CS 240 – Data Structures and Data Management

Module 5: Other Dictionary Implementations

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Based on lecture notes by many previous cs240 instructors

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Winter 2023

version *2023-02-05 15:18*

Outline

5. Dictionaries with Lists revisited
   - Dictionary ADT: Implementations thus far
   - Skip Lists
   - Re-ordering Items

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Dictionary ADT: Implementations thus far

A *dictionary* is a collection of key-value pairs (KVPs), supporting operations *search*, *insert*, and *delete*.

Realizations we have seen so far:

- **Unordered array or linked list**: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- **Ordered array**: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- **Binary search trees**: $\Theta(\mathit{height})$ search, insert and delete
- **Balanced BST** (AVL trees):
  $\Theta(\log n)$ search, insert, and delete

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Improvements/Simplifications?

- **Can show:** If the KVPs were inserted in random order, then the expected height of the binary search tree would be $O(\log n)$.
- How can we use randomization within the data structure to mirror what would happen on random input?

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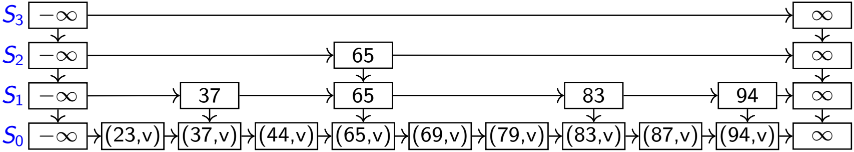
Skip Lists

- A hierarchy S of ordered linked lists (*levels*) $S_0, S_1, \cdots, S_h$:
  - Each list S_i contains the special keys $-\infty$ and $+\infty$ (sentinels)
  - List S_0 contains the KVPs of S in non-decreasing order.
    (The other lists store only keys, or links to nodes in S_0.)
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List S_h contains only the sentinels; the left sentinel is the *root*



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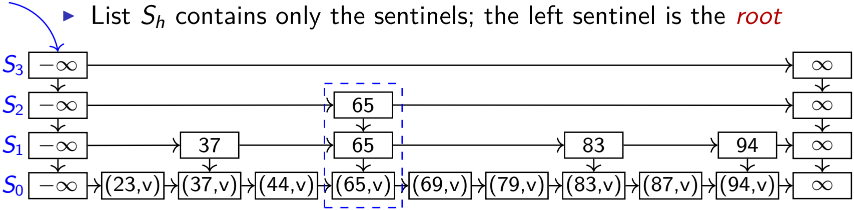


- Each KVP belongs to a **tower** of nodes
- There are (usually) more *nodes* than *keys*
- The skip list consists of a reference to the topmost left node.
- Each node p has references *p.after* and *p.below*

Search in Skip Lists

For each level, find **predecessor** (node before where k would be).
This will also be useful for *insert*/*delete*.

getPredecessors (k)

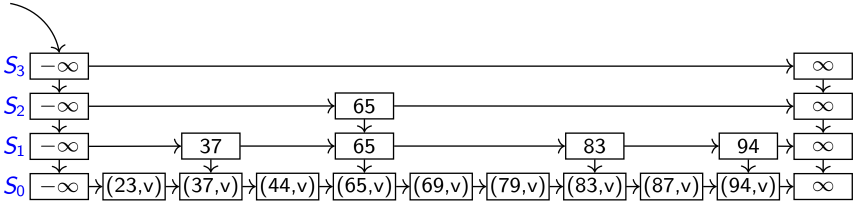
$p \leftarrow$ root
$P \leftarrow$ stack of nodes, initially containing p
while $p.below \neq \texttt{NIL}$ **do**
    $p \leftarrow p.below$
    while $p.after.key < k$ **do** $p \leftarrow p.after$
    $P.push(p)$
return P

skipList::search (k)

$P \leftarrow$ *getPredecessors*(k)
$p_0 \leftarrow P.top()$ // predecessor of k in S_0
if $p_0.after.key = k$ **return** $p_0.after$
else return "not found, but would be after p_0"

Example: Search in Skip Lists

Example: *search*(87)



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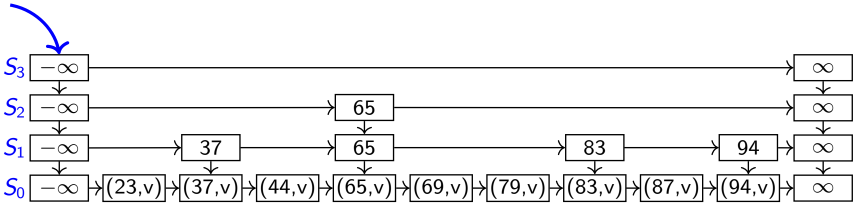

