CS 240 – Data Structures and Data Management

Module 2E: Priority Queues - Enriched

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Based on lecture notes by many previous cs240 instructors

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Outline

2. Merging heaps
   - More PQ operations
   - Meldable Heaps
   - Binomial Heaps

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Merging Priority Queues

New operation: *merge*(P_1, P_2)

- Given: two priority queues P_1, P_2 of size n_1 and n_2.
- Want: One priority queue P that contains all their items

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Can we do it *faster* if PQs are stored as trees?

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Can we do it *faster* if PQs are stored as trees?

Three approaches (where $n = n_1 + n_2$):

- Merge binary heaps (stored as trees).
  $O(\log^3 n)$ worst-case time (no details)
- Merge *meldable heaps* that have heap-property (but not structural property). $O(\log n)$ *expected* run-time.
- Merge *binomial heaps* that have a different structural property.
  $O(\log n)$ *worst-case* run-time.

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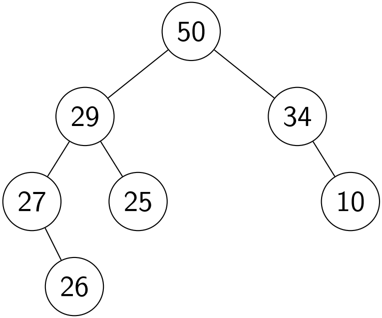
Meldable Heaps

- Priority queue stored as binary tree
- Heap-order-property: Parent no smaller than child.
- No structural property; any binary tree allowed.
- *Tree-based:* Store nodes and references to *left*/*right*



PQ-operations in Meldable Heaps

Both *insert* and *deleteMax* can be done by *reduction* to *merge*.

P.*insert*(k, v):

- Create a 1-node meldable heap P' that stores (k, v).
- Merge P' with P.

P.*deleteMax*():

- Stash item that is at root.
- Let P_ℓ and P_r be left and right sub-heap of root.
- Update $P \leftarrow merge(P_\ell, P_r)$
- Return stashed item.

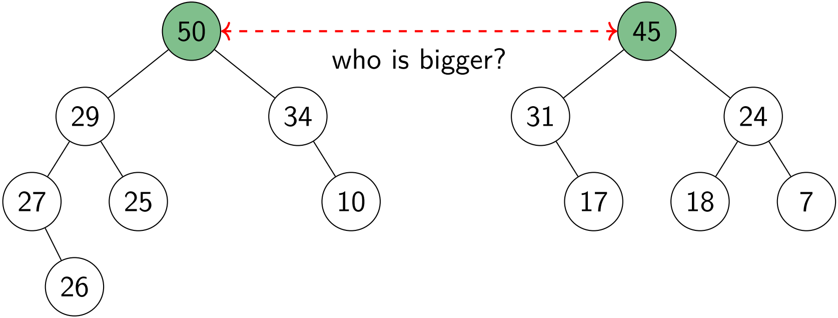
Both operations have run-time $O(merge)$.

Merging Meldable Heaps

- Idea: Merge heap with smaller root into other one, *randomly* choose into which sub-heap to merge.
- Structural property not maintained

meldableHeap::merge(r_1, r_2)
r_1, r_2: *roots of two heaps (possibly* `NIL`*)*
returns root of merged heap
1. **if** r_1 is `NIL` **return** r_2
2. **if** r_2 is `NIL` **return** r_1
3. **if** $r_1.key < r_2.key$ *swap*(r_1, r_2)
4. `//` `now` r_1 `has max-key and becomes the root.`
5. *randomly* pick one child c of r_1
6. replace subheap at c by *heapMerge*(c, r_2)
7. **return** r_1

Merge Example



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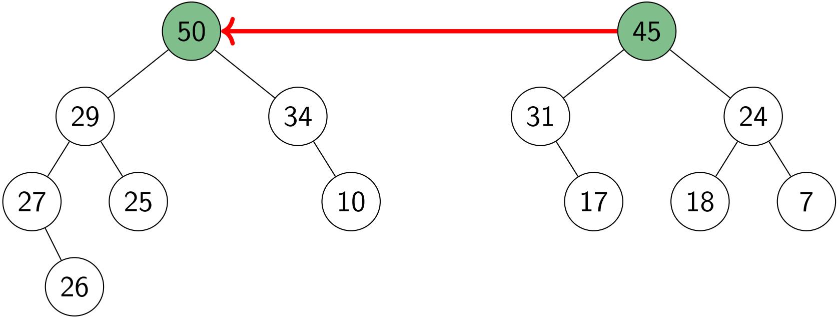

