CS 240 – Data Structures and Data Management

Module 11E: External Memory - enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

11 External Memory
- Red-black trees
- Pre-emptive splitting/merging
- B^+-trees
- LSM-trees
- Extendible Hashing

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Towards red-black-tree

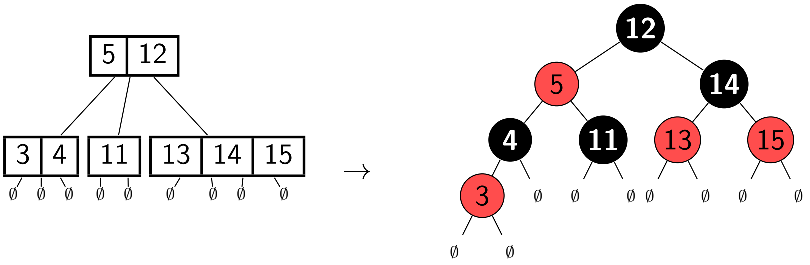
(We currently only consider run-time in RAM. We will return to the EMM shortly.)

- Recall: All operations in 2-4 trees have $O(\log n)$ worst-case run-time.
- The height is much smaller than for AVL-trees ($\log_2(\frac{n+1}{2})$ vs. $\log_\Phi(n) \approx 1.44 \log_2 n$.)
- So they might be more efficient, depending on implementation details.

- But: Handling three kinds of nodes is cumbersome.
  (We either need a list for KVPs and subtrees, or waste space at nodes to have space for links always available.)

Better idea: Design a class of binary search trees that mirrors 2-4-trees!

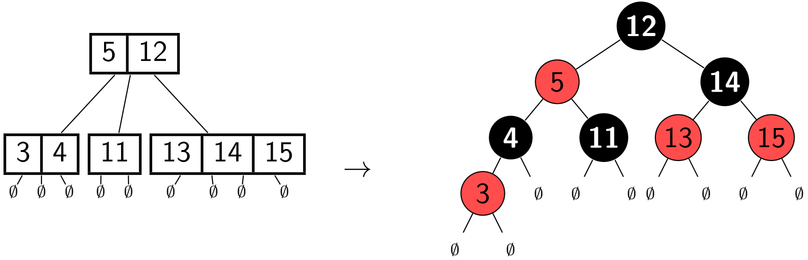
2-4-tree to red-black-tree



Converting a 2-4-tree:

- A d-node becomes a black node with $d-1$ red children
  (Assembled so that they form a BST of height at most 1.)

2-4-tree to red-black-tree



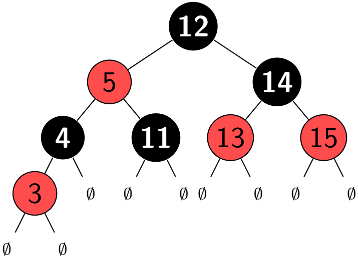
Converting a 2-4-tree:

- A d-node becomes a black node with $d-1$ red children (Assembled so that they form a BST of height at most 1.)

Resulting properties:

- Any red node has a black parent.
- Any empty subtree T has the same **black-depth** (number of black nodes on path from root to T)

Red-black-trees



Definition: A **red-black tree** is a binary search tree such that

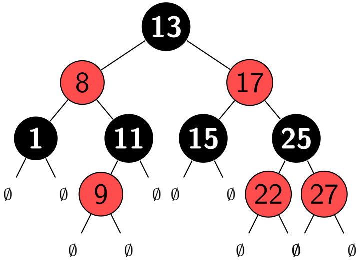
- Every node has a color (red or black)
- Every red node has a black parent.
  (In particular the root is black.)
- Any empty subtree T has the same black-depth.

Note: Can store this with one bit overhead per node.

Red-black tree

Rather than proving properties directly, we re-use properties of 2-4-trees.

Lemma: Any red-black tree T can be converted into a 2-4-tree T' where $\mathit{height}(T') = \mathit{black\text{-}depth}(T) - 1$.



