CS 240 – Data Structures and Data Management

Module 5: Other Dictionary Implementations - Enriched

T. Biedl E. Kondratovsky M. Petrick O. Veksler

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Outline

1. Even more Dictionary implementations
   - Expected height of a BST
   - Treaps
   - Optimal static binary search trees
   - MTF-heuristic in a BST
   - Splay Trees

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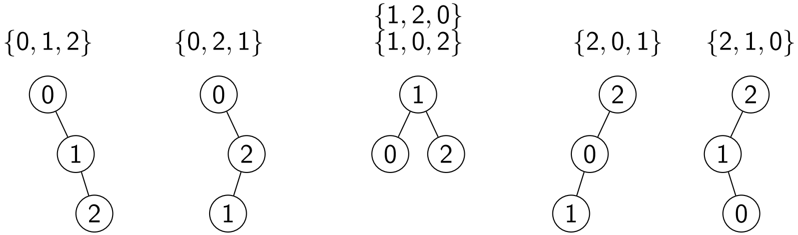
Expected height of BSTs

Assume we *randomly* choose a permutation of $\{0, \ldots, n-1\}$ and build a binary search tree in this order:



Theorem: The expected height of the tree is $O(\log n)$.
Proof:

Expected height vs. average height

This does *not* imply that the average height of a BST is $O(\log n)$.

- Can show: Average height is $\Theta(\sqrt{n})$ (no details).
- Average height (over all BSTs)
  $\neq$ expected height (over all randomly built BSTs)

Expected height vs. average height

This does *not* imply that the average height of a BST is $O(\log n)$.

- Can show: Average height is $\Theta(\sqrt{n})$ (no details).
- Average height (over all BSTs)
  $\neq$ expected height (over all randomly built BSTs)
- Difference already obvious for $n = 3$:
  - Expected height is $\frac{1}{6}(2+2+1+1+2+2) \approx 1.66$.
    6 possible permutations.
  - Average height is $\frac{1}{5}(2+2+1+2+2) = 1.8$.
    5 possible binary search trees.

- Message: Randomization does *not* automatically imply an average-case bound.
  (It depends on what we average over and how we randomize.)

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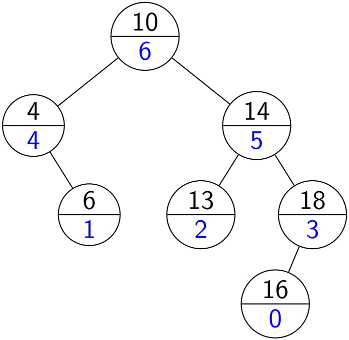
Treaps

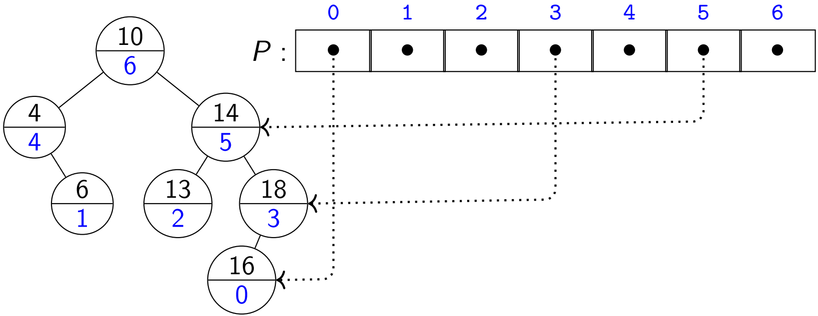
Goal: Build a binary search tree that acts as if it had been build in randomly picked insertion order.

Idea: Use binary search tree, but store a priority with each node.

- Priorities are a permutation of $\{0, \ldots, n-1\}$.
- Permutation has been picked *randomly*
- All permutations should be equally likely.
- Priorities are *decreasing* when going downwards (similar to a heap).

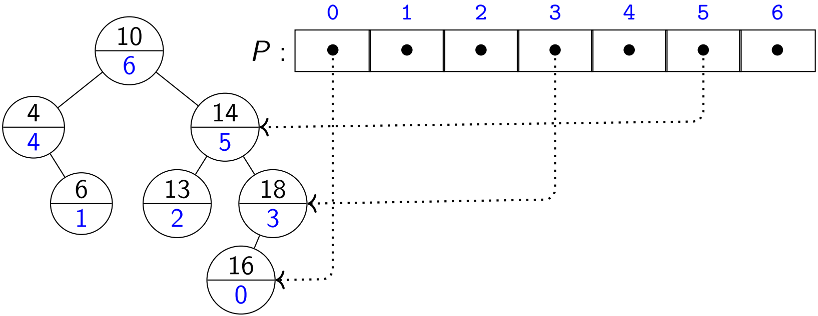


Treaps



- We will also need an array P where $P[i]$ stores node with priority i.
- We call this a **treap** (= tree + heap).