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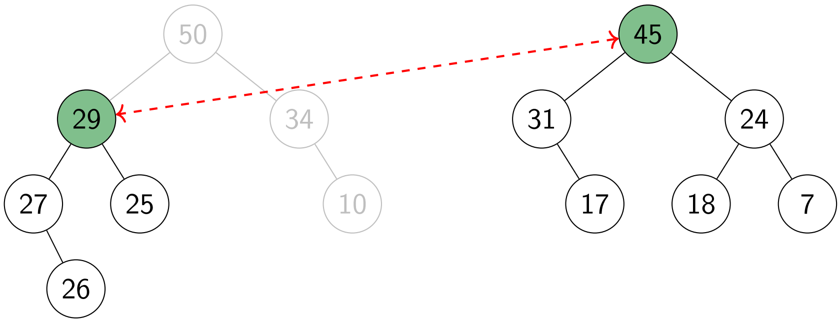

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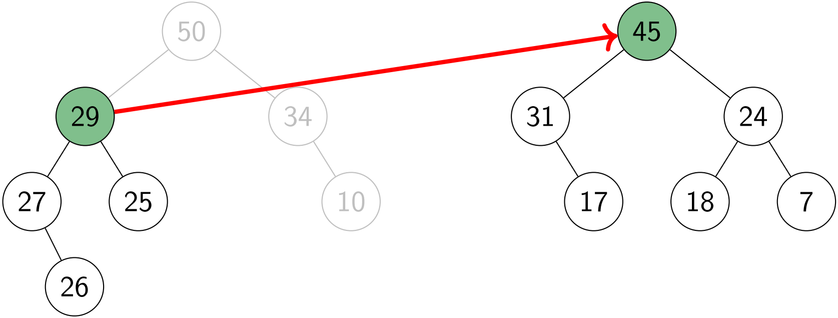

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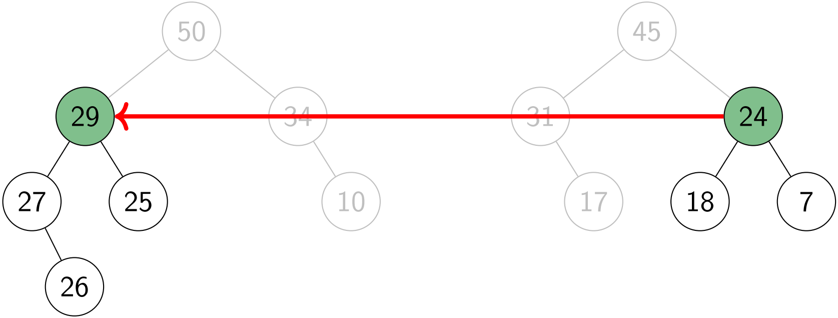

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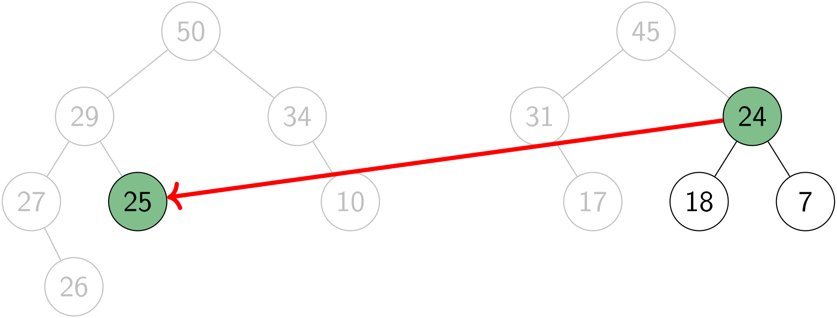