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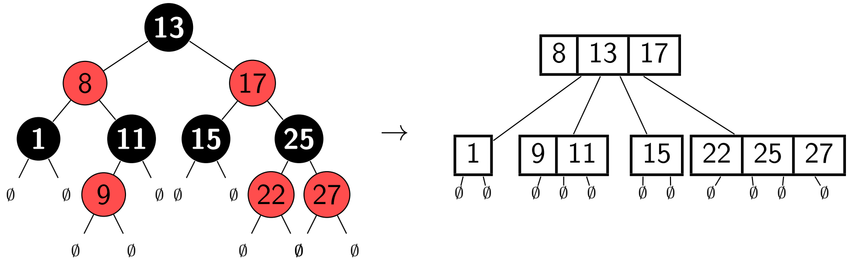

Proof:

- Black node with $0 \le d \le 2$ red children becomes a $(d{+}1)$-node

Red-black tree properties

- Red-black trees have height $\leq 2\log(\frac{n+1}{2}) + 1$
  - black-depth $\leq \log(\frac{n+1}{2}) + 1$ by 2-4-tree height.
  - At least half of the nodes on the path to deepest nodes are black (recall: red nodes have black parents)

$\Rightarrow$ height$=$# nodes on path - 1 $\leq$ 2 black-depth - 1

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- *insert*/*delete* can be done as for 2-4-trees.
  - One can "translate" the code directly to red-black trees.
  - The transfer/split/merge operations become rotations.
- So all operations take $\Theta(\log n)$ worst-case time.
- In the worst case, $\Theta(\log n)$ rotations are required for *insert*/*delete*.
- But experiments show that few rotations usually suffice, and updates are faster in red-black trees than in AVL-trees.

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This is a very efficient balanced binary search tree.

(There are even better balanced binary search trees. No details.)

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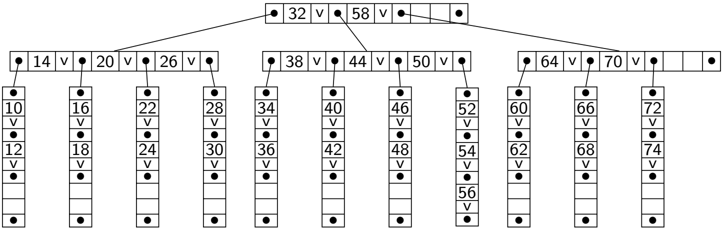
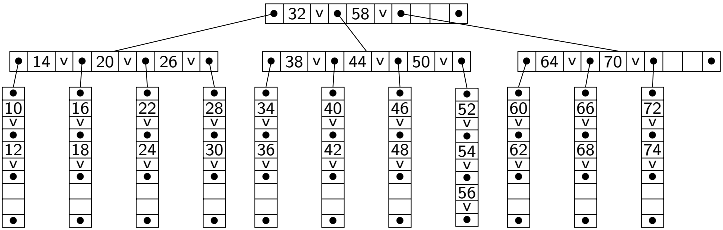
Pre-emptive splitting/merging



- Observe: *BTree::insert*(k, v) traverses tree twice:
  - Search down on a path to the leaf where we add (k, v).
  - Go back up on the path to fix overflow, if needed.
- So the number of block-transfers could be twice the height.
- How can we avoid this?

Pre-emptive splitting/merging



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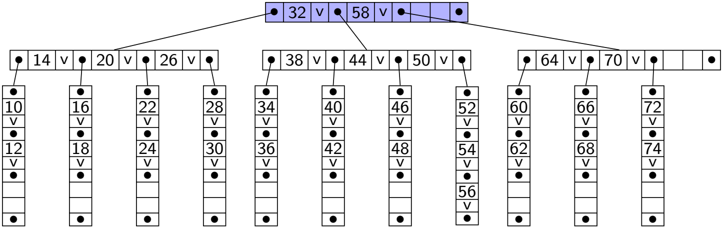
- **Idea:** During the search, *always* split if the node is full.
- Then a node split at the leaf does not create an overfull parent.

Pre-emptive splitting/merging example

PreemptiveBTree::insert(49):



- If node is not full, keep searching.

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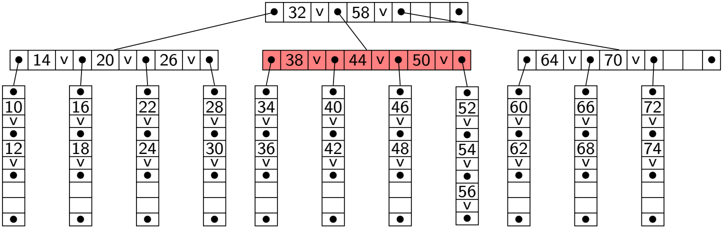

- If node is not full, keep searching.
- If node is full, immediately split.

