

CS 240 – Data Structures and Data Management

Module 2E: Priority Queues - Enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

- 1 Merging heaps
 - More PQ operations
 - Meldable Heaps
 - Binomial Heaps

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Merging Priority Queues

New operation: *merge*(P_1, P_2)

- Given: two priority queues P_1, P_2 of size n_1 and n_2 .
- Want: One priority queue P that contains all their items

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Can we do it *faster* if PQs are stored as trees?

Three approaches (where $n = n_1 + n_2$):

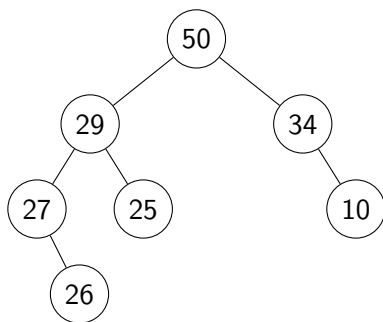
- Merge binary heaps (stored as trees).
 $O(\log^3 n)$ worst-case time (no details)
- Merge *meldable heaps* that have heap-property (but not structural property). $O(\log n)$ *expected* run-time.
- Merge *binomial heaps* that have a different structural property.
 $O(\log n)$ *worst-case* run-time.

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Meldable Heaps

- Priority queue stored as binary tree
- Heap-order-property: Parent no smaller than child.
- No structural property; any binary tree allowed.
- *Tree-based*: Store nodes and references to *left/right*



PQ-operations in Meldable Heaps

Both *insert* and *deleteMax* can be done by *reduction* to *merge*.

P.insert(k, v):

- Create a 1-node meldable heap P' that stores (k, v) .
- Merge P' with P .

P.deleteMax():

- Stash item that is at root.
- Let P_ℓ and P_r be left and right sub-heap of root.
- Update $P \leftarrow \text{merge}(P_\ell, P_r)$
- Return stashed item.

Both operations have run-time $O(\text{merge})$.

Merging Meldable Heaps

- Idea: Merge heap with smaller root into other one, *randomly* choose into which sub-heap to merge.
- Structural property not maintained

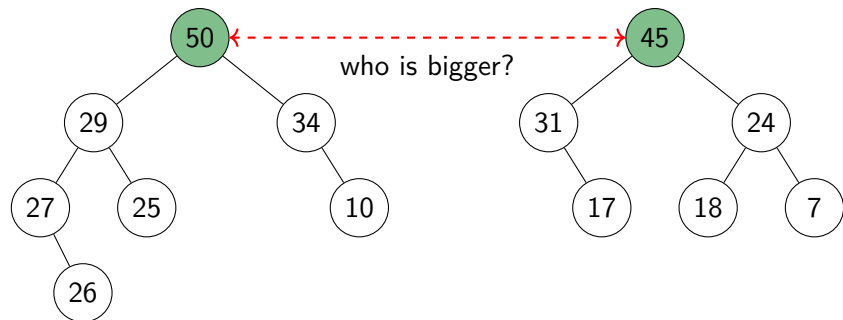
```
meldableHeap::merge( $r_1, r_2$ )
```

r_1, r_2 : roots of two heaps (possibly NIL)

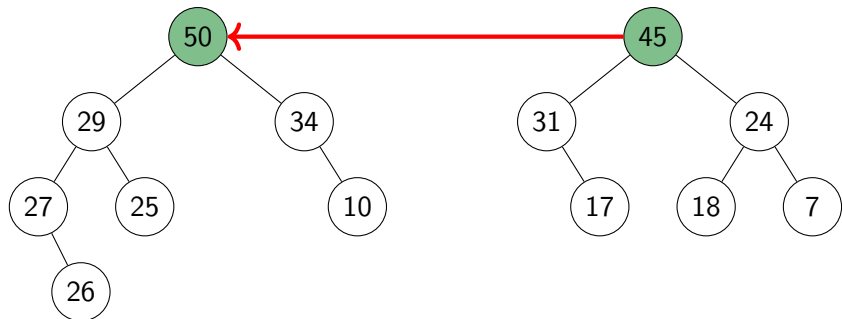
returns root of merged heap

1. **if** r_1 is NIL **return** r_2
2. **if** r_2 is NIL **return** r_1
3. **if** $r_1.key < r_2.key$ *swap*(r_1, r_2)
4. // now r_1 has max-key and becomes the root.
5. *randomly* pick one child c of r_1
6. replace subheap at c by *heapMerge*(c, r_2)
7. **return** r_1

