CS 240 – Data Structures and Data Management

Module 6E: Dictionaries for special keys - Enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

- A tighter lower bound
- Improving binary search
- More on interpolation search
- More on pruned tries

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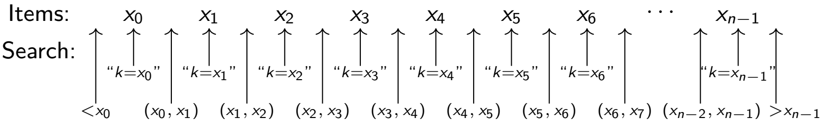
A tighter lower bound

- Create $2n+1$ instances:



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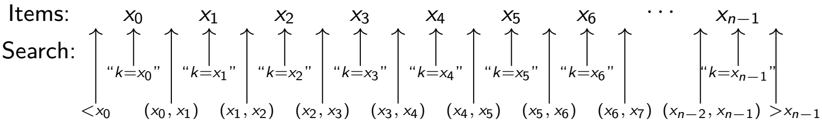

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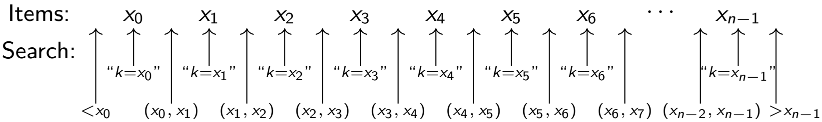

- **Claim:** These instances must lead to distinct leaves (assuming no equality-comparison).

- So we require at least $\lceil \log(2n + 1) \rceil$ comparisons.

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Improving binary search

- *binary-search* uses $\approx 2 \log n$ comparisons.
- **Goal:** Improve it to use $\lceil \log(2n+1) \rceil \approx \log n + 1$ comparisons.
- Main ingredient: Do only *one* comparison per round.

binary-search-optimized(A, n, k)
A: Sorted array of size n, k: key

```
ℓ ← 0, r ← n − 1, χ ← 0
while (ℓ < r)
    m ← ⌊(ℓ+r)/2⌋
    if (A[m] < k) then ℓ ← m + 1
    else r ← m, χ ← 1                          // this is different!
if (k < A[ℓ]) then return "not found, between A[ℓ−1] and A[ℓ]"
else if χ = 1 or (k ≤ A[ℓ]) then return "found at A[ℓ]"
else "not found, between A[ℓ] and A[ℓ+1]"
```

(χ needed for optimum # of comparisons, but not normally used)

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  Sub-array has size $\leq \lceil n/2 \rceil$, so $\lceil \log n \rceil$ rounds.
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- **Claim 4:** If χ is used, then # comparisons $\leq \lceil \log(2n+1) \rceil$.
  (Straightforward but tedious cases. See textbook for details.)

- This uses the *optimum* number of comparisons and also in practice performs better than *binary-search*.
  - But normally omit χ (only needed in Claim 4)
  - Can replace two comparisons in lines 6-7 by equality-comparison.

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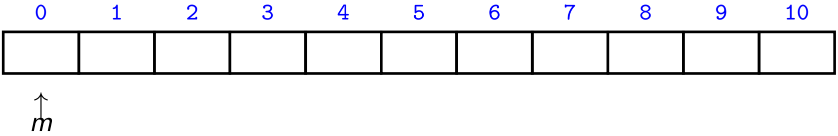
Improving Interpolation Search

- Had: Average-case run-time of *interpolation-search* is $O(\log\log n)$.
- This is very complicated to prove!

  (
  - Study error, i.e., distance between index of k and where we probed.
  - Argue that error is in $O(\sqrt{n})$ in first round.
  - Argue that error is in $O(\frac{1}{2^i}n)$ after i rounds.
  - Study the martingale formed by the errors in the rounds.
  - Argue that its expected length is $O(\log\log n)$.

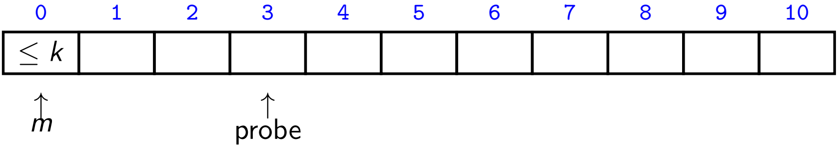
)
- Instead: Define a variant of *interpolatation-search*
  - Better worst-case run-time.
  - Easier to analyze.
- Idea: *Force* the sub-array to have size $\sqrt{n}$
- To do so, search for suitable sub-array with probes.
- Crucial question: how many probes are needed?

Improving Interpolation Search



- First compare ("probe") at m as before.

Improving Interpolation Search



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- If $A[m] \leq k$, probe rightward.
- Probes always go $\lceil \sqrt{N} \rceil$ indices rightward
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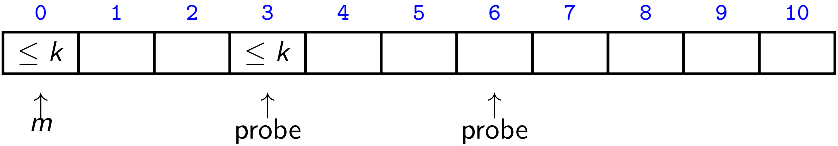

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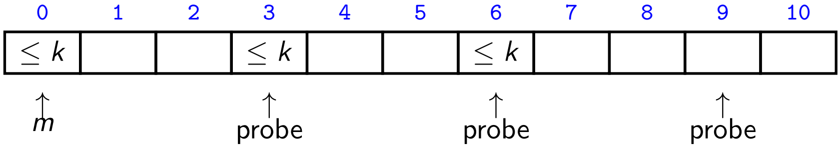