Pre-emptive splitting/merging example

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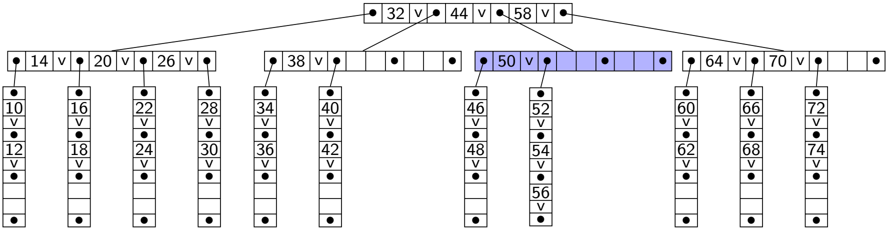

- If node is not full, keep searching.
- If node is full, immediately split.
- Then keep searching in appropriate new node.

Pre-emptive splitting/merging example

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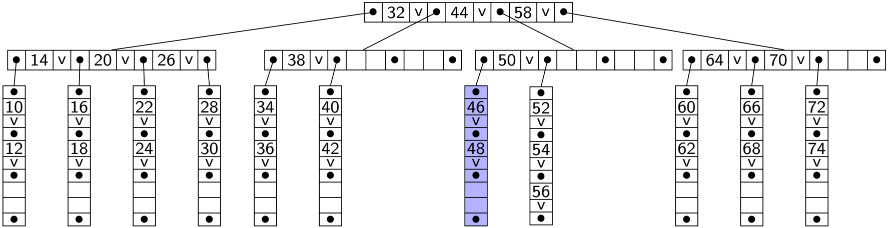

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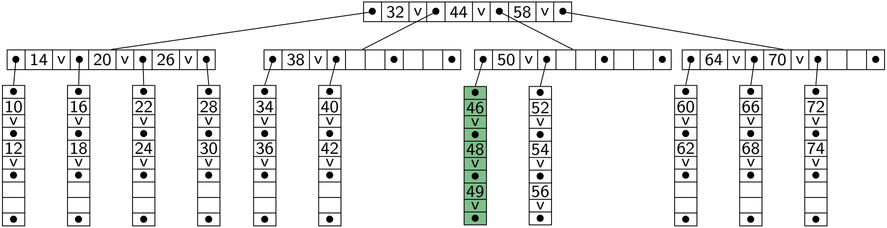

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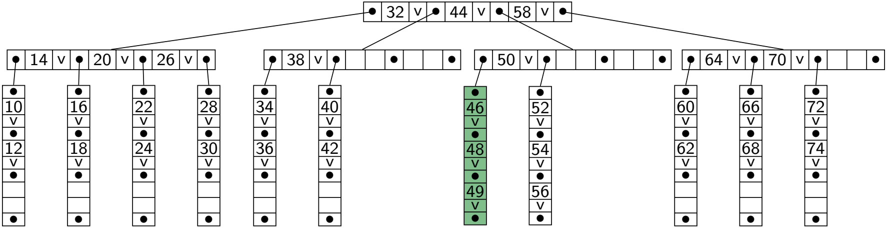

- If node is not full, keep searching.
- If node is full, immediately split.
- Then keep searching in appropriate new node.
- We may have split unnecessarily. (But space is cheap.)

- Similarly *delete* should pre-emptively merge. (No details.)
- With this, we no longer need parent-references.

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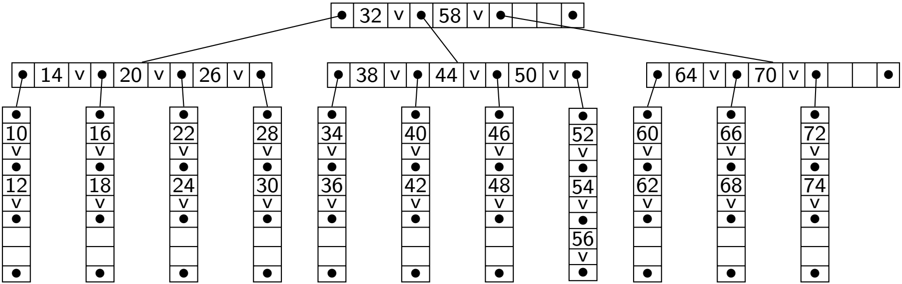
Towards B^+-trees

In a B-tree, each node is one block of memory. In this example, up to 10 keys/references fit into one block, so the order is 4.