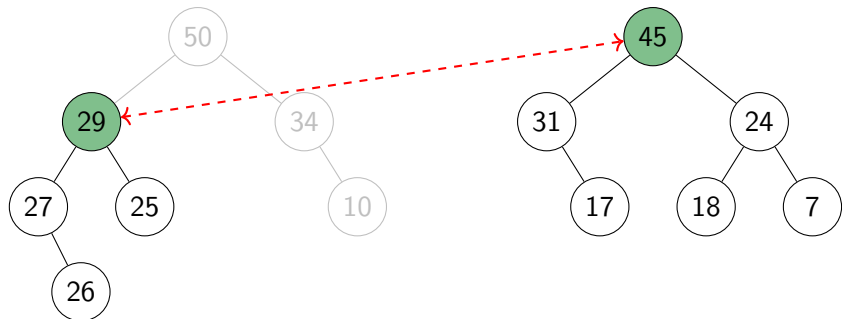
Merge Example



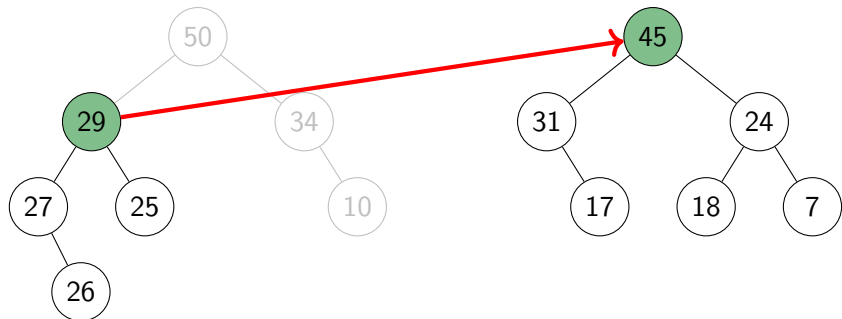
Merge Example



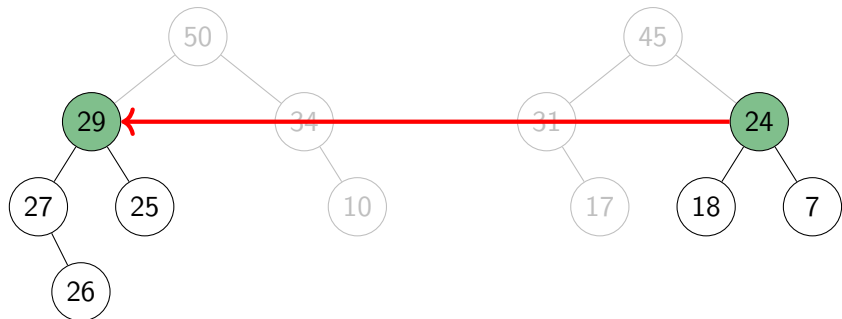
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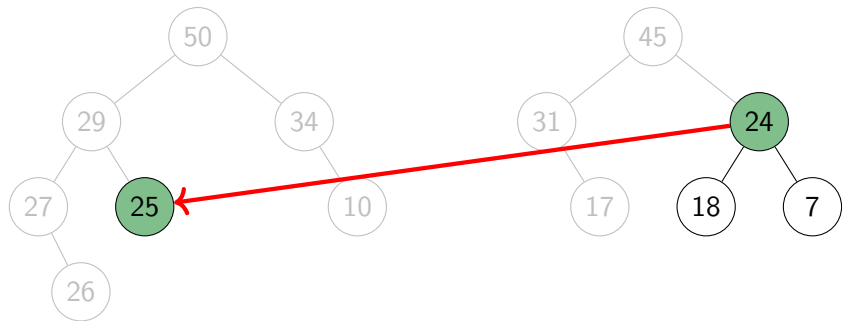
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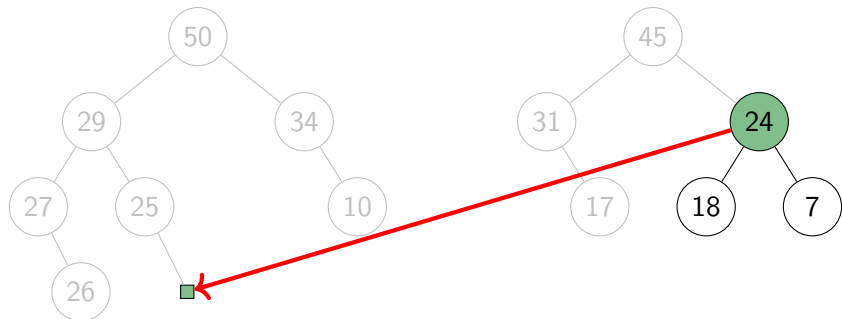
Merge Example