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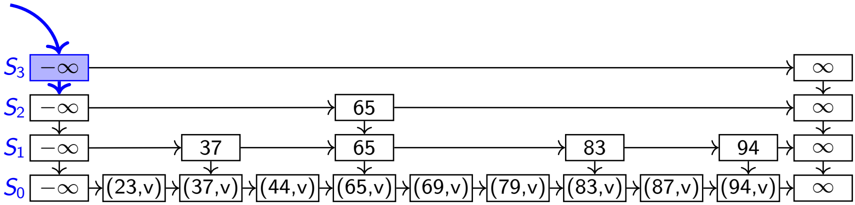

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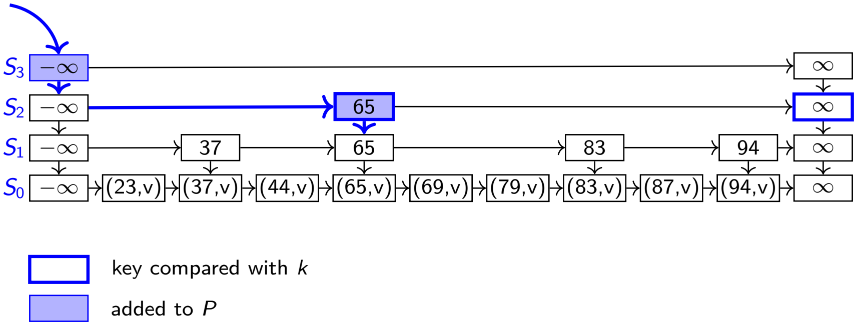

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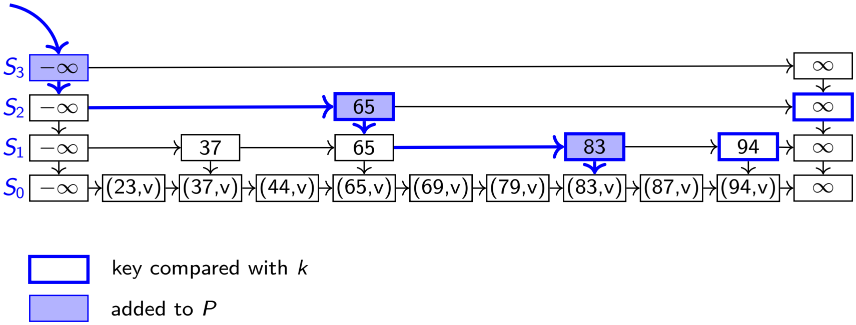

Example: Search in Skip Lists

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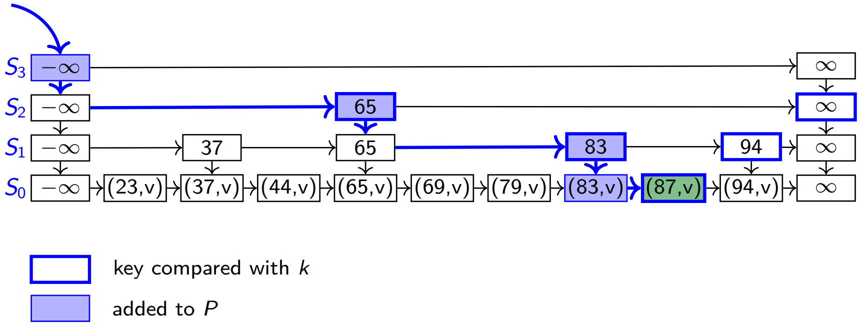


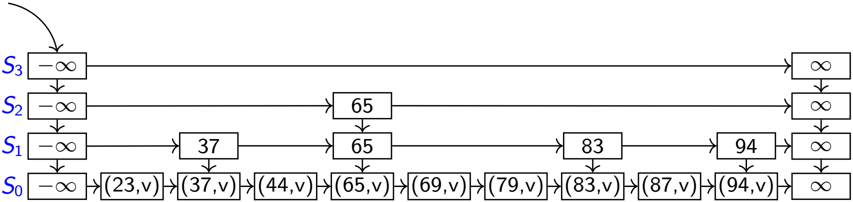
Insert in Skip Lists

skipList::insert(k, v)