Treaps



- We will also need an array P where $P[i]$ stores node with priority i.
- We call this a **treap** (= tree + heap).

Theorem: The expected height of a treap is $O(\log n)$.

Proof: Root-item has priority $n-1$. This is picked randomly, so proof for expected height of BST applies.

Treap Insertion

Consider adding a new KVP. What priority should it get?

- We need a random permutation of $\{0, \ldots, n-1\}$
  - Currently we had a random permutation of $\{0, \ldots, n-2\}$.

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- Recall *shuffle* from long ago:

```
shuffle(A)
A: array of size n stores ⟨0, ...n−1⟩
1.    for i ← 1 to n − 1 do
2.        swap( A[i], A[random(i + 1)] )
```

- In ith round,
  - have random permutation of $\{0, \ldots, i-1\}$
  - build random permutation of $\{0, \ldots, i\}$ in $O(1)$ time
  - key insight: swap with randomly chosen item

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    for $i \leftarrow 1$ to $n-1$ **do**
        swap($A[i], A[random(i+1)]$)

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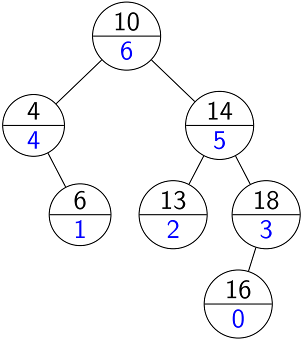
  - key insight: swap with randomly chosen item

We can do the same by *randomly* picking priority p for new item.

- The item that had priority p previously now has priority $n-1$.
- If this violates the heap-property, then rotate to fix it.

Treap Insertions Example

Example: *treap::insert*(17)



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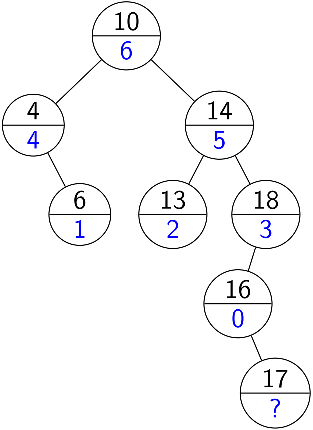

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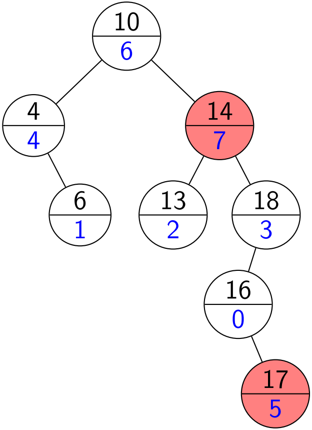
Randomly pick priority $5 \in \{0, \ldots, 7\}$



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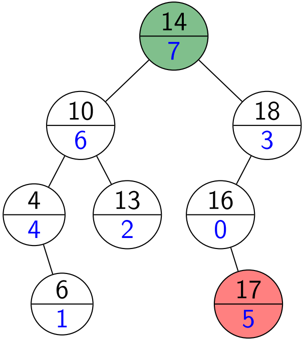

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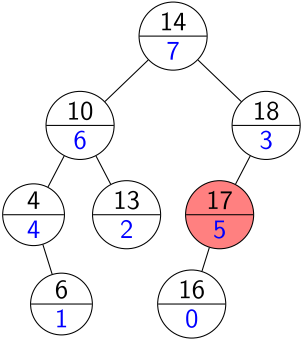

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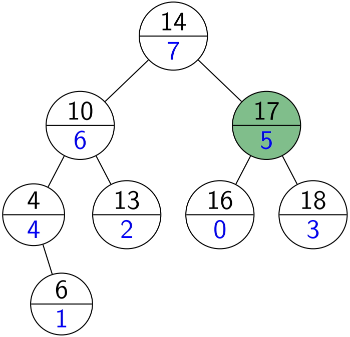

Treap Insertion Code

We assume that the treap stores array where $P[i]$ = node with priority i.