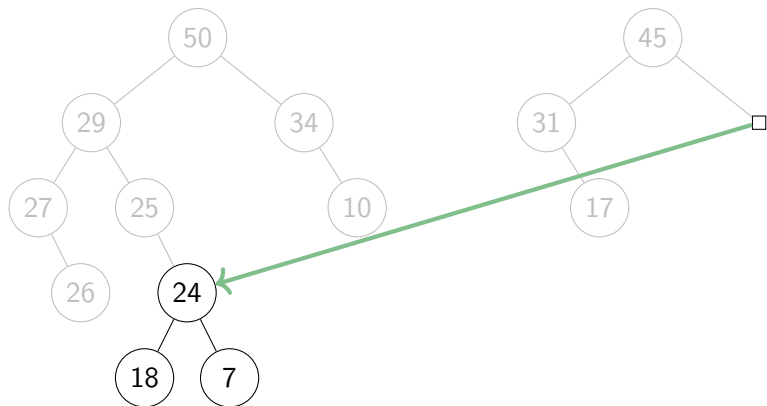
Merge Example



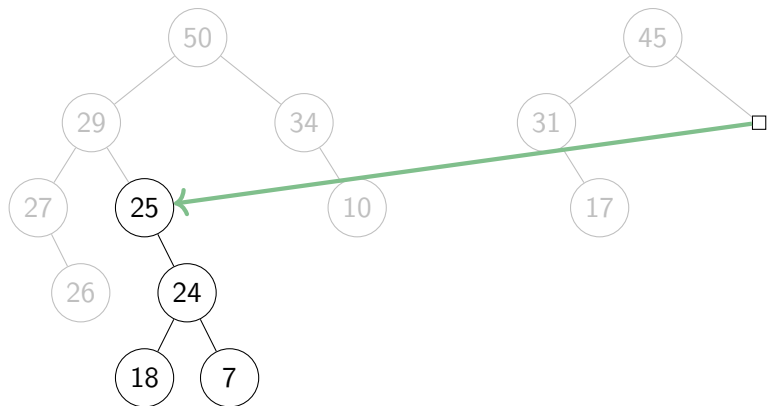
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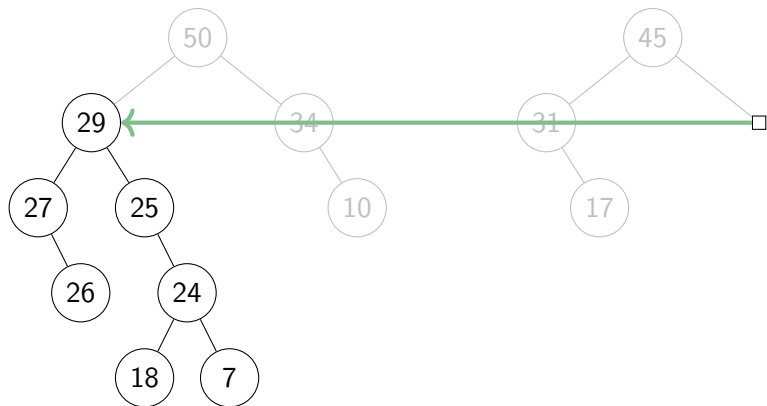
Merge Example