- Randomly repeatedly toss a coin until you get tails
  - Negative binomial distribution
- Let i the number of times the coin came up heads
  - we want k to be in lists $S_0, \ldots, S_i$.
  - $i \rightarrow$ *height* of tower of k
  - $P(\text{tower of key } k \text{ has height} \geq i) = \left(\frac{1}{2}\right)^i$
- Increase height h of skip list, if needed, to have $h > i$ levels.
- Use *getPredecessors*(k) to get stack P.
  The top i items of P are the predecessors $p_0, p_1, \cdots, p_i$ of where k should be in each list $S_0, S_1, \cdots, S_i$
- Insert (k, v) after p_0 in S_0, and k after p_j in S_j for $1 \leq j \leq i$

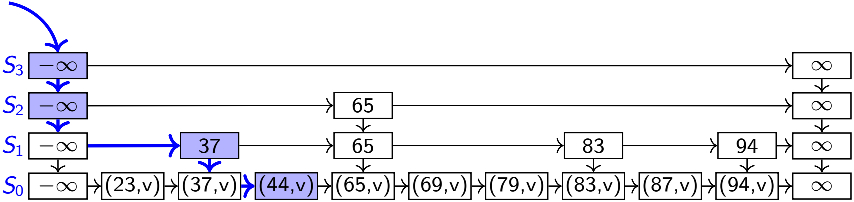
Example: Insert in Skip Lists

Example: *skipList::insert*$(52, v)$
Coin tosses: H,T $\Rightarrow i = 1$



Example: Insert in Skip Lists

Example: *skipList::insert*$(52, v)$
Coin tosses: H,T $\Rightarrow i = 1$
getPredecessors(52)



Example: Insert in Skip Lists

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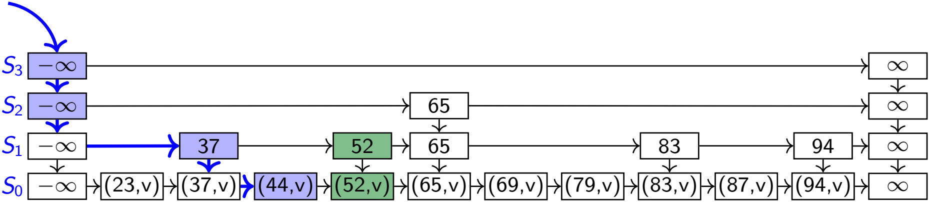
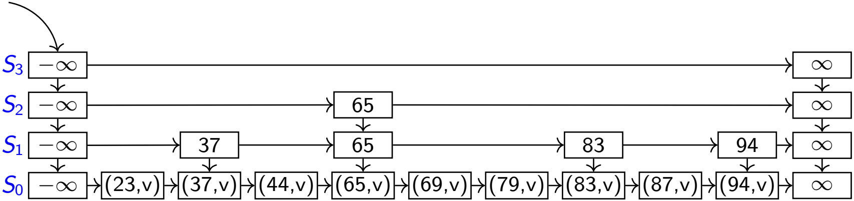


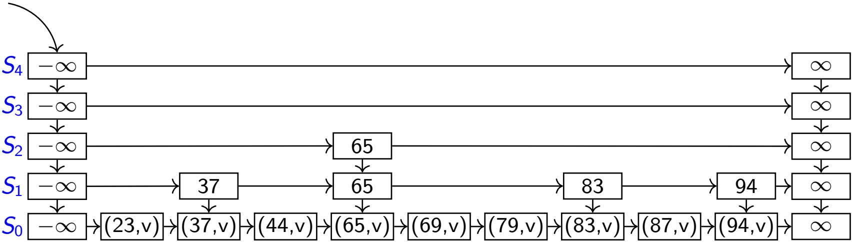
Example 2: Insert in Skip Lists

Example: *skipList::insert*$(100, v)$
Coin tosses: H,H,H,T $\Rightarrow i = 3$



Example 2: Insert in Skip Lists

Example: *skipList::insert*$(100, v)$
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Height increase



