CS 240 – Data Structures and Data Management

Module 7E: Hashing - Enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

- Rehashing
- Multiplication method
- Randomly chosen hash-functions

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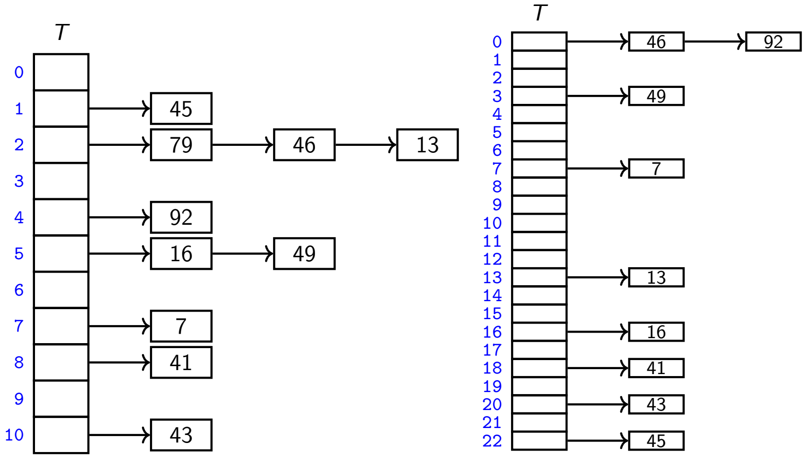
Rehashing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Multiplication method

- Pick some number $A \in (0,1)$ (preferably an irrational)
- $h(k) = \Bigg\lfloor \underbrace{M \cdot \big(\underbrace{\underbrace{A \cdot k}_{\text{multiply}} - \underbrace{\lfloor A \cdot k \rfloor}_{\text{integral part}}}_{\text{fractional part, in } [0,1)} \big)}_{\text{integer in } [0,M)} \Bigg\rfloor$
- Example:

  $A = 0.a_1 a_2 a_3 \ldots$
  $k = b_1 b_2 \ldots b_6$
  (both in base 2)

	(leading bits)	(bits of fractional part)	
$A \cdot k \quad =$	0 0 0 .. 0 0		
$+a_1 \cdot$	0 $b_1 b_2 b_3$ .. b_5	b_6	
$+a_2 \cdot$	0 0 $b_1 b_2 b_3$ ..	b_5 b_6	
$+a_3 \cdot$	0 0 0 b_1 b_2 b_3	.. b_5 b_6	
$\vdots$	$\ddots$	$\ddots$	$\ddots$
$+a_5 \cdot$	0 0 0 0 0 b_1	b_2 b_3 ..	b_5 b_6
$+a_6 \cdot$	0 0 0 0 0 0	b_1 b_2 b_3	.. b_5 b_6
$+a_7 \cdot$	0 0 0 0 0 0	0 b_1 b_2	b_3 .. b_5 b_6
$+a_8 \cdot$	0 0 0 0 0 0	0 0 b_1	b_2 b_3 .. b_5 b
$\vdots$		$\leftarrow h(k) \rightarrow$	$\ddots$

- Should use at least $\log|U| + \log|M|$ bits of A.

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Randomly chosen hash-functions

- There are too many possible hash-functions ⇒ we cannot compute hash-value quickly if we choose randomly among them.
- Idea: fix a family $\mathcal{H}$ of hash-functions that are easy to compute.
- But what should critera be for $\mathcal{H}$?
- Uniform hash-values, i.e., $P(h(k) = i) = \frac{1}{M}$, is *not* enough.

$\mathcal{H}_1$	keys		
	x	y	z
h_0	0	0	0
h_1	1	1	1

  - $M = 2$ in this example.
  - $P(h(k) = i) = \frac{1}{2}$ for $i = 0, 1$ and $k = x, y, z$
  - But these hash-functions are terrible!

- Goal: Small probability of collisions (**universal hashing**):

$$P(h(k) = h(k')) = \frac{1}{M} \quad \text{for any two keys } k \neq k'$$

- This is enough for hashing-analysis for chaining to hold.

Carter-Wegman hash-function

$$h_{a,b}(k) = \Big(\underbrace{a \cdot k + b \bmod p}_{f_{a,b}(k)}\Big) \bmod M$$

(where $k \in \mathbb{Z}_p$, p prime, $a, b \in \mathbb{Z}_p$ chosen randomly, $a \neq 0$)

Example: ($p = 5, M = 2$):

	keys				
	0	1	2	3	4
$f_{1,0}$	0	1	2	3	4
$f_{2,0}$	0	2	4	1	3
$f_{1,2}$	2	3	4	0	1
$f_{2,1}$	1	3	0	2	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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Example: $(p = 5, M = 2)$:

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$f_{2,0}$	0	2	4	1	3
$f_{1,2}$	2	3	4	0	1
$f_{2,1}$	1	3	0	2	4
⋮	⋮	⋮	⋮	⋮	⋮

	keys				
	0	1	2	3	4
$h_{1,0}$	0	1	0	1	0
$h_{2,0}$	0	0	0	1	1
$h_{1,2}$	0	1	0	0	1
$h_{2,1}$	1	1	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮

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⋮	⋮	⋮	⋮	⋮	⋮

	keys				
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$h_{1,0}$	0	1	0	1	0
$h_{2,0}$	0	0	0	1	1
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$h_{2,1}$	1	1	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮

Claim: $f_{a,b}$ is a permutation of $\mathbb{Z}_p$.