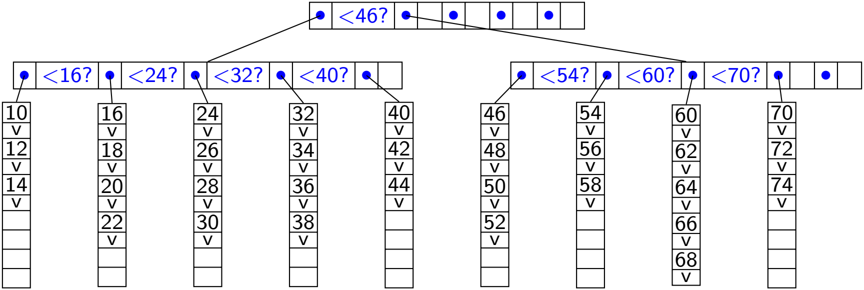
This B-tree could store up to 63 KVPs with height 2.

Two ideas to achieve smaller height:

1. The leaves are wasting space for references that will never be used.
2. Use a *decision-tree version* $\Rightarrow$ inner nodes can have more children.

B^+-trees

- Each node is one block of memory.
- All KVPs are stored at *leaves*. Each leaf is at least half full.
- *Interior nodes* store only keys for comparison during search.
- Interior (non-root) nodes have at least half of the possible subtrees.
- *insert*/*delete* use pre-emptive splitting/merging.



This B^+-tree could store up to 125 KVPs with height 2.

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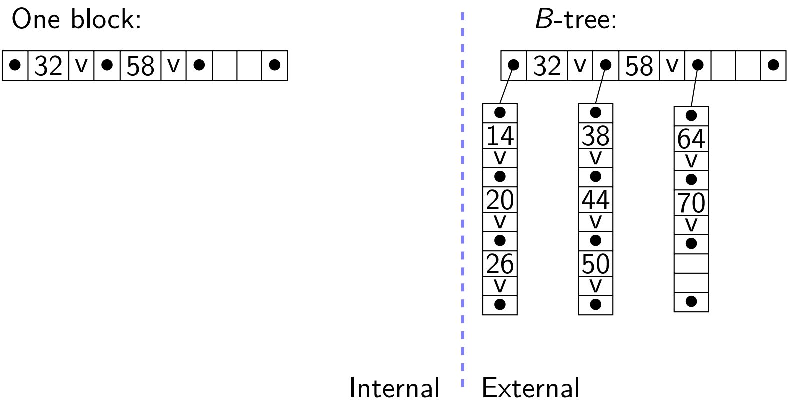
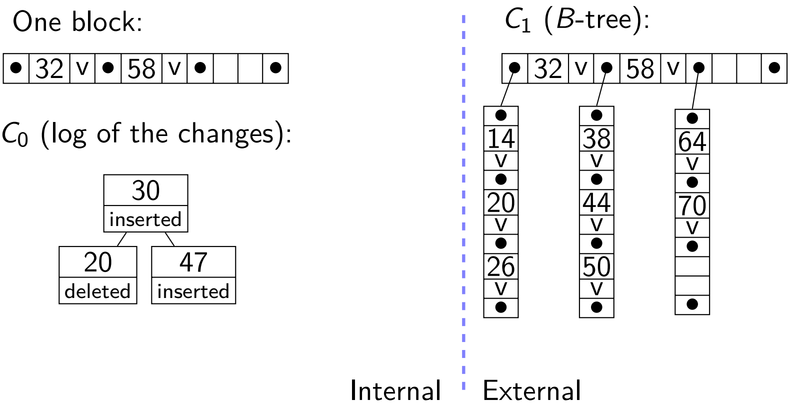
Towards LSM-trees



- Main memory only requires 1-2 blocks at a time.
- Roughly $M - 2B$ space free.
- How can we use this to increase speed for updates?

