CS 240 – Data Structures and Data Management

Module 1: Introduction and Asymptotic Analysis

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Based on lecture notes by many previous cs240 instructors

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Outline

Outline

1. Introduction and Asymptotic Analysis
   - CS240 Overview
   - Algorithm Design
   - Analysis of Algorithms I
   - Asymptotic Notation
   - Analysis of Algorithms II
   - Example: Analysis of MergeSort
   - Helpful Formulas

Course Objectives: What is this course about?

- Much of Computer Science is *problem solving*: Write a program that converts the given input to the expected output.
- When first learning to program, we emphasize *correctness*: does your program output the expected results?
- Starting with this course, we will also be concerned with *efficiency*: is your program using the computer's resources (typically processor time) efficiently?
- We will study efficient methods of *storing*, *accessing*, and *organizing* large collections of data.

  Motivating examples: Digital Music Collection, English Dictionary

  Typical operations include: *inserting* new data items, *deleting* data items, *searching* for specific data items, *sorting*.

Course Objectives: What is this course about?

- We will consider various **abstract data types** (ADTs) and how to realize them efficiently using appropriate **data structures**.
- There is a strong emphasis on mathematical analysis in the course.
- Algorithms are presented using pseudo-code and analyzed using order notation (big-Oh, etc.).

Course Topics

- big-Oh analysis
- priority queues and heaps
- sorting, selection
- binary search trees, AVL trees
- skip lists
- hashing
- quadtrees, kd-trees
- range search
- tries
- string matching
- data compression

CS Background

Topics covered in previous courses with relevant sections in [Sedgewick]:

- arrays, linked lists (Sec. 3.2–3.4)
- strings (Sec. 3.6)
- stacks, queues (Sec. 4.2–4.6)
- abstract data types (Sec. 4-intro, 4.1, 4.8–4.9)
- recursive algorithms (5.1)
- binary trees (5.4–5.7)
- sorting (6.1–6.4)
- binary search (12.4)
- binary search trees (12.5)
- probability and expectations (Goodrich & Tamassia, Section 1.3.4)

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Problems (terminology)

First, we must introduce terminology so that we can precisely characterize what we mean by efficiency.

Problem: Given a problem instance, carry out a particular computational task.

Problem Instance: *Input* for the specified problem.

Problem Solution: *Output* (correct answer) for the specified problem instance.

Size of a problem instance: *Size*(I) is a positive integer which is a measure of the size of the instance I.

Example: Sorting problem

Algorithms and Programs

Algorithm: An algorithm is a *step-by-step process* (e.g., described in pseudo-code) for carrying out a series of computations, given an arbitrary problem instance I.

Solving a problem: An Algorithm A *solves* a problem Π if, for every instance I of Π, A finds (computes) a valid solution for the instance I in finite time.

Program: A program is an *implementation* of an algorithm using a specified computer language.

In this course, our emphasis is on algorithms (as opposed to programs or programming).

Algorithms and Programs

Pseudocode: a method of communicating an algorithm to another person.

In contrast, a program is a method of communicating an algorithm to a computer.

Pseudocode

- omits obvious details, e.g. variable declarations,
- has limited if any error detection,
- sometimes uses English descriptions,
- sometimes uses mathematical notation.

Algorithms and Programs

For a problem Π, we can have several algorithms.

For an algorithm $\mathcal{A}$ solving Π, we can have several programs (implementations).

Algorithms in practice: Given a problem Π

1. Design an algorithm $\mathcal{A}$ that solves Π. $\rightarrow$ **Algorithm Design**
2. Assess *correctness* and *efficiency* of $\mathcal{A}$. $\rightarrow$ **Algorithm Analysis**
3. If acceptable (correct and efficient), implement $\mathcal{A}$.

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Efficiency of Algorithms/Programs

- How do we decide which algorithm or program is the most efficient solution to a given problem?
- In this course, we are primarily concerned with the *amount of time* a program takes to run. $\rightarrow$ **Running Time**
- We also may be interested in the *amount of additional memory* the program requires. $\rightarrow$ **Auxiliary space**
- The amount of time and/or memory required by a program will depend on *Size(I)*, the size of the given problem instance I.