treap::insert(k, v)
- $n \leftarrow P.size$ // current size
- $z \leftarrow$ *BST::insert*(k, v); $n{+}{+}$
- $p \leftarrow$ *random*(n)
- **if** $p < n-1$ **do**
  - $z' \leftarrow P[p]$, $z'.priority \leftarrow n-1$, $P[n-1] \leftarrow z'$
  - *fixUpWithRotations*(z')
- $z.priority \leftarrow p$; $P[p] \leftarrow z$
- *fixUpWithRotations*(z)

treap::fixUpWithRotations(z)
- **while** ($y \leftarrow z.parent$ is not `NIL` and $z.priority > y.priority$) **do**
  - **if** z is the left child of y **do** *rotate-right*(y)
  - **else** *rotate-left*(y)

Treaps summary

- Randomized binary search tree, so expected height is $O(\log n)$
- Achieves $O(\log n)$ expected time for *search* and *insert*
- *delete* can be handled similar (but even more exchanges)

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- Randomized binary search tree, so expected height is $O(\log n)$
- Achieves $O(\log n)$ expected time for *search* and *insert*
- *delete* can be handled similar (but even more exchanges)

- Large space overhead (parent-pointers, priorities, P)
- Not particularly efficient in practice
(except when priorities have meaning $\rightsquigarrow$ later)
- There are ways to avoid some of the space overhead, but in general randomized binary search trees are rarely used.
- We will soon see a randomization that works better (but is not a binary search tree)

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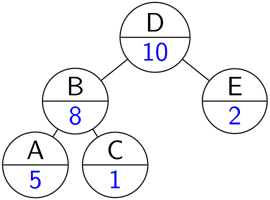
Optimal static binary search trees

- Can we find the optimal static order for a binary search tree?

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$



- Access-cost is now $\sum_k P(k) \cdot (1 + \text{depth of } k)$

  since we use $(1 + \text{depth of } k)$ comparisons to search for key k.

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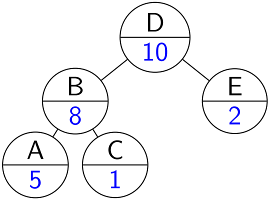

$1 \cdot \frac{10}{26} + 2 \cdot \frac{8}{26} + 2 \cdot \frac{2}{26} + 3 \cdot \frac{5}{26} + 3 \cdot \frac{1}{26} = \frac{48}{26}$

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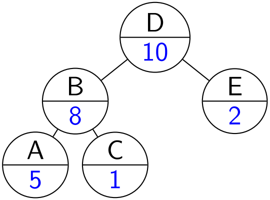

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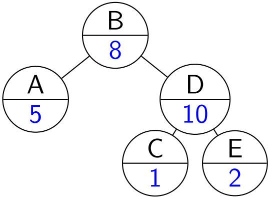
- Natural greedy-algorithm:
  - Put item with highest access-probability at the root.
  - Split keys into left/right as dictated by the order-property.
  - Recurse in the subtree.

Optimal static binary search trees

The greedy-algorithm does *not* give the optimum!

k_i	A	B	C	D	E
$P(k_i)$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{2}{26}$



$$1 \cdot \frac{8}{26} + 2 \cdot \frac{5}{26} + 2 \cdot \frac{10}{26} + 3 \cdot \frac{1}{26} + 3 \cdot \frac{2}{26} = \frac{47}{26}$$

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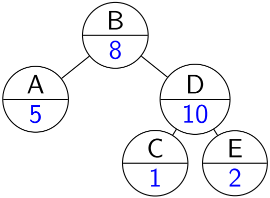

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- To find the optimum, use "dynamic programming":
  - Effectively try *all* possible binary search trees
  - This would take exponential time if done in a straightfoward way.
  - Key idea: We can store and re-use solutions of subproblems to achieve polynomial run-time
- Many more details in cs341 (though not perhaps for this problem)

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MTF-heuristic for binary search trees

What does 'move-to-front' mean in a binary search tree?

- Front = the place that is easiest to access
- In a binary search tree, that's the root.