LSM-trees



- Store dictionary in internal memory that logs all changes
- To search: first search in C_0, then (if needed) in C_1
- If internal memory full: do lots of updates in C_1 at once

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Dictionaries for Integers in External Memory

Recall Hashing:

- Direct Addressing allowed for $O(1)$ insert and delete if keys are small integers.
- If keys are too big, use hash-function to map them to (smaller) integers.
- Expected run-time of operations is $O(1)$ if load factor α is kept small

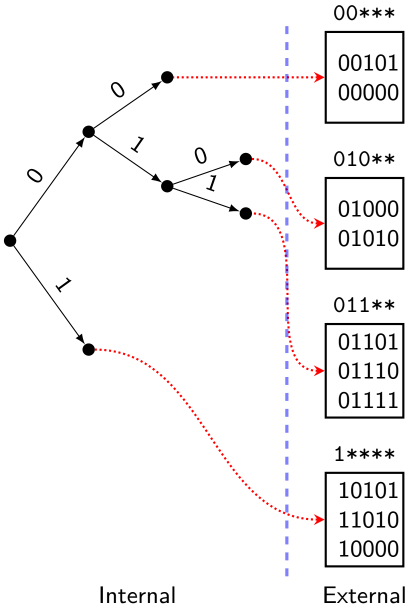
This does not adapt well to external memory.

- We must occasionally re-hash to keep α small.
- And re-hashing must load *all* n/B blocks.
- This is unacceptably slow.

Goal: Data structure for integers that typically uses $O(1)$ block transfers, and never needs to load all blocks.

Idea: Hash-values = bitstrings. Store trie of links to blocks of integers.

Trie of blocks – Overview



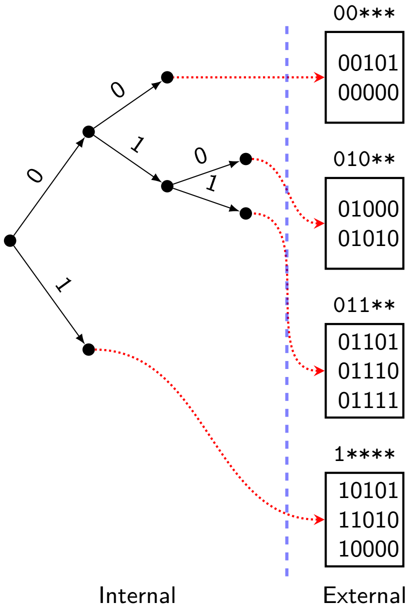
Assumption: We store non-negative integers (here written as bitstrings). [Typically these are hash-values.]

Build trie D (the **directory**) of integers in internal memory.

Stop splitting in trie when remaining items fit in one block. (∼ pruned trie, but stop earlier)

Each leaf of D refers to block of external memory that stores the items.

Trie of blocks – operations



search(k): Search for k in D until we reach leaf ℓ. Load block at ℓ and search in it.
1 block transfer.

insert(k): Search for k, load block, then insert k. If this exceeds block-capacity, split at trie-node and split blocks (possibly repeatedly).
Typically 2 **block transfers**.

delete(k): Search for k, load block, then delete k.
Optional: combine underfull blocks.
2 block transfers.

Trie of blocks: Insert

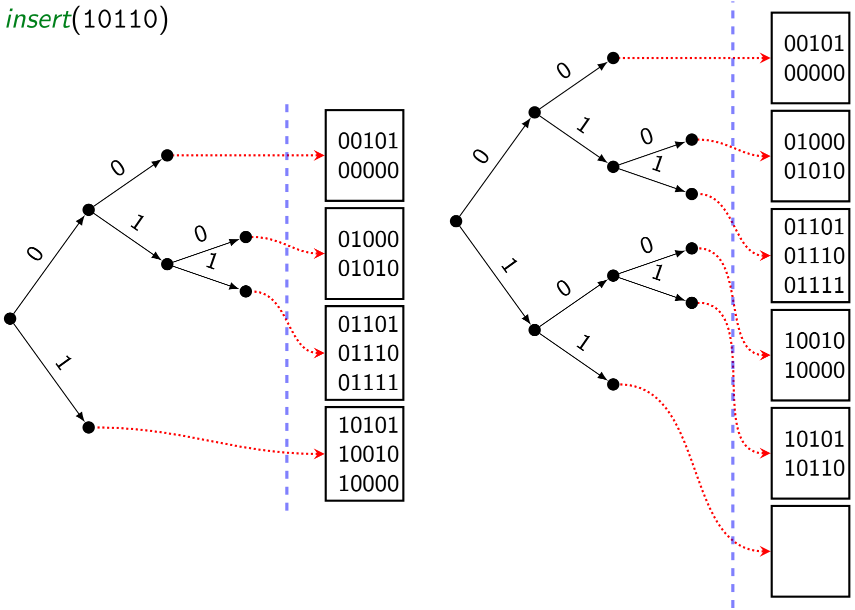
TrieOfBlocks::insert(k, v)
(k, v): key-value pair, k is a bit-string

