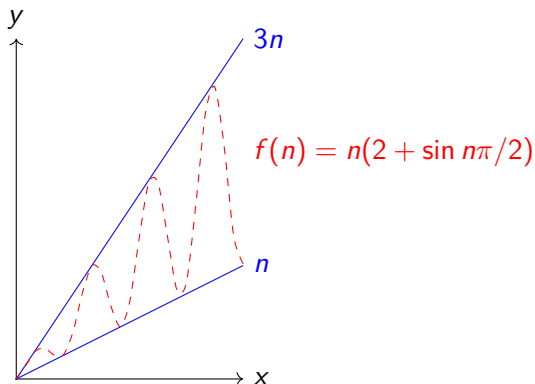
Then  $f(n) \in \Theta(n^d)$ :

## Example 2

Prove that  $n(2 + \sin n\pi/2)$  is  $\Theta(n)$ . Note that  $\lim_{n \rightarrow \infty} (2 + \sin n\pi/2)$  does not exist.

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## Example 3

Compare the growth rates of  $\log n$  and  $n$ .

Now compare the growth rates of  $(\log n)^c$  and  $n^d$  (where  $c > 0$  and  $d > 0$  are arbitrary numbers).

# Growth rates

- If  $f(n) \in \Theta(g(n))$ , then the *growth rates* of  $f(n)$  and  $g(n)$  are the *same*.
- If  $f(n) \in o(g(n))$ , then we say that the growth rate of  $f(n)$  is *less than* the growth rate of  $g(n)$ .
- If  $f(n) \in \omega(g(n))$ , then we say that the growth rate of  $f(n)$  is *greater than* the growth rate of  $g(n)$ .
- Typically,  $f(n)$  may be “complicated” and  $g(n)$  is chosen to be a very simple function.



# Common Growth Rates

Commonly encountered growth rates in analysis of algorithms include the following (in increasing order of growth rate):

- $\Theta(1)$  (*constant complexity*),
- $\Theta(\log n)$  (*logarithmic complexity*),
- $\Theta(n)$  (*linear complexity*),
- $\Theta(n \log n)$  (*linearithmic*),
- $\Theta(n \log^k n)$ , for some constant  $k$  (*quasi-linear*),
- $\Theta(n^2)$  (*quadratic complexity*),
- $\Theta(n^3)$  (*cubic complexity*),
- $\Theta(2^n)$  (*exponential complexity*).

# How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e.,  $n \rightarrow 2n$ ).

- constant complexity:  $T(n) = c$
- logarithmic complexity:  $T(n) = c \log n$
- linear complexity:  $T(n) = cn$
- linearithmic  $\Theta(n \log n)$ :  $T(n) = cn \log n$
- quadratic complexity:  $T(n) = cn^2$
- cubic complexity:  $T(n) = cn^3$
- exponential complexity:  $T(n) = c2^n$

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- constant complexity:  $T(n) = c \quad \rightsquigarrow T(2n) = c.$
- logarithmic complexity:  $T(n) = c \log n \quad \rightsquigarrow T(2n) = T(n) + c.$
- linear complexity:  $T(n) = cn$
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- linear complexity:  $T(n) = cn$   $\rightsquigarrow T(2n) = 2T(n).$
- linearithmic  $\Theta(n \log n)$ :  $T(n) = cn \log n$   $\rightsquigarrow T(2n) = 2T(n) + 2cn.$
- quadratic complexity:  $T(n) = cn^2$
- cubic complexity:  $T(n) = cn^3$
- exponential complexity:  $T(n) = c2^n$

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- quadratic complexity:  $T(n) = cn^2$   $\rightsquigarrow T(2n) = 4T(n).$
- cubic complexity:  $T(n) = cn^3$
- exponential complexity:  $T(n) = c2^n$

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- quadratic complexity:  $T(n) = cn^2$   $\rightsquigarrow T(2n) = 4T(n).$
- cubic complexity:  $T(n) = cn^3$   $\rightsquigarrow T(2n) = 8T(n).$
- exponential complexity:  $T(n) = c2^n$



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- cubic complexity:  $T(n) = cn^3$   $\rightsquigarrow T(2n) = 8T(n).$
- exponential complexity:  $T(n) = c2^n$   $\rightsquigarrow T(2n) = (T(n))^2/c.$

# Relationships between Order Notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$
  
- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

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## Techniques for Run-time Analysis

- Goal: Use asymptotic notation to simplify run-time analysis.
- Running time of an algorithm depends on the *input size*  $n$ .