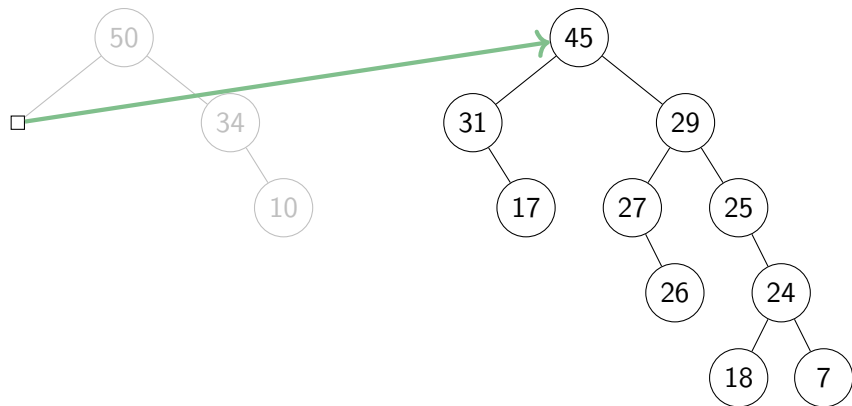
Merge Example



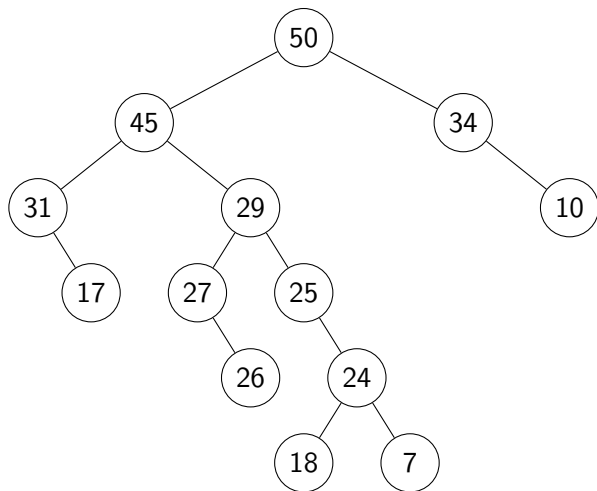
Merge Example



Merge Example



Merge Example



Merging meldable heaps

Run-time? Not more than two *random downward walks* in a binary tree.

Let $T(n)$ = expected length of a random downward walk.

Theorem: $T(n) \in O(\log n)$.

Proof:

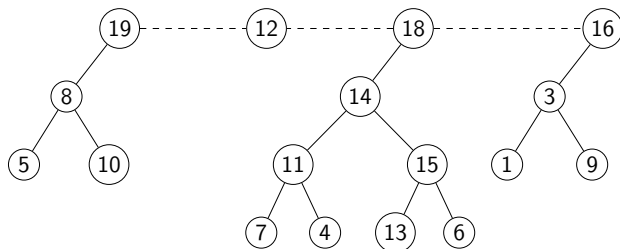
So *merge* (and also *insert* and *deleteMax*) takes $O(\log n)$ expected time.

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Binomial Heaps

Very different structure from binary heaps and meldable heaps:



- List L of binary trees.
- Each binary tree is a **flagged tree**:
Complete binary tree T plus root r that has T as left subtree
 - ▶ Flagged tree of height h has 2^h nodes.
 - ▶ So $h \leq \log n$ for all flagged trees.
- Order-property: Nodes in *left* subtree have no-smaller keys.
(No restrictions on nodes in the right subtree.)

Binomial Heap Operations

- *insert*: Reduce to *merge* as before.
- *findMax*:
 - ▶ At each flag tree, root contains the maximum.
 - ▶ Search roots in $L \Rightarrow O(|L|)$ time.
- We want L to be short.