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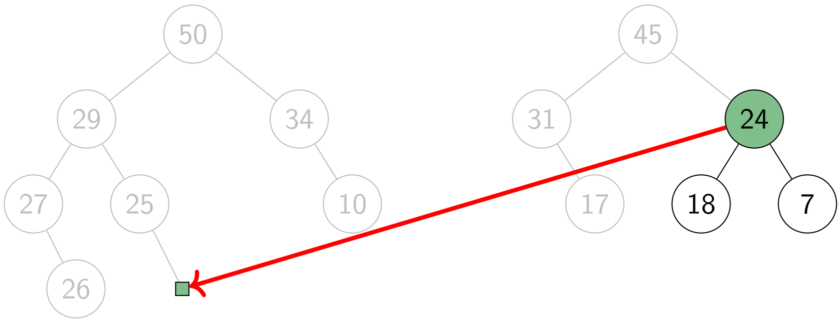

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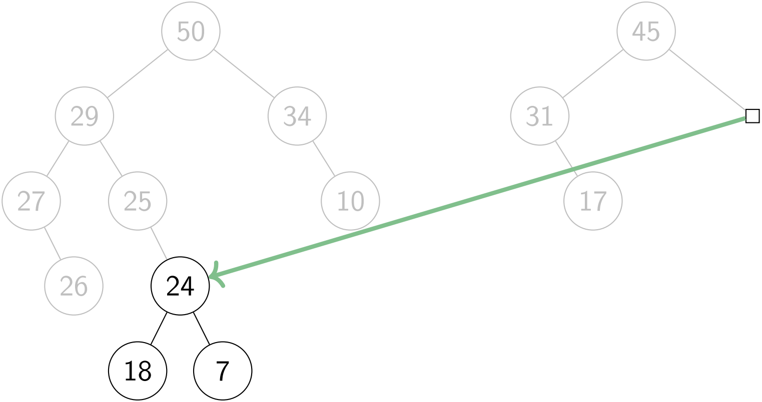

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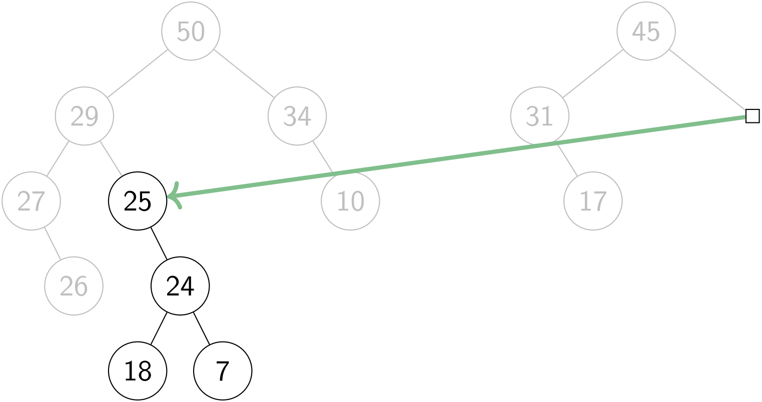

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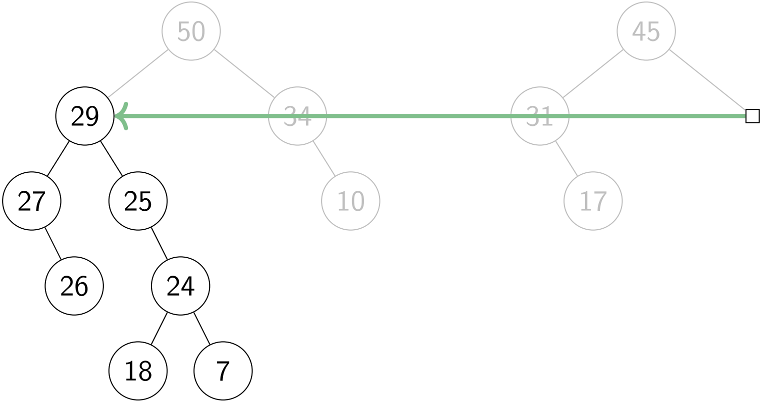