*Test1*( $n$ )

```
1.   $sum \leftarrow 0$ 
2.  for  $i \leftarrow 1$  to  $n$  do
3.      for  $j \leftarrow i$  to  $n$  do
4.           $sum \leftarrow sum + (i - j)^2$ 
5.  return  $sum$ 
```

- Identify *primitive operations* that require  $\Theta(1)$  time.
- The complexity of a loop is expressed as the *sum* of the complexities of each iteration of the loop.
- Nested loops: start with the innermost loop and proceed outwards. This gives *nested summations*.

## Techniques for Run-time Analysis

Two general strategies are as follows.

**Strategy I:** Use  $\Theta$ -bounds *throughout the analysis* and obtain a  $\Theta$ -bound for the complexity of the algorithm.

**Strategy II:** Prove a  $O$ -bound and a *matching*  $\Omega$ -bound *separately*. Use upper bounds (for  $O$ -bounds) and lower bounds (for  $\Omega$ -bound) early and frequently.

This may be easier because upper/lower bounds are easier to sum.

```
Test2(A, n)
1.   max ← 0
2.   for i ← 1 to n do
3.       for j ← i to n do
4.           sum ← 0
5.           for k ← i to j do
6.               sum ← A[k]
7.   return max
```

# Complexity of Algorithms

- Algorithm can have different running times on two instances of the same size.

```
Test3(A, n)
A: array of size n
1.   for i ← 1 to n - 1 do
2.       j ← i
3.       while j > 0 and A[j] < A[j - 1] do
4.           swap A[j] and A[j - 1]
5.           j ← j - 1
```

Let  $T_{\mathcal{A}}(I)$  denote the running time of an algorithm  $\mathcal{A}$  on instance  $I$ .

**Worst-case complexity** of an algorithm: take the worst  $I$

**Average-case complexity** of an algorithm: average over  $I$

# Complexity of Algorithms

**Worst-case complexity of an algorithm:** The worst-case running time of an algorithm  $\mathcal{A}$  is a function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$  mapping  $n$  (the input size) to the *longest* running time for any input instance of size  $n$ :

$$T_{\mathcal{A}}(n) = \max\{T_{\mathcal{A}}(I) : \text{Size}(I) = n\}.$$

**Average-case complexity of an algorithm:** The average-case running time of an algorithm  $\mathcal{A}$  is a function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$  mapping  $n$  (the input size) to the *average* running time of  $\mathcal{A}$  over all instances of size  $n$ :

$$T_{\mathcal{A}}^{\text{avg}}(n) = \frac{1}{|\{I : \text{Size}(I) = n\}|} \sum_{\{I : \text{Size}(I) = n\}} T_{\mathcal{A}}(I).$$

# O-notation and Complexity of Algorithms

- It is important not to try and make *comparisons* between algorithms using O-notation.
- For example, suppose algorithm  $\mathcal{A}_1$  and  $\mathcal{A}_2$  both solve the same problem,  $\mathcal{A}_1$  has worst-case run-time  $O(n^3)$  and  $\mathcal{A}_2$  has worst-case run-time  $O(n^2)$ .
- Observe that we *cannot* conclude that  $\mathcal{A}_2$  is more efficient than  $\mathcal{A}_1$  for all input!
  - 1 The worst-case run-time may only be achieved on some instances.
  - 2 O-notation is an upper bound.  $\mathcal{A}_1$  may well have worst-case run-time  $O(n)$ . If we want to be able to compare algorithms, we should always use  $\Theta$ -notation.