$\Rightarrow$ After every access, bring item to the root of BST

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⇒ After every access, bring item to the root of BST

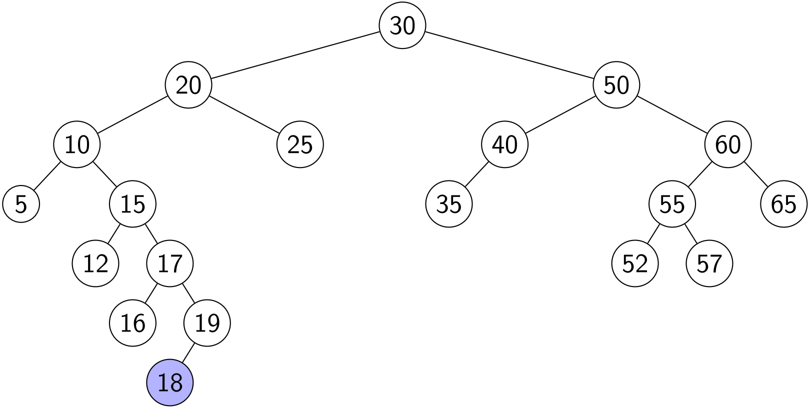
- But: order-property must be maintained!

⇒ Use *rotations*!

(This should remind you of treaps.)

MTF-heuristic for binary search trees

Example: *BST-MTF::search*(18)



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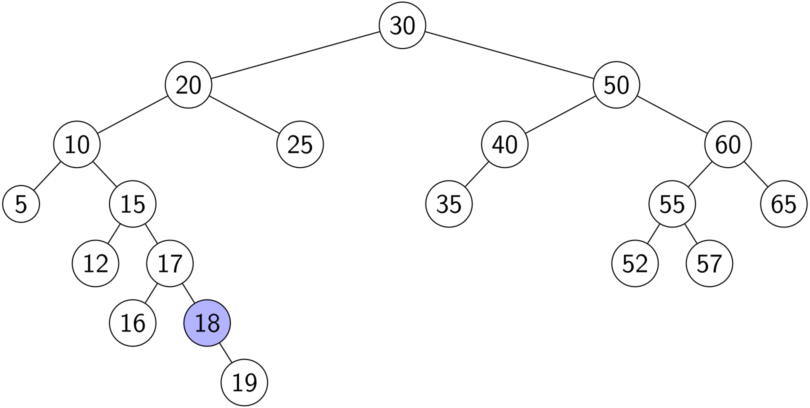

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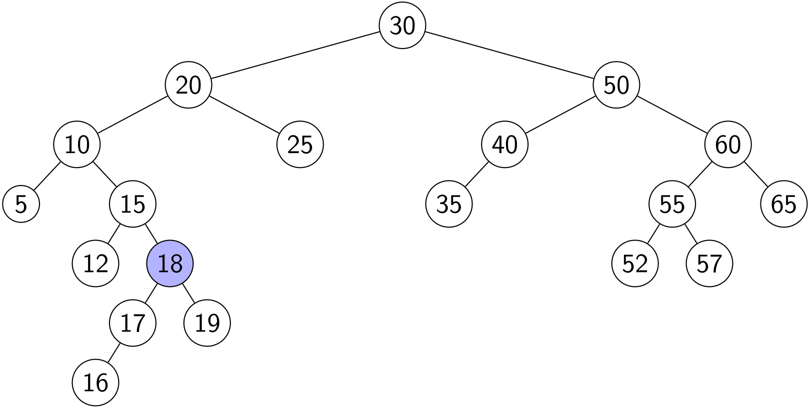

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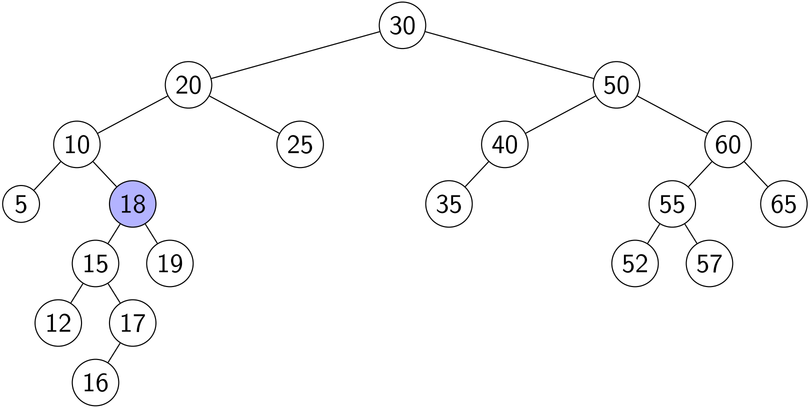