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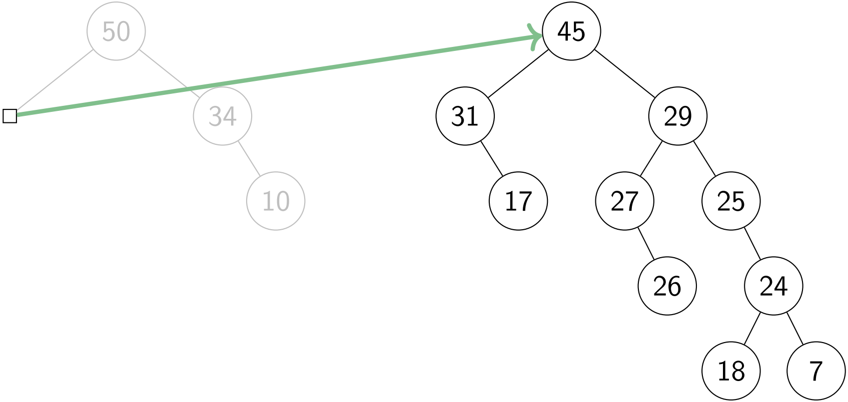

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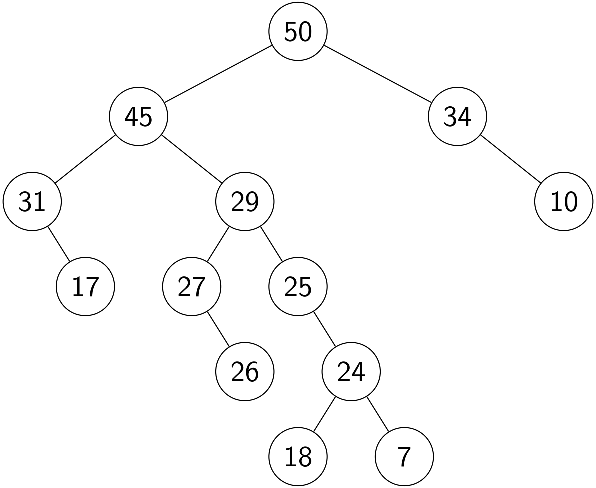

Merging meldable heaps

Run-time? Not more than two *random downward walks* in a binary tree.

Let $T(n)$ = expected length of a random downward walk.

Theorem: $T(n) \in O(\log n)$.
Proof:

So *merge* (and also *insert* and *deleteMax*) takes $O(\log n)$ expected time.

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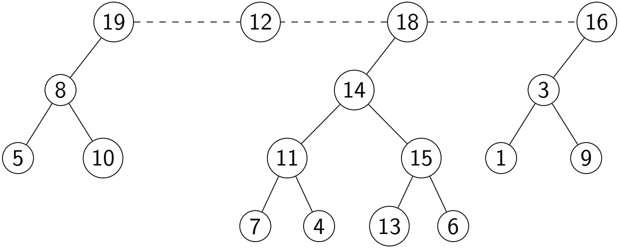
Binomial Heaps

Very different structure from binary heaps and meldable heaps:



- List L of binary trees.
- Each binary tree is a **flagged tree**:
  Complete binary tree T plus root r that has T as left subtree
  - Flagged tree of height h has 2^h nodes.
  - So $h \leq \log n$ for all flagged trees.
- Order-property: Nodes in *left* subtree have no-smaller keys.
  (No restrictions on nodes in the right subtree.)

Binomial Heap Operations

- *insert*: Reduce to *merge* as before.
- *findMax*:
  - At each flag tree, root contains the maximum.
  - Search roots in $L \Rightarrow O(|L|)$ time.
- We want L to be short.

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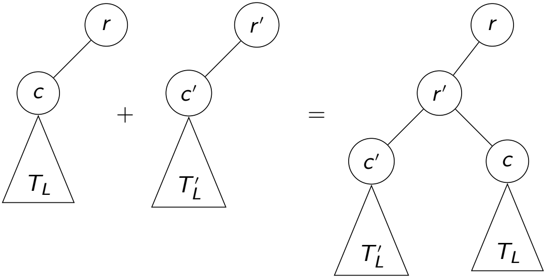
- **Proper binomial heap:** No two flagged trees have the same height.
- **Observation:** A proper binomial heap has $|L| \leq \log n + 1$.
  - The flagged tree of largest height h has $h \leq \log n$.
  - Can have only one flagged tree of each height in $\{0, \ldots, h\}$.