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# Design Idea for MergeSort

**Input:** Array  $A$  of  $n$  integers

- **Step 1:** We split  $A$  into two subarrays:  $A_L$  consists of the first  $\lceil \frac{n}{2} \rceil$  elements in  $A$  and  $A_R$  consists of the last  $\lfloor \frac{n}{2} \rfloor$  elements in  $A$ .
- **Step 2:** *Recursively* run *MergeSort* on  $A_L$  and  $A_R$ .
- **Step 3:** After  $A_L$  and  $A_R$  have been sorted, use a function *Merge* to merge them into a single sorted array.

# MergeSort

```
MergeSort( $A, n, \ell \leftarrow 0, r \leftarrow n - 1, S \leftarrow \text{NIL}$ )
A: array of size  $n, 0 \leq \ell \leq r \leq n - 1$ 
1.   if  $S$  is NIL initialize it as array  $S[0..n - 1]$ 
2.   if ( $r \leq \ell$ ) then
3.       return
4.   else
5.        $m = (r + \ell)/2$ 
6.       MergeSort( $A, n, \ell, m, S$ )
7.       MergeSort( $A, n, m + 1, r, S$ )
8.       Merge( $A, \ell, m, r, S$ )
```

Two tricks to reduce run-time and auxiliary space:

- The recursion uses parameters that indicate the range of the array that needs to be sorted.
- The array used for copying is passed along as parameter.

# Merge

*Merge*( $A, \ell, m, r, S$ )

$A[0..n-1]$  is an array,  $A[\ell..m]$  is sorted,  $A[m+1..r]$  is sorted  
 $S[0..n-1]$  is an array

1. copy  $A[\ell..r]$  into  $S[\ell..r]$
2.  $\text{int } i_L \leftarrow \ell; \text{int } i_R \leftarrow m + 1;$
3. **for** ( $k \leftarrow \ell; k \leq r; k++$ ) **do**
4.     **if** ( $i_L > m$ )  $A[k] \leftarrow S[i_R++]$
5.     **else if** ( $i_R > r$ )  $A[k] \leftarrow S[i_L++]$
6.     **else if** ( $S[i_L] \leq S[i_R]$ )  $A[k] \leftarrow S[i_L++]$
7.     **else**  $A[k] \leftarrow S[i_R++]$

*Merge* takes time  $\Theta(r - \ell + 1)$ , i.e.,  $\Theta(n)$  time for merging  $n$  elements.

## Analysis of MergeSort

Let  $T(n)$  denote the time to run *MergeSort* on an array of length  $n$ .

- Step 1 takes time  $\Theta(n)$
- Step 2 takes time  $T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$
- Step 3 takes time  $\Theta(n)$

The **recurrence relation** for  $T(n)$  is as follows:

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + \Theta(n) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1. \end{cases}$$

It suffices to consider the following *exact recurrence*, with constant factor  $c$  replacing  $\Theta$ 's:

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

# Analysis of MergeSort

- The following is the corresponding **sloppy recurrence** (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2 T(\frac{n}{2}) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- The exact and sloppy recurrences are *identical* when  $n$  is a power of 2.
- The recurrence can easily be solved by various methods when  $n = 2^j$ . The solution has growth rate  $T(n) \in \Theta(n \log n)$ .
- It is possible to show that  $T(n) \in \Theta(n \log n)$  *for all  $n$*  by analyzing the exact recurrence.

## Some Recurrence Relations

Recursion	resolves to	example
$T(n) = T(n/2) + \Theta(1)$	$T(n) \in \Theta(\log n)$	Binary search
$T(n) = 2T(n/2) + \Theta(n)$	$T(n) \in \Theta(n \log n)$	Mergesort
$T(n) = 2T(n/2) + \Theta(\log n)$	$T(n) \in \Theta(n)$	Heapify ( $\rightarrow$ later)
$T(n) = T(cn) + \Theta(n)$ for some $0 < c < 1$	$T(n) \in \Theta(n)$	Selection ( $\rightarrow$ later)
$T(n) = 2T(n/4) + \Theta(1)$	$T(n) \in \Theta(\sqrt{n})$	Range Search ( $\rightarrow$ later)
$T(n) = T(\sqrt{n}) + \Theta(1)$	$T(n) \in \Theta(\log \log n)$	Interpolation Search ( $\rightarrow$ later)

- Once you know the result, it is (usually) easy to prove by induction.
- Many more recursions, and some methods to find the result, in cs341.