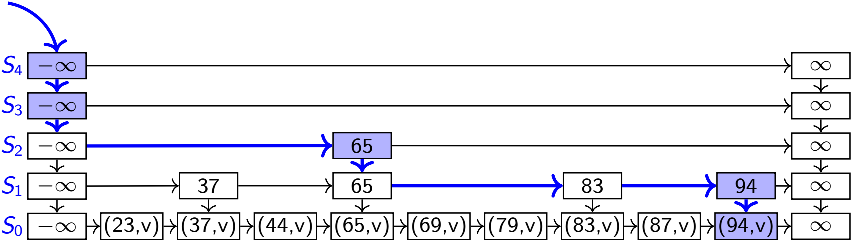
Example 2: Insert in Skip Lists

Example: *skipList::insert*$(100, v)$

Coin tosses: H,H,H,T $\Rightarrow i = 3$

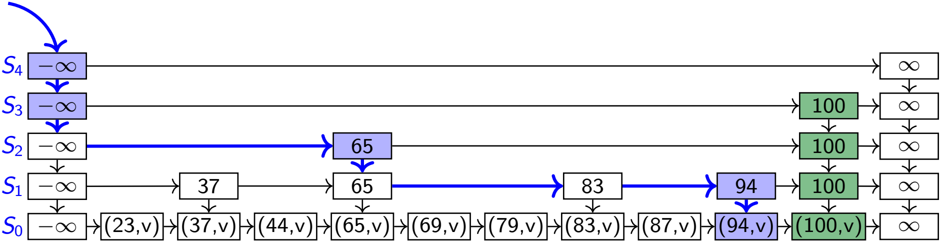
Height increase

getPredecessors(100)



Example 2: Insert in Skip Lists

Example: *skipList::insert*(100, v)
Coin tosses: H,H,H,T $\Rightarrow i = 3$
Height increase
getPredecessors(100)



Insert in Skip Lists

skipList::insert(k, v)

$P \leftarrow$ *getPredecessors*(k)
for $(i \leftarrow 0;\ random(2) = 1; i \leftarrow i{+}1)\ \{\}$ // random tower height
while $i \geq P.size()$ // increase skip-list height?
  root $\leftarrow$ new sentinel-only list, linked in appropriately
  add left sentinel of root at bottom of stack P
$p \leftarrow P.pop()$ // insert (k, v) in S_0
$z_{below} \leftarrow$ new node with (k, v), inserted after p
while $i > 0$ // insert k in $S_1, \ldots, S_i$
  $p \leftarrow P.pop()$
  $z \leftarrow$ new node with k added after p
  $z.below \leftarrow z_{below}$; $z_{below} \leftarrow z$
  $i \leftarrow i - 1$

Delete in Skip Lists

It is easy to remove a key since we can find all predecessors.
Then eliminate layers if there are multiple ones with only sentinels.

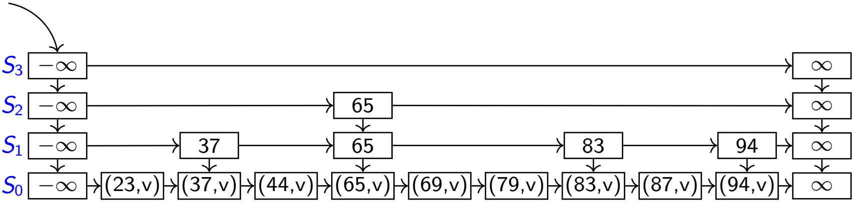
skipList::delete(k)

```
P ← getPredecessors(k)
while P is non-empty
    p ← P.pop()          // predecessor of k in some layer
    if p.after.key = k
        p.after ← p.after.after
    else break           // no more copies of k

p ← left sentinel of the root-list
while p.below.after is the ∞-sentinel
    // the two top lists are both only sentinels, remove one
    p.below ← p.below.below
    p.after.below ← p.after.below.below
```

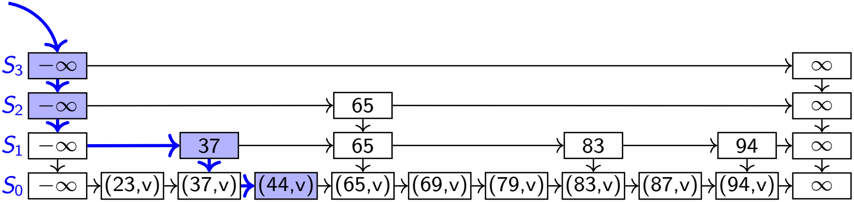
Example: Delete in Skip Lists

Example: *skipList::delete*(65)



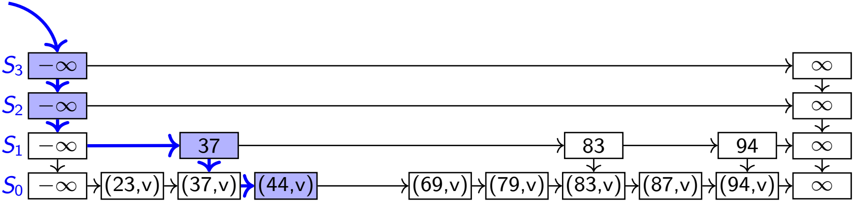
Example: Delete in Skip Lists

Example: *skipList::delete*(65)
getPredecessors(65)