$\ell \leftarrow$ *Trie::search*(D, k) // leaf with prefix of k
$d \leftarrow$ depth of ℓ in D
transfer block P that ℓ refers to
while P has no room for additional items
  Split P into two blocks P_0 and P_1 by $k[d]$
  Create two children ℓ_0 and ℓ_1 of ℓ, linked to P_0 and P_1
  $d \leftarrow d{+}1, \ell \leftarrow \ell_{k[d]}, P \leftarrow P_{k[d]}$
insert (k, v) into P

Note: This may create empty blocks, but this should be rare.

Trie of blocks: Insert

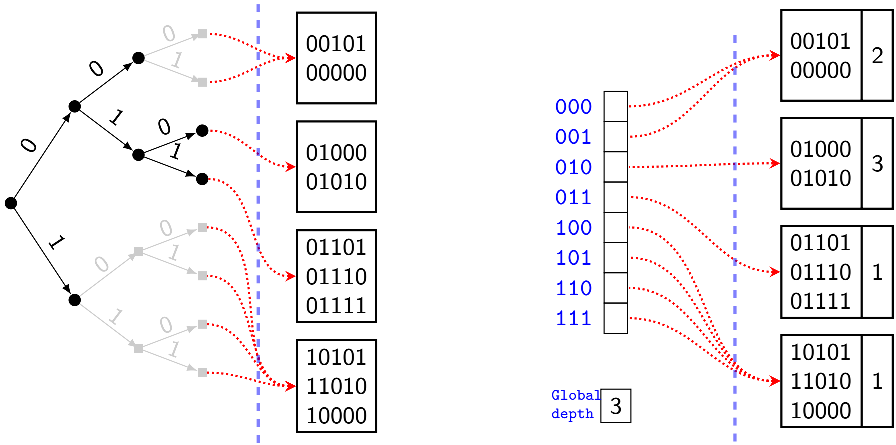
insert(10110)



Extendible hashing

We can save links (hence space in internal memory) with two tricks:

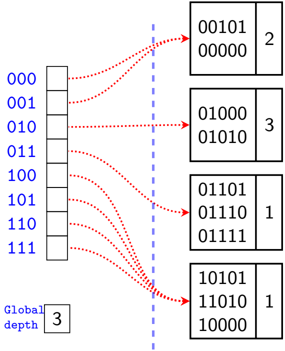
- Expand the trie so that all leaves have the same **global depth** d_D.
- Store *only* the leaves, and in an array D of size 2^{d_D}.

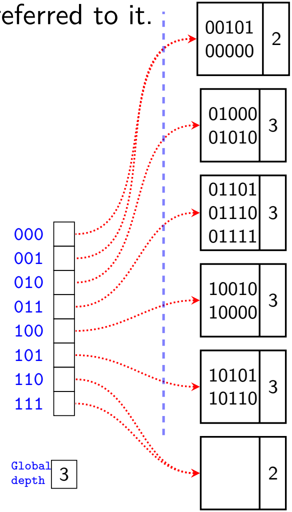


Extendible hashing operations

- Conceptually: convert table to trie, do operation, convert trie to table
- But work directly on table if each block stores its **local depth**, i.e., the depth of the original trie-node that referred to it.