Running Time of Algorithms/Programs

First option: *experimental studies*

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition.
- Use a method like `clock()` (from `time.h`) to get an accurate measure of the actual running time.
- Plot/compare the results.

Running Time of Algorithms/Programs

Shortcomings of experimental studies

- Implementation may be complicated/costly.
- Timings are affected by many factors: *hardware* (processor, memory), *software environment* (OS, compiler, programming language), and *human factors* (programmer).
- We cannot test all inputs; what are good *sample inputs*?
- We cannot easily compare two algorithms/programs.

We want a framework that:

- Does not require implementing the algorithm.
- Is independent of the hardware/software environment.
- Takes into account all input instances.

We need some *simplifications*.

Overview of Algorithm Analysis

We will develop several aspects of algorithm analysis in the next slides.
To overcome dependency on hardware/software:

- Algorithms are presented in structured high-level *pseudo-code* which is language-independent.
- Analysis of algorithms is based on an *idealized computer model*.
- Instead of time, count the number of *primitive operations*
- The efficiency of an algorithm (with respect to time) is measured in terms of its *growth rate* (this is called the *complexity* of the algorithm).

Random Access Machine

Random Access Machine (RAM) model:

- A set of memory cells, each of which stores one item (word) of data. Implicit assumption: memory cells are big enough to hold the items that we store.
- Any *access to a memory location* takes constant time.
- Any *primitive operation* takes constant time. Implicit assumption: primitive operations have fairly similar, though different, running time on different systems
- The *running time* of a program is proportional to the number of memory accesses plus the number of primitive operations.

This is an idealized model, so these assumptions may not be valid for a "real" computer.

Computational model: word RAM

Rough definition:

- memory locations contain integer words of b bits each
- assume $b \geq \log(n)$ for input size n
- Random Access Memory: can access any memory location at unit cost
- basic operations on words have unit costs

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Sum$(A[1..n])$
1. $s \leftarrow 0$
2. **for** $i = 1, \ldots, n$
3. $s \leftarrow s + A[i]$

Padlet

If all entries of A fit in a word, the cost is ...

`https://padlet.com/arminjamshidpey/cs240`

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- assume $b \geq \log(n)$ for input size n
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Product($A[1..n]$)
1. $s \leftarrow 1$
2. **for** $i = 1, \ldots, n$
3. $\quad s \leftarrow s \times A[i]$

Padlet

All entries of A fit in a word. Runtime?
`https://padlet.com/arminjamshidpey/cs240`

Computational model: word RAM

Rough definition:

- memory locations contain integer words of b bits each
- assume $b \geq \log(n)$ for input size n
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More examples

- matrix multiplication algorithms (with word-size inputs) are OK
- other matrix algorithms (Gaussian elimination) need more care
- (weighted) graph algorithms (weights fit in a word) are usually OK

Running Time Simplifications

We will simplify our analysis by considering the behaviour of algorithms for large inputs sizes.

- **Example 1:** What is larger, $100n$ or $10n^2$?
- **Example 2:** What is larger, $1000000n + 200000000000000$ or $0.01n^2$?

- To simplify comparisons, use **order notation**
- Informally: ignore constants and lower order terms

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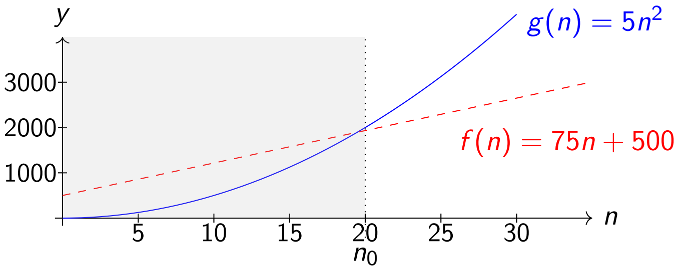
Order Notation

***O*-notation:** $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

Example: $f(n) = 75n + 500$ and $g(n) = 5n^2$ (e.g. $c = 1, n_0 = 20$)



Note: The absolute value signs in the definition are irrelevant for analysis of run-time or space, but are useful in other applications of asymptotic notation.