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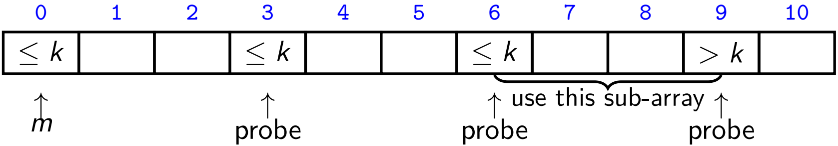

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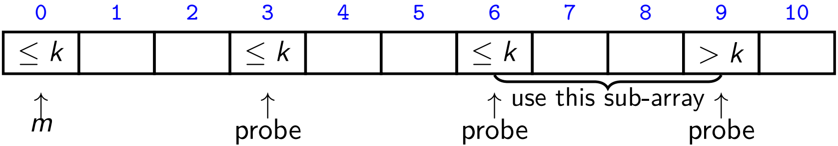

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- Continue probing until $> k$ or out-of-bounds
- Recurse in the only sub-array where k can be; it has size $O(\sqrt{N})$.
- Observe: $\#$ probes $\in O(\sqrt{N})$

Improving Interpolation Search

Interpolation-search-modified(A, n, k)
A: sorted array of size n, k: key

if $(k < A[0]$ or $k > A[n-1])$ **return** "not found"
if $(k = A[n-1])$ **return** "found at index $n-1$"
$\ell \leftarrow 0$, $r \leftarrow n-1$ // have $A[\ell] \leq k < A[r]$
while $(N \leftarrow (r - \ell - 1) \geq 1)$
  $m \leftarrow \ell + \lceil \frac{k - A[\ell])}{A[r] - A[\ell]} \cdot (r - \ell - 1) \rceil$
  if $(A[m] \leq k)$ // probe rightward
    for $h = 1, 2, \ldots$
      $\ell \leftarrow m + (h-1)\lceil\sqrt{N}\rceil$, $r' \leftarrow \min\{r, m + h\lceil\sqrt{N}\rceil\}$
      if $(r' = r$ or $A[r'] > k)$ **then** $r \leftarrow r'$ and **break**
  else ... // symmetrically probe leftward
if $(k = A[\ell])$ **return** "found at index ℓ"
else return "not found"

Analysis of *interpolation-search-improved*

- $T(n) \leq T(\text{size of sub-array}) + O(\#\text{probes})$

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- size of sub-array $\leq \sqrt{n} + O(1)$, $\#$ probes $\leq \sqrt{n} + O(1)$
- Use a sloppy recursion:

$$T^{\text{worst}}(n) \leq \begin{cases} c & n \leq 15 \\ T^{\text{worst}}(\sqrt{n}) + c \cdot \sqrt{n} & \text{otherwise} \end{cases}$$

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- Easy induction proof: $T^{\text{worst}}(n) \leq 2c\sqrt{n}$.
- Therefore worst-case run-time is $O(\sqrt{n})$.

Analysis of *interpolation-search-improved*

- What is the number of probes on average?
- Rephrase: If numbers are chosen uniformly at random, what is the expected number of probes?
- **Claim:** Expected number of probes is $c \leq 2.5$.

Analysis of *interpolation-search-improved*

- Sloppy recursion: $T^{\text{avg}}(n) \leq \begin{cases} T^{\text{avg}}(\sqrt{n}) + c & n \geq 4 \\ c & \text{otherwise} \end{cases}$

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- **Claim:** This resolves to $T^{\text{avg}}(n) \leq c\lceil \log\log n \rceil$.

Key ingredient: $\log\log\sqrt{n} \leq \lceil \log\log n \rceil - 1$.

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Key ingredient: $\log\log\sqrt{n} \leq \lceil \log\log n \rceil - 1$.

- Therefore the average-case # comparisons is $\leq 2.5\lceil \log\log n \rceil$.
- Fewer than *binary-search-optimized*'s $\lceil \log n \rceil + 1$ for $n \geq 16$.

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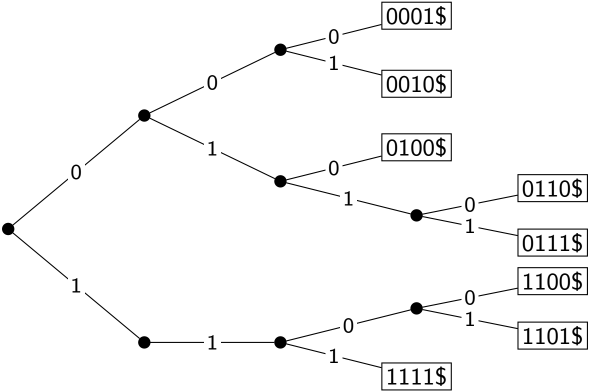
Pruned tries and MSD-radix sort

For bitstrings: Pruned trie equals recursion tree of MSD radix-sort.



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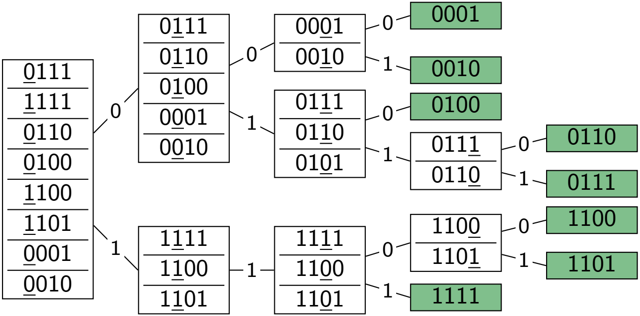

Pruned tries can store real numbers

If we have a generator for each bit of a real number, then we can store them in a pruned trie.