# Outline

## 1 Introduction and Asymptotic Analysis

- CS240 Overview
- Algorithm Design
- Analysis of Algorithms I
- Asymptotic Notation
- Analysis of Algorithms II
- Example: Analysis of MergeSort
- **Helpful Formulas**



# Order Notation Summary

**$O$ -notation:**  $f(n) \in O(g(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that  $|f(n)| \leq c |g(n)|$  for all  $n \geq n_0$ .

**$\Omega$ -notation:**  $f(n) \in \Omega(g(n))$  if there exist constants  $c > 0$  and  $n_0 \geq 0$  such that  $c |g(n)| \leq |f(n)|$  for all  $n \geq n_0$ .

**$\Theta$ -notation:**  $f(n) \in \Theta(g(n))$  if there exist constants  $c_1, c_2 > 0$  and  $n_0 \geq 0$  such that  $c_1 |g(n)| \leq |f(n)| \leq c_2 |g(n)|$  for all  $n \geq n_0$ .

**$o$ -notation:**  $f(n) \in o(g(n))$  if for all constants  $c > 0$ , there exists a constant  $n_0 \geq 0$  such that  $|f(n)| \leq c |g(n)|$  for all  $n \geq n_0$ .

**$\omega$ -notation:**  $f(n) \in \omega(g(n))$  if for all constants  $c > 0$ , there exists a constant  $n_0 \geq 0$  such that  $c |g(n)| \leq |f(n)|$  for all  $n \geq n_0$ .

# Useful Sums

## Arithmetic sequence:

$$\sum_{i=0}^{n-1} i = ??? \qquad \sum_{i=0}^{n-1} (a + di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2) \quad \text{if } d \neq 0.$$

## Geometric sequence:

$$\sum_{i=0}^{n-1} 2^i = ??? \qquad \sum_{i=0}^{n-1} ar^i = \begin{cases} a \frac{r^n - 1}{r - 1} \in \Theta(r^{n-1}) & \text{if } r > 1 \\ na \in \Theta(n) & \text{if } r = 1 \\ a \frac{1 - r^n}{1 - r} \in \Theta(1) & \text{if } 0 < r < 1. \end{cases}$$

## Harmonic sequence:

$$\sum_{i=1}^n \frac{1}{i} = ??? \qquad H_n := \sum_{i=1}^n \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$$

## A few more:

$$\sum_{i=1}^n \frac{1}{i^2} = ??? \qquad \sum_{i=1}^n \frac{1}{i^2} = \frac{\pi^2}{6} \in \Theta(1)$$

$$\sum_{i=1}^n i^k = ??? \qquad \sum_{i=1}^n i^k \in \Theta(n^{k+1}) \quad \text{for } k \geq 0$$

# Useful Math Facts

## Logarithms:

- $c = \log_b(a)$  means  $b^c = a$ . E.g.  $n = 2^{\log n}$ .
- $\log(a)$  (in this course) means  $\log_2(a)$
- $\log(a \cdot c) = \log(a) + \log(c)$ ,  $\log(a^c) = c \log(a)$
- $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$ ,  $a^{\log_b c} = c^{\log_b a}$
- $\ln(x) = \text{natural log} = \log_e(x)$ ,  $\frac{d}{dx} \ln x = \frac{1}{x}$
- concavity:  $\alpha \log x + (1-\alpha) \log y \leq \log(\alpha x + (1-\alpha)y)$  for  $0 \leq \alpha \leq 1$

## Factorial:

- $n! := n(n-1)(n-2) \cdots 2 \cdot 1 = \#$  ways to permute  $n$  elements
- $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$

## Probability and moments:

- $E[aX] = aE[X]$ ,  $E[X + Y] = E[X] + E[Y]$  (linearity of expectation)