Example of Order Notation

In order to prove that $2n^2 + 3n + 11 \in O(n^2)$ from first principles, we need to find c and n_0 such that the following condition is satisfied:

$$0 \leq 2n^2 + 3n + 11 \leq c\, n^2 \text{ for all } n \geq n_0.$$

note that not all choices of c and n_0 will work.

Aymptotic Lower Bound

- We have $2n^2 + 3n + 11 \in O(n^2)$.
- But we also have $2n^2 + 3n + 11 \in O(n^{10})$.
- We want a *tight* asymptotic bound.

Ω-notation: $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$.

Θ-notation: $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that $c_1\,|g(n)| \leq |f(n)| \leq c_2\,|g(n)|$ for all $n \geq n_0$.

$$f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))$$

Example of Order Notation

Prove that $f(n) = 2n^2 + 3n + 11 \in \Omega(n^2)$ from first principles.

Prove that $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$ from first principles.

Prove that $\log_b(n) \in \Theta(\log n)$ for all $b > 1$ from first principles.

Strictly smaller/larger asymptotic bounds

- We have $f(n) = 2n^2 + 3n + 11 \in \Theta(n^2)$.
- How to express that $f(n)$ is *asymptotically strictly smaller* than n^3?

***o*-notation:** $f(n) \in o(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

ω-notation: $f(n) \in \omega(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(n)| \geq c\,|g(n)|$ for all $n \geq n_0$.

- Main difference to O, Ω is the quantifier for c.
- Rarely proved from first principles.

Example 1

Using first principles, prove that $2000n^2 \in o(n^n)$.

Algebra of Order Notations

Identity rule: $f(n) \in \Theta(f(n))$

Transitivity:

- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.
- If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.

Maximum rules: Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Then:

- $f(n) + g(n) \in O(\max\{f(n), g(n)\})$
- $f(n) + g(n) \in \Omega(\max\{f(n), g(n)\})$

Proof: $\max\{f(n), g(n)\} \leq f(n) + g(n) \leq 2\max\{f(n), g(n)\}$

Example 2

$f(n) \in O(g(n))$ and $g(n) \in O(h(n)) \Rightarrow f(n) \in O(h(n))$.

Techniques for Order Notation

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Suppose that

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \text{(in particular, the limit exists).}$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \omega(g(n)) & \text{if } L = \infty. \end{cases}$$

The required limit can often be computed using *l'Hôpital's rule*. Note that this result gives *sufficient* (but not necessary) conditions for the stated conclusions to hold.

Example 1

Let $f(n)$ be a polynomial of degree $d \geq 0$:

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

for some $c_d > 0$.

Then $f(n) \in \Theta(n^d)$:

Example 2

Prove that $n(2 + \sin n\pi/2)$ is $\Theta(n)$. Note that $\lim_{n\to\infty}(2 + \sin n\pi/2)$ does not exist.

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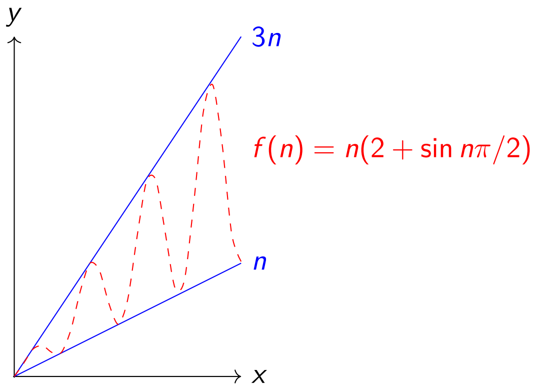

Example 3

Compare the growth rates of $\log n$ and n.

Now compare the growth rates of $(\log n)^c$ and n^d (where $c > 0$ and $d > 0$ are arbitrary numbers).

Growth rates

- If $f(n) \in \Theta(g(n))$, then the *growth rates* of $f(n)$ and $g(n)$ are the *same*.
- If $f(n) \in o(g(n))$, then we say that the growth rate of $f(n)$ is *less than* the growth rate of $g(n)$.
- If $f(n) \in \omega(g(n))$, then we say that the growth rate of $f(n)$ is *greater than* the growth rate of $g(n)$.
- Typically, $f(n)$ may be "complicated" and $g(n)$ is chosen to be a very simple function.