Example: *insert*(10110)

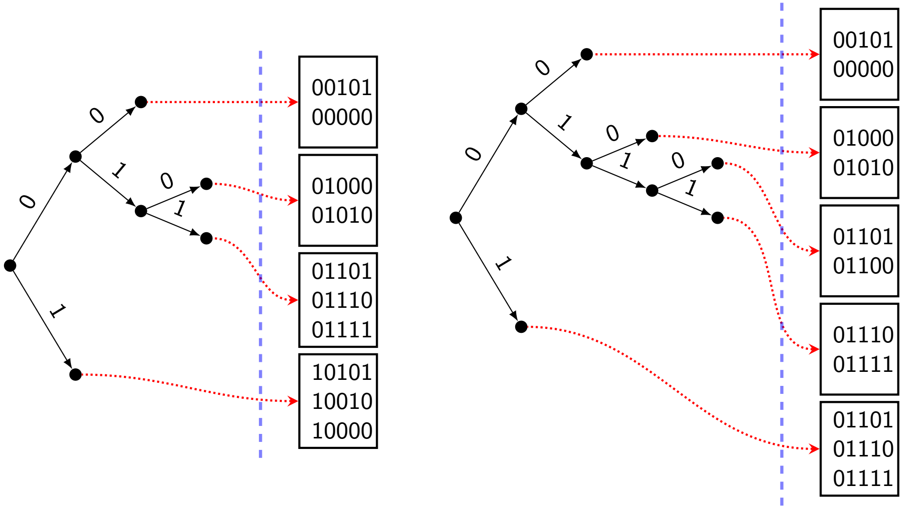




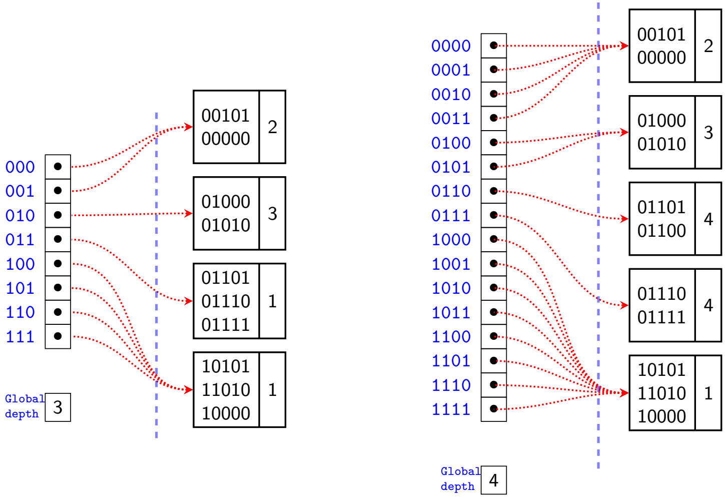
Extendible hashing operations

If *insert* increased the trie-height, then the array-size now doubles.

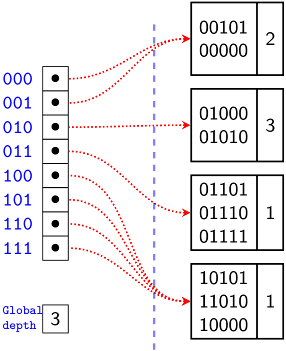
Example: *insert*(01100) in trie of blocks

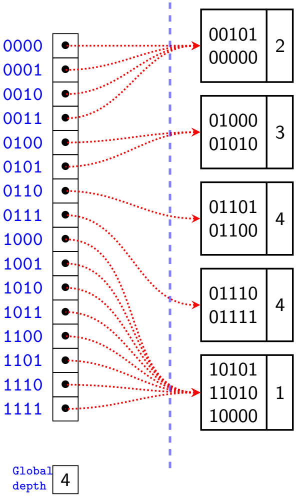


insert(01100) in extendible hash-table



insert(01100) in extendible hash-table





But notice: We do *not* need to load extra blocks for this.
The number of block-transfers is exactly the same as with the trie of blocks, but the space used by the dictionary is much better.

Extendible hashing discussion

- Hashing collisions (= duplicate keys) are resolved within the block and do not affect the block transfers.
  If more items collide than can fit into a block we *extend the hash-function*, i.e., make bit-strings longer without changing the initial bits.
- Directory typically fits into in internal memory.
  If it does not, then strategies similar to B-trees can be applied.
- Only 1 or 2 block transfers expected for *any* operation.
- To make more space, we only add one block.
  Rarely change the size of the directory.
  Never have to move all items. (in contrast to re-hashing!)
- Space usage is not too inefficient: one can show that under uniform distribution assumption each block is expected to be 69% full.