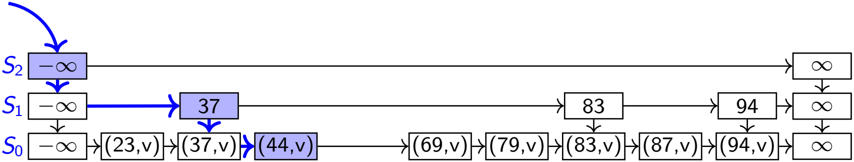
Example: Delete in Skip Lists

Example: *skipList::delete*(65)
getPredecessors(65)



Example: Delete in Skip Lists

Example: *skipList::delete*(65)
getPredecessors(65)
Height decrease

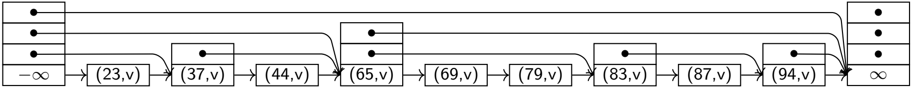


Analysis of Skip Lists

- Expected **space** usage: $O(n)$
- Expected **height**: $O(\log n)$
- Crucial for all operations:
  - How often do we *drop down* (execute $p \leftarrow p.below$)?
  - How often do we *step forward* (execute $p \leftarrow p.after$)?
- *skipList::search*: $O(\log n)$ expected time
  - $\#$ drop-downs $=$ height
  - expected $\#$ forward-steps is ≤ 1 in each level
  - expected total $\#$ forward-steps is in $O(\log n)$
- *skipList::insert*: $O(\log n)$ expected time
- *skipList::delete*: $O(\log n)$ expected time

Summary of Skip Lists

- $O(n)$ expected space, all operations take $O(\log n)$ expected time.
- As described they are no faster than randomized binary search trees.
- Can show: A biased coin-flip to determine tower-height gives smaller expected run-times.
- Can save links (hence space) by implementing towers as array.



- Then skip lists are simple to implement. They are fast with good cache locality but can still suffer from cache misses.

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   - Re-ordering Items

Re-ordering Items

- Recall: Unordered list/array implementation of ADT Dictionary
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Lists/arrays are a very simple and popular implementation. Can we do something to make search more effective in practice?

Re-ordering Items

- Recall: Unordered list/array implementation of ADT Dictionary
  search: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Lists/arrays are a very simple and popular implementation. Can we do something to make search more effective in practice?
- No: if items are accessed equally likely
- Yes: otherwise (we have a probability distribution of the items)
  - Intuition: Frequently accessed items should be in the front.
  - Two cases: Do we know the access distribution beforehand or not?
  - For short lists or extremely unbalanced distributions this may be faster than AVL trees or Skip Lists, and much easier to implement.

Optimal Static Ordering

Example:

key	A	B	C	D	E
frequency of access	2	8	1	10	5
access-probability	$\frac{2}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{5}{26}$

- We count cost i for accessing the key in the ith position.
- Order A, B, C, D, E has expected access cost
  $\frac{2}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{1}{26} \cdot 3 + \frac{10}{26} \cdot 4 + \frac{5}{26} \cdot 5 = \frac{86}{26} \approx 3.31$
- Order D, B, E, A, C has expected access cost
  $\frac{10}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{5}{26} \cdot 3 + \frac{2}{26} \cdot 4 + \frac{1}{26} \cdot 5 = \frac{66}{26} \approx 2.54$

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- **Claim:** Over all possible static orderings, the one that sorts items by non-increasing access-probability minimizes the expected access cost.
- **Proof Idea:** For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Dynamic Ordering: MTF

- What if we do *not know the access probabilities* ahead of time?
- Rule of thumb (**temporal locality**): A recently accessed item is likely to be used soon again.
- In list: Always insert at the front
- **Move-To-Front heuristic** (MTF): Upon a successful search, move the accessed item to the front of the list

$$A \to B \to C \to D \to E$$

$$\downarrow \text{search(D)}$$

$$D \to A \to B \to C \to E$$

$$\downarrow \text{insert(F)}$$

$$F \to D \to A \to B \to C \to E$$

- We can also do MTF on an array, but should then insert and search from the *back* so that we have room to grow.

Dynamic Ordering: Transpose

Transpose heuristic: Upon a successful search, swap the accessed item with the item immediately preceding it

A → B → C → D → E

↓ search(D)

A → B → D → C → E

↓ insert(F)

F → A → B → D → C → E

Dynamic Ordering: Transpose

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$$\downarrow \text{search}(D)$$

$$A \to B \to D \to C \to E$$

$$\downarrow \text{insert}(F)$$

$$F \to A \to B \to D \to C \to E$$

Performance of dynamic ordering:

- Transpose does not adapt quickly to changing access patterns.
- MTF works well in practice.
- **Can show:** MTF is "2-competitive":
  No more than twice as bad as the optimal static ordering.