Making Binomial Heaps Proper

- Goal: Given a binomial heap, make it proper.
- Need subroutine: combine two flagged trees of the same height. This can be done in constant time. If $r.key \geq r'.key$:

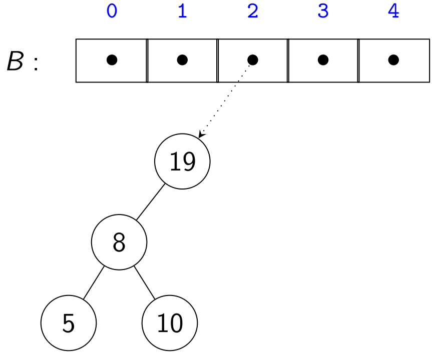


- Idea: Do this whenever two flagged trees have same height.
- Run-time to make proper: $O(|L| + \log n)$ if implemented suitably.

Making Binomial Heaps Proper

	0	1	2	3	4
B :	•	•	•	•	•

Making Binomial Heaps Proper



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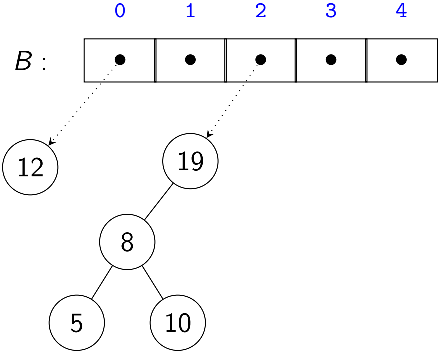

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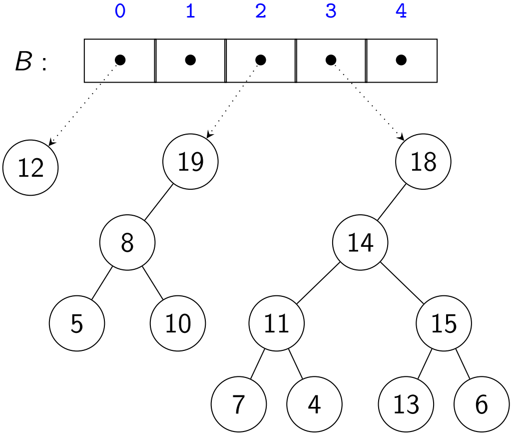

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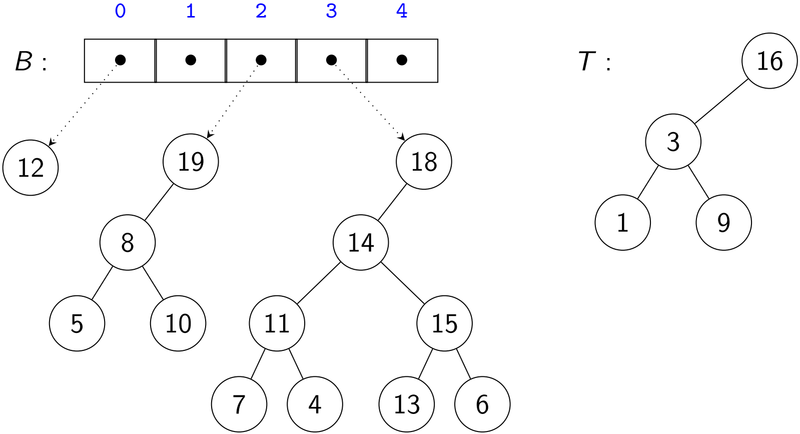