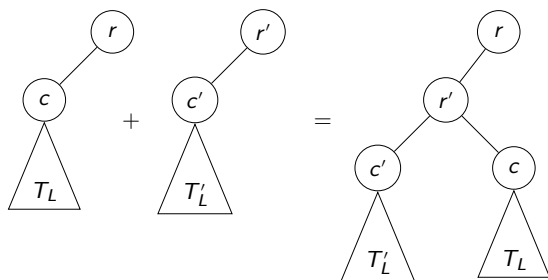
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- **Proper binomial heap**: No two flagged trees have the same height.
- **Observation**: A proper binomial heap has $|L| \leq \log n + 1$.
 - ▶ The flagged tree of largest height h has $h \leq \log n$.
 - ▶ Can have only one flagged tree of each height in $\{0, \dots, h\}$.

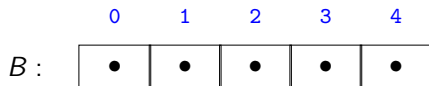
Making Binomial Heaps Proper

- Goal: Given a binomial heap, make it proper.
- Need subroutine: combine two flagged trees of the same height. This can be done in constant time. If $r.\text{key} \geq r'.\text{key}$:

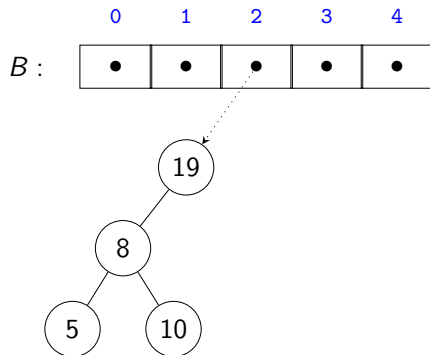


- Idea: Do this whenever two flagged trees have same height.
- Run-time to make proper: $O(|L| + \log n)$ if implemented suitably.

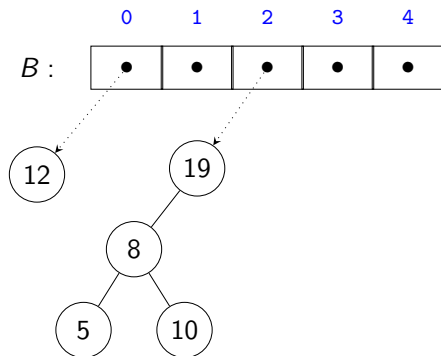
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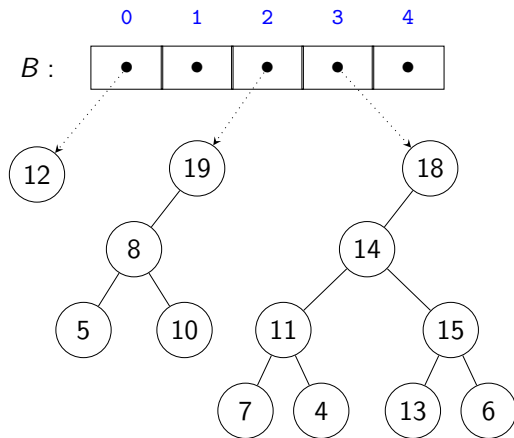
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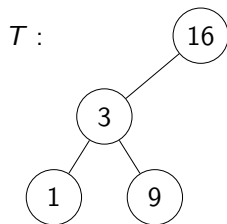
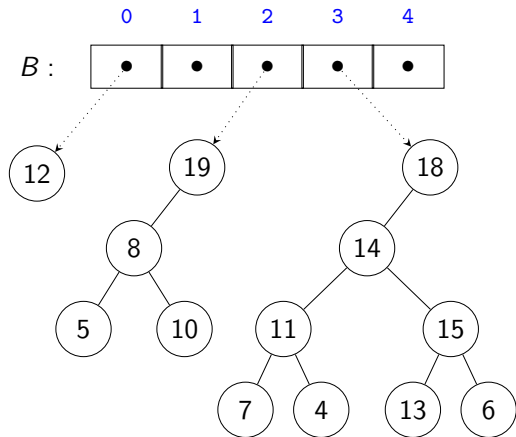
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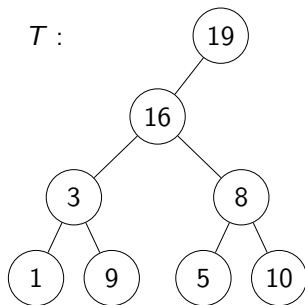
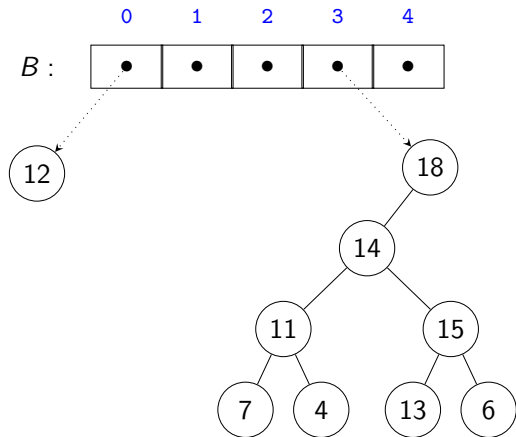
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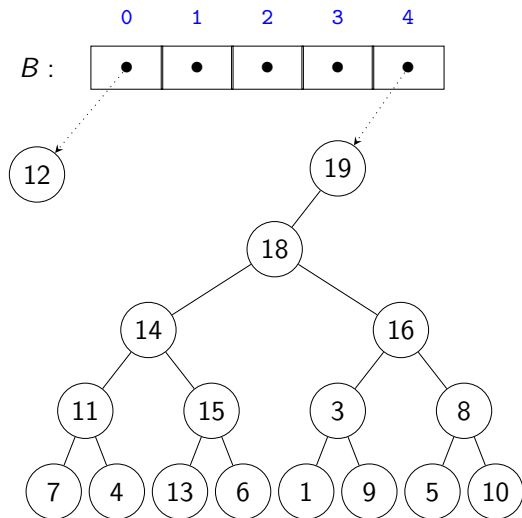
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binomialHeap::makeProper()