Common Growth Rates

Commonly encountered growth rates in analysis of algorithms include the following (in increasing order of growth rate):

- $\Theta(1)$ (*constant complexity*),
- $\Theta(\log n)$ (*logarithmic complexity*),
- $\Theta(n)$ (*linear complexity*),
- $\Theta(n \log n)$(*linearithmic*),
- $\Theta(n \log^k n)$, for some constant k (*quasi-linear*),
- $\Theta(n^2)$ (*quadratic complexity*),
- $\Theta(n^3)$ (*cubic complexity*),
- $\Theta(2^n)$ (*exponential complexity*).

How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$
- logarithmic complexity: $T(n) = c \log n$
- linear complexity: $T(n) = cn$
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$
- quadratic complexity: $T(n) = cn^2$
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- exponential complexity: $T(n) = c2^n$

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- logarithmic complexity: $T(n) = c \log n$ $\quad \rightsquigarrow \quad T(2n) = T(n) + c$.
- linear complexity: $T(n) = cn$
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- linear complexity: $T(n) = cn$ $\rightsquigarrow$ $T(2n) = 2T(n)$.
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$ $\rightsquigarrow$ $T(2n) = 2T(n) + 2cn$.
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- cubic complexity: $T(n) = cn^3$
- exponential complexity: $T(n) = c2^n$

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- exponential complexity: $T(n) = c2^n$

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How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$ $\leadsto$ $T(2n) = c$.
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- cubic complexity: $T(n) = cn^3$ $\leadsto$ $T(2n) = 8T(n)$.
- exponential complexity: $T(n) = c2^n$ $\leadsto$ $T(2n) = (T(n))^2/c$.

Relationships between Order Notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$

- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

Definitions for several parameters

Consider two functions $f(n), g(n)$ with values in $\mathbb{R}_{>0}$

O-**notation** (several parameters): $f(n, m)$ is in $O(g(n, m))$ if there exist C, n_0, m_0 such that $f(n, m) \leq Cg(n, m)$ for $n \geq n_0$ **or** $m \geq m_0$

Remark:

- less strict definition: there exist C, n_0, m_0 such that $f(n, m) \leq Cg(n, m)$ for $n \geq n_0$ **and** $m \geq m_0$
- will not matter too much which one we choose

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Techniques for Run-time Analysis

- Goal: Use asymptotic notation to simplify run-time analysis.
- Running time of an algorithm depends on the *input size* n.

```
Test1(n)
    sum ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← sum + (i − j)^2
    return sum
```

- Identify *primitive operations* that require $\Theta(1)$ time.
- The complexity of a loop is expressed as the *sum* of the complexities of each iteration of the loop.
- Nested loops: start with the innermost loop and proceed outwards. This gives *nested summations*.

Techniques for Run-time Analysis

Two general strategies are as follows.

Strategy I: Use Θ-bounds *throughout the analysis* and obtain a Θ-bound for the complexity of the algorithm.

Strategy II: Prove a O-bound and a *matching* Ω-bound *separately*.
Use upper bounds (for O-bounds) and lower bounds (for Ω-bound) early and frequently.
This may be easier because upper/lower bounds are easier to sum.

```
Test2(A, n)
    max ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← 0
            for k ← i to j do
                sum ← A[k]
    return max
```

Complexity of Algorithms

- Algorithm can have different running times on two instances of the same size.

```
Test3(A, n)
A: array of size n
    for i ← 1 to n − 1 do
        j ← i
        while j > 0 and A[j] < A[j − 1] do
            swap A[j] and A[j − 1]
            j ← j − 1
```

Let $T_{\mathcal{A}}(I)$ denote the running time of an algorithm $\mathcal{A}$ on instance I.