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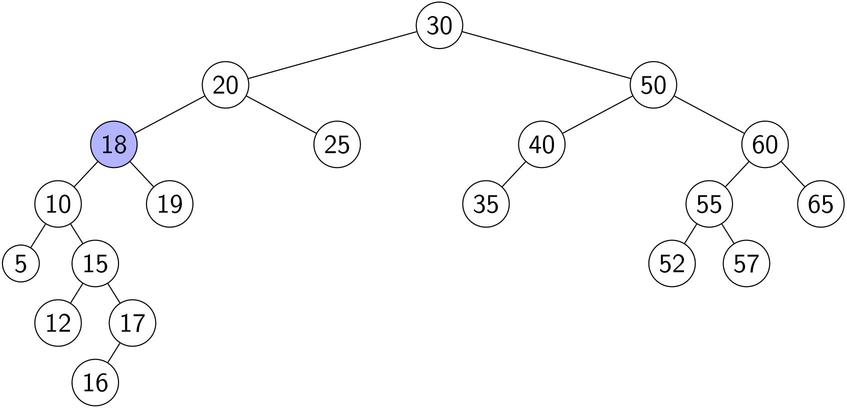

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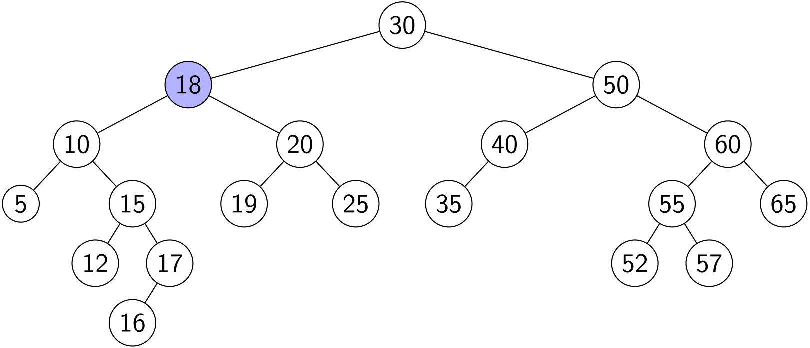

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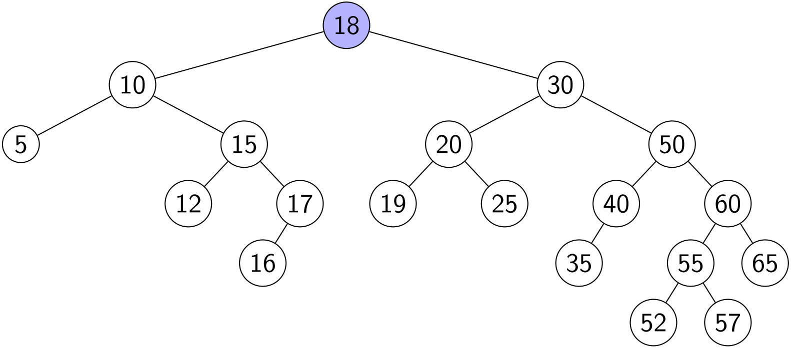

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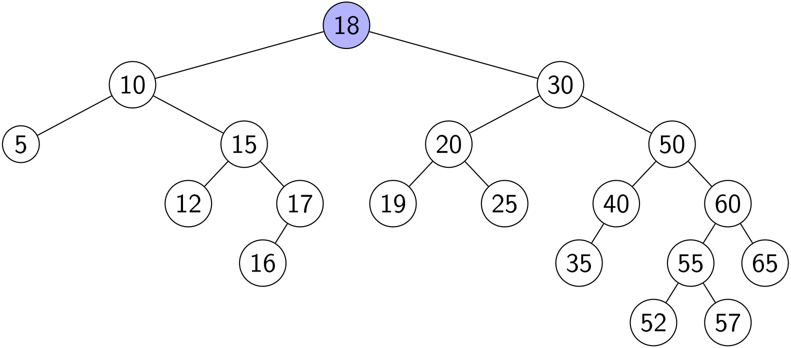

This should work well, but we can do better by moving two level at a time.