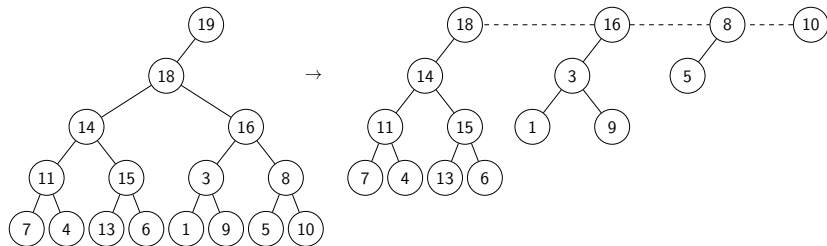
1. $n \leftarrow$ size of the binomial heap
2. compute $\ell \leftarrow \lfloor \log n \rfloor$
3. $B \leftarrow$ array of size $\ell + 1$, initialized all-NIL
4. $L \leftarrow$ list of flagged trees
5. **while** L is non-empty **do**
6. $T \leftarrow L.pop()$, $h \leftarrow T.height$
7. **while** $T' \leftarrow B[h]$ is not NIL **do**
8. **if** $T.root.key < T'.root.key$ **do** swap T and T'
9. // combine T with T'
10. $T'.right \leftarrow T.left$, $T.left \leftarrow T'$, $T.height \leftarrow h+1$
11. $B[h] \leftarrow$ NIL, $h++$
12. $B[h] \leftarrow T$
13. // copy B back to list
14. **for** ($h = 0$; $h \leq \ell$; $h++$) **do**
15. **if** $B[h] \neq$ NIL **do** $L.append(B[h])$

Binomial Heap Operations

- **Idea:** Make binomial heap proper after *every* operation.
 - ⇒ L always has length $O(\log n)$
 - ⇒ Each *makeProper* takes $O(\log n)$ time
- *findMax*: $O(\log n)$ worst-case time.
- *merge*: $O(\log n)$ worst-case time.
 - ▶ Concatenate the two lists.
 - ▶ Call *makeProper*.
- *insert*: $O(\log n)$ worst-case time via *merge*.
- *deleteMax*?

deleteMax in a binomial heap

- Search for maximum among roots, say it is in tree T
- Split $T \setminus \{\text{root}\}$ into into flagged trees T_1, \dots, T_k



- Merge $L \setminus T$ with $\{T_1, \dots, T_k\}$
- Have $k \leq \log n \Rightarrow O(\log n)$ worst-case time.

Summary: All operations have $O(\log n)$ worst-case run-time.