Worst-case complexity of an algorithm: take the worst I

Average-case complexity of an algorithm: average over I

Complexity of Algorithms

Worst-case complexity of an algorithm: The worst-case running time of an algorithm $\mathcal{A}$ is a function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ mapping n (the input size) to the *longest* running time for any input instance of size n:

$$T_{\mathcal{A}}(n) = \max\{T_{\mathcal{A}}(I) : Size(I) = n\}.$$

Average-case complexity of an algorithm: The average-case running time of an algorithm $\mathcal{A}$ is a function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ mapping n (the input size) to the *average* running time of $\mathcal{A}$ over all instances of size n:

$$T_{\mathcal{A}}^{avg}(n) = \frac{1}{|\{I : Size(I) = n\}|} \sum_{\{I : Size(I) = n\}} T_{\mathcal{A}}(I).$$

O-notation and Complexity of Algorithms

- It is important not to try and make *comparisons* between algorithms using O-notation.
- For example, suppose algorithm $\mathcal{A}_1$ and $\mathcal{A}_2$ both solve the same problem, $\mathcal{A}_1$ has worst-case run-time $O(n^3)$ and $\mathcal{A}_2$ has worst-case run-time $O(n^2)$.
- Observe that we *cannot* conclude that $\mathcal{A}_2$ is more efficient than $\mathcal{A}_1$ for all input!
  1. The worst-case run-time may only be achieved on some instances.
  2. O-notation is an upper bound. $\mathcal{A}_1$ may well have worst-case run-time $O(n)$. If we want to be able to compare algorithms, we should always use Θ-notation.

Outline

Design Idea for MergeSort

Input: Array A of n integers

- **Step 1:** We split A into two subarrays: A_L consists of the first $\lceil \frac{n}{2} \rceil$ elements in A and A_R consists of the last $\lfloor \frac{n}{2} \rfloor$ elements in A.
- **Step 2:** *Recursively* run *MergeSort* on A_L and A_R.
- **Step 3:** After A_L and A_R have been sorted, use a function *Merge* to merge them into a single sorted array.

MergeSort

```
MergeSort(A, n, ℓ ← 0, r ← n − 1, S ← NIL)
A: array of size n, 0 ≤ ℓ ≤ r ≤ n − 1
1.    if S is NIL   initialize it as array S[0..n − 1]
2.    if (r ≤ ℓ) then
3.         return
4.    else
5.         m = (r + ℓ)/2
6.         MergeSort(A, n, ℓ, m, S)
7.         MergeSort(A, n, m + 1, r, S)
8.         Merge(A, ℓ, m, r, S)
```

Two tricks to reduce run-time and auxiliary space:

- The recursion uses parameters that indicate the range of the array that needs to be sorted.
- The array used for copying is passed along as parameter.

Merge

Merge(A, ℓ, m, r, S)
$A[0..n-1]$ is an array, $A[\ell..m]$ is sorted, $A[m+1..r]$ is sorted
$S[0..n-1]$ is an array

    copy $A[\ell..r]$ into $S[\ell..r]$
    int $i_L \leftarrow \ell$; int $i_R \leftarrow m+1$;
    for $(k \leftarrow \ell; k \leq r; k{+}{+})$ **do**
        if $(i_L > m)$ $A[k] \leftarrow S[i_R{+}{+}]$
        else if $(i_R > r)$ $A[k] \leftarrow S[i_L{+}{+}]$
        else if $(S[i_L] \leq S[i_R])$ $A[k] \leftarrow S[i_L{+}{+}]$
        else $A[k] \leftarrow S[i_R{+}{+}]$

Merge takes time $\Theta(r - \ell + 1)$, i.e., $\Theta(n)$ time for merging n elements.

Analysis of MergeSort

Let $T(n)$ denote the time to run *MergeSort* on an array of length n.

- Step 1 takes time $\Theta(n)$
- Step 2 takes time $T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$
- Step 3 takes time $\Theta(n)$

The **recurrence relation** for $T(n)$ is as follows:

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + \Theta(n) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1. \end{cases}$$

It suffices to consider the following *exact recurrence*, with constant factor c replacing Θ's:

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

Analysis of MergeSort

- The following is the corresponding **sloppy recurrence** (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- The exact and sloppy recurrences are *identical* when n is a power of 2.
- The recurrence can easily be solved by various methods when $n = 2^j$. The solution has growth rate $T(n) \in \Theta(n \log n)$.
- It is possible to show that $T(n) \in \Theta(n \log n)$ *for all n* by analyzing the exact recurrence.