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Splay trees

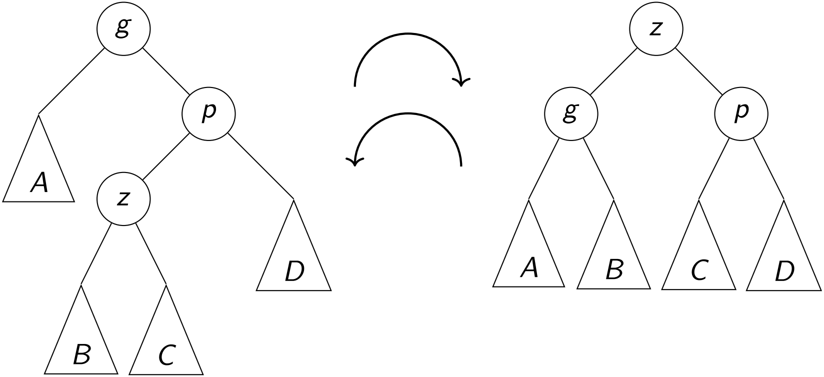
Splay tree overview:

- Binary search tree
- *No* extra information (such as height, balance, size) needed at nodes
- After search/insert, bring accessed node to the root with rotations
- Move node up two layers at a time (except when near root)
  - Use **zig-zig-rotation** or **zig-zag-rotation** to move up two levels.

Goal: This has amortized run-time $O(\log n)$.

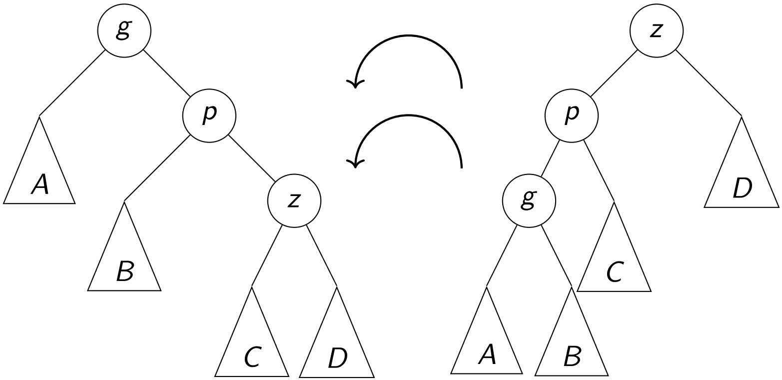
Zig-zag Rotation = Double Rotation

- Let z be the node that we want to move up.
- Let p and g be its parent and grandparent.
- If they are in zig-zag formation, apply a double-rotation.



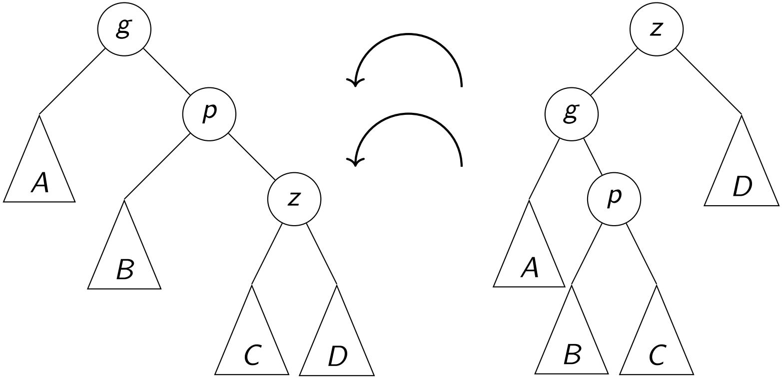
Zig-zig Rotation

- If they are in zig-zig formation, apply a new kind of rotation.



First, a left rotation at g. Second, a left rotation at p.

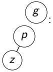
Compare to doing two single rotations

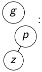


- Both operations bring z two levels higher.
- But using the zig-zig rotation allows to do amortized analysis.

Splay Tree Operations

SplayTree::insert(k, v)

$z \leftarrow$ *BST::insert*(k, v)
while (z is not the root)
  $p \leftarrow z.parent$
  if (z is the left child of p)
    if (p is the root) *rotate-right*(p)
    else $g \leftarrow p.parent$
    case 

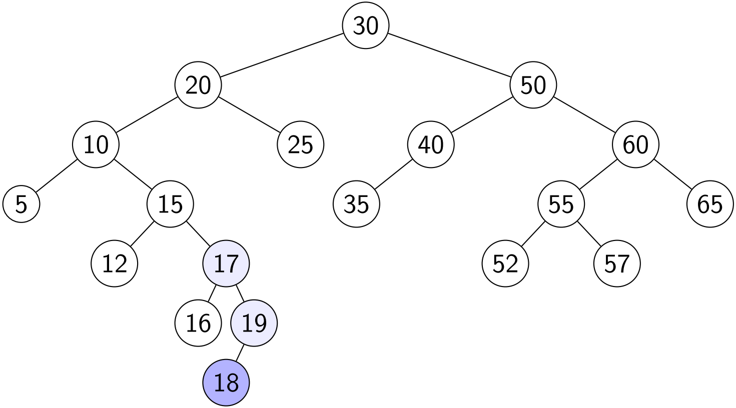
: // Zig-zig rotation
      rotate-right(g)
      rotate-right(p)
    

: // Zig-zag rotation
      rotate-right(p)
      rotate-left(g)
  else ... // symmetric case, z is right child

search and *delete* use corresponding BST-method
Then rotate the lowest visited node up.