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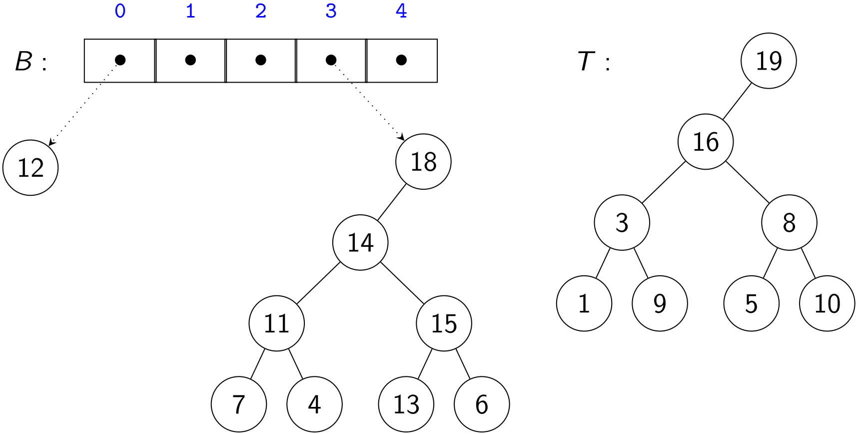

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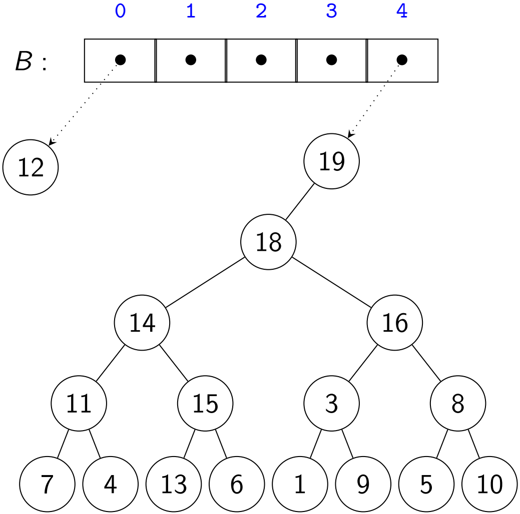


Making Binomial Heaps Proper

binomialHeap::makeProper()

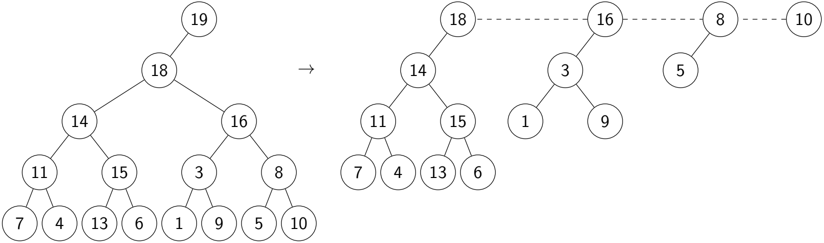
```
n ← size of the binomial heap
compute ℓ ← ⌊log n⌋
B ← array of size ℓ + 1, initialized all-NIL
L ← list of flagged trees
while L is non-empty do
    T ← L.pop(), h ← T.height
    while T' ← B[h] is not NIL do
        if T.root.key < T'.root.key do swap T and T'
        // combine T with T'
        T'.right ← T.left, T.left ← T', T.height ← h+1
        B[h] ← NIL, h++
    B[h] ← T
// copy B back to list
for (h = 0; h ≤ ℓ; h++) do
    if B[h] ≠ NIL do L.append(B[h])
```

Binomial Heap Operations

- **Idea:** Make binomial heap proper after *every* opration.
  - $\Rightarrow$ L always has length $O(\log n)$
  - $\Rightarrow$ Each *makeProper* takes $O(\log n)$ time
- *findMax*: $O(\log n)$ worst-case time.
- *merge*: $O(\log n)$ worst-case time.
  - Concatenate the two lists.
  - Call *makeProper*.
- *insert*: $O(\log n)$ worst-case time via *merge*.
- *deleteMax*?

deleteMax in a binomial heap

- Search for maximum among roots, say it is in tree T
- Split $T \setminus \{\text{root}\}$ into into flagged trees $T_1, \ldots, T_k$



- Merge $L \setminus T$ with $\{T_1, \ldots, T_k\}$
- Have $k \leq \log n \Rightarrow O(\log n)$ worst-case time.

Summary: All operations have $O(\log n)$ worst-case run-time.