The Master Method

The "Master Theorem" provides a formula for the solution of many recurrence relations typically encountered in the analysis of divide-and-conquer algorithms.

The following is a simplified version:

Theorem (Master theorem)

Suppose that $a \geq 1$ *and* $b > 1$*. Consider the recurrence*

$$T(n) = a\,T\left(\frac{n}{b}\right) + \Theta(n^y)$$

in sloppy or exact form. Denote $x = \log_b a$*. Then*

$$T(n) \in \begin{cases} \Theta(n^x) & \text{if } y < x \\ \Theta(n^x \log n) & \text{if } y = x \\ \Theta(n^y) & \text{if } y > x. \end{cases}$$

Some Recurrence Relations

Recursion	resolves to	example
$T(n) = T(n/2) + \Theta(1)$	$T(n) \in \Theta(\log n)$	Binary search
$T(n) = 2T(n/2) + \Theta(n)$	$T(n) \in \Theta(n \log n)$	Mergesort
$T(n) = 2T(n/2) + \Theta(\log n)$	$T(n) \in \Theta(n)$	Heapify (*)
$T(n) = T(cn) + \Theta(n)$ for some $0 < c < 1$	$T(n) \in \Theta(n)$	Selection (*)
$T(n) = 2T(n/4) + \Theta(1)$	$T(n) \in \Theta(\sqrt{n})$	Range Search (*)
$T(n) = T(\sqrt{n}) + \Theta(\sqrt{n})$	$T(n) \in \Theta(\sqrt{n})$	Interpol. Search (*)
$T(n) = T(\sqrt{n}) + \Theta(1)$	$T(n) \in \Theta(\log \log n)$	Interpol. Search (*)

- Once you know the result, it is (usually) easy to prove by induction.
- Many more recursions, and some methods to find the result, in cs341.

(*) These will be studied later in the course.

Outline

Order Notation Summary

***O*-notation:** $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

Ω-notation: $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$.

Θ-notation: $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that $c_1\,|g(n)| \leq |f(n)| \leq c_2\,|g(n)|$ for all $n \geq n_0$.

***o*-notation:** $f(n) \in o(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

ω-notation: $f(n) \in \omega(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$.

Useful Sums

Arithmetic sequence:

$\sum_{i=0}^{n-1} i = ???$

$\sum_{i=0}^{n-1}(a + di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2)$ if $d \neq 0$.

Geometric sequence:

$\sum_{i=0}^{n-1} 2^i = ???$

$$\sum_{i=0}^{n-1} a\,r^i = \begin{cases} a\dfrac{r^n - 1}{r - 1} & \in \Theta(r^{n-1}) & \text{if } r > 1 \\ na & \in \Theta(n) & \text{if } r = 1 \\ a\dfrac{1 - r^n}{1 - r} & \in \Theta(1) & \text{if } 0 < r < 1. \end{cases}$$

Harmonic sequence:

$\sum_{i=1}^{n} \frac{1}{i} = ???$

$H_n := \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$

A few more:

$\sum_{i=1}^{n} \frac{1}{i^2} = ???$

$\sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6} \in \Theta(1)$

$\sum_{i=1}^{n} i^k = ???$

$\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$ for $k \geq 0$

Useful Math Facts

Logarithms:

- $c = \log_b(a)$ means $b^c = a$. E.g. $n = 2^{\log n}$.
- $\log(a)$ (in this course) means $\log_2(a)$
- $\log(a \cdot c) = \log(a) + \log(c)$, $\log(a^c) = c\log(a)$, $\log x \leq x$
- $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$, $a^{\log_b c} = c^{\log_b a}$
- $\ln(x) =$ natural log $= \log_e(x)$, $\frac{d}{dx}\ln x = \frac{1}{x}$
- concavity: $\alpha \log x + (1-\alpha)\log y \leq \log(\alpha x + (1-\alpha)y)$ for $0 \leq \alpha \leq 1$

Factorial:

- $n! := n(n-1)(n-2)\cdots 2 \cdot 1 =$ # ways to permute n elements
- $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$

Probability and moments:

- $E[aX] = aE[X]$, $E[X+Y] = E[X] + E[Y]$ (linearity of expectation)