Splay Tree Insert

Example: *SplayTree::search*(18)



Splay Tree Insert

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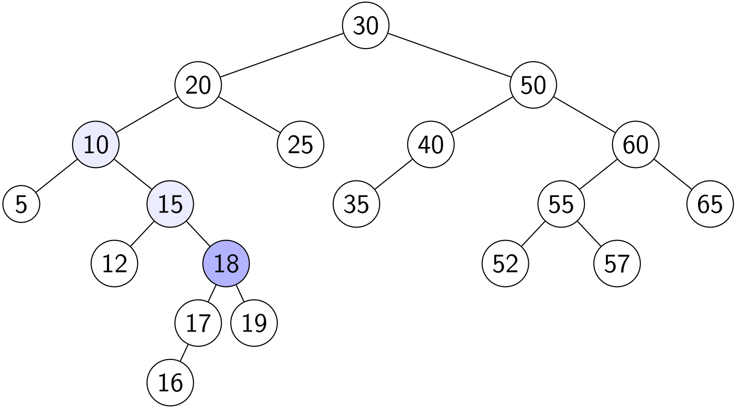

Splay Tree Insert

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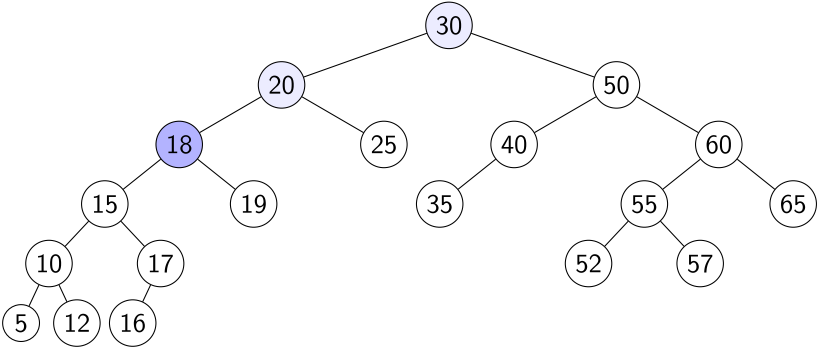

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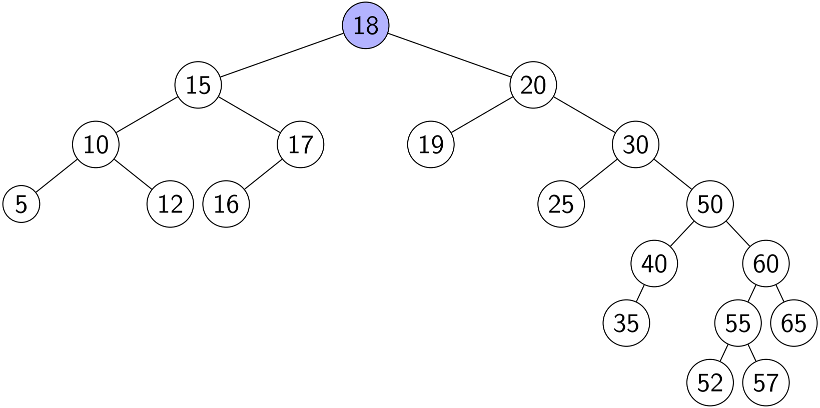
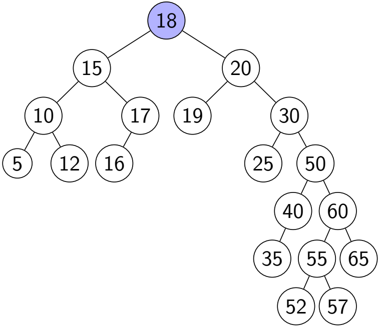


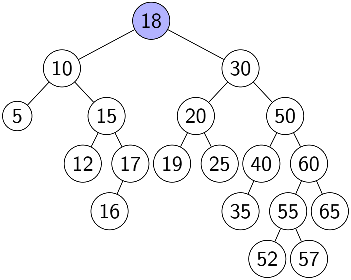
Zig-zig rotations vs. single rotations

Compare the resulting trees:

With zig-zig rotations:



With single rotations:

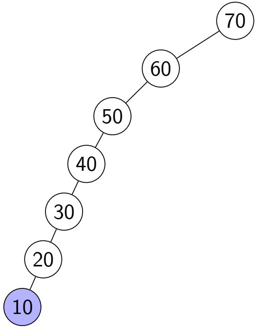


This is *not* more balanced, why do we apply zig-zig-rotations?

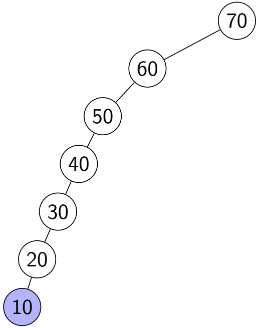
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



With single rotations:



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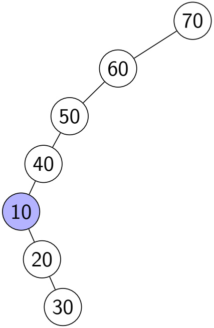
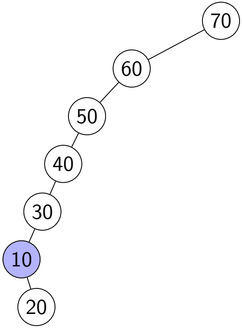


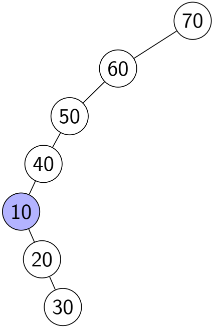
With single rotations:



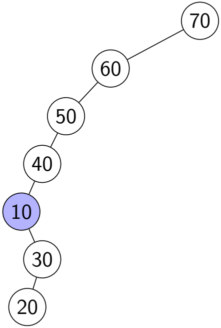
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



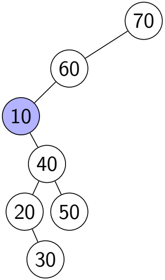
With single rotations:



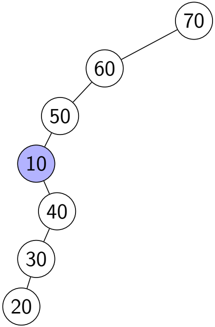
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



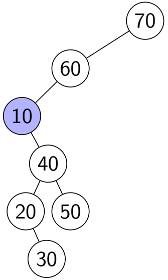
With single rotations:



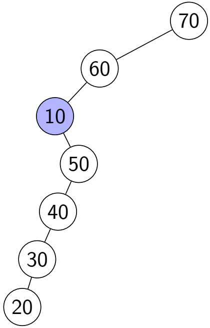
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



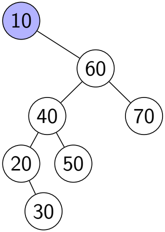
With single rotations:



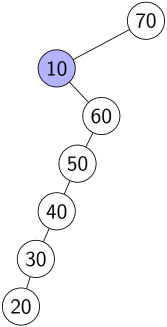
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



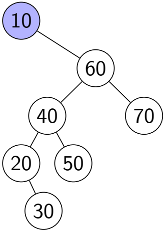
With single rotations:



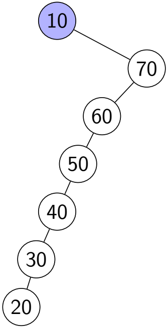
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



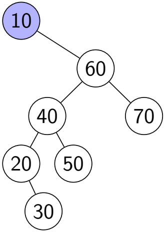
With single rotations:



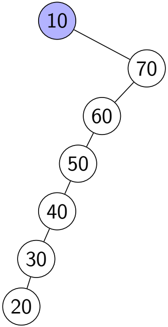
Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

With zig-zig rotations:



With single rotations:



Splay tree intuition:

- For any node on search-path, the depth (roughly) halves
- For all nodes, the depth increases by at most 2

Splay tree summary

Theorem: In a splay tree, all operations take $O(\log n)$ amortized time.

(The formal proof does not follow the intuition and uses a potential function.)

In summary:

- Needs *no* extra information (such as height or size) needed at nodes
- Our pseudo-code assumed parent-references; this can be avoided by temporarily storing search-path.