# CS 240 – Data Structures and Data Management

## Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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# Outline

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# Lower bound for search

The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among $n$ items. Is this the best possible?

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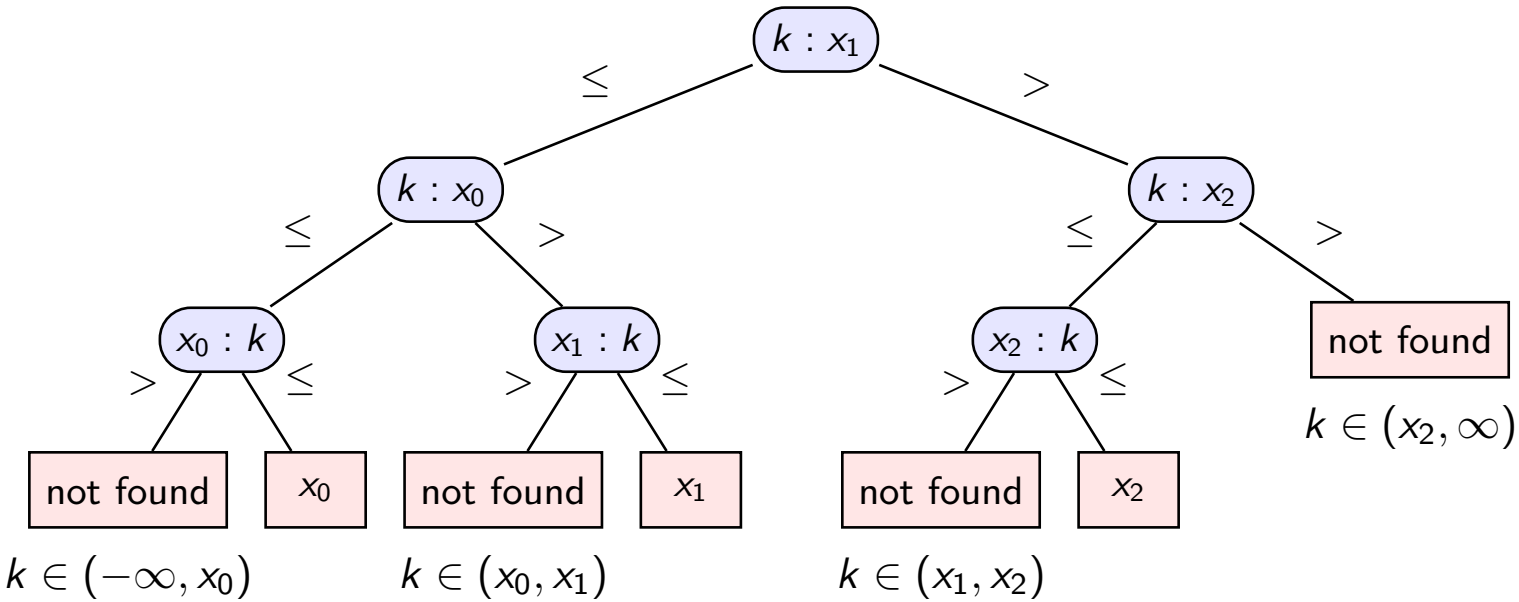
The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among $n$ items. Is this the best possible?

**Theorem**: In the comparison model (on the keys),
$\Omega(\log n)$ comparisons are required to search a size-$n$ dictionary.

**Proof**: via decision tree for items $x_0, \ldots, x_{n-1}$




But can we beat the lower bound for special keys?

# Outline

# Binary Search

Recall the run-times in a *sorted array*:

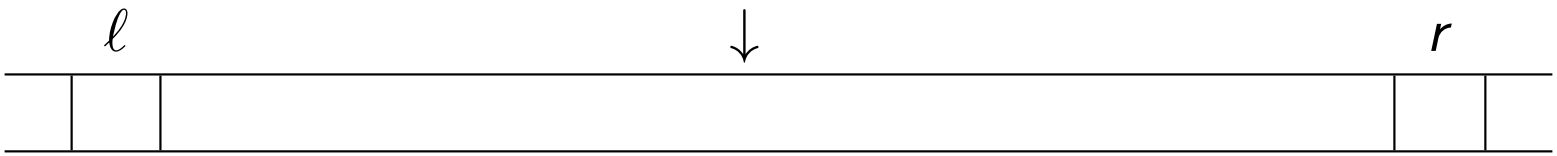
- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

*binary-search*$(A, n, k)$
$A$: Sorted array of size $n$, $k$: key

1. $\ell \leftarrow 0$, $r \leftarrow n-1$
2. **while** $(\ell \leq r)$
3. $\quad m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
4. $\quad$ **if** $(A[m] == k)$ **then return** "found at $A[m]$"
5. $\quad$ **else if** $(A[m] < k)$ **then** $\ell \leftarrow m+1$
6. $\quad$ **else** $r \leftarrow m-1$
7. **return** "not found, but would be between $A[\ell-1]$ and $A[\ell]$"

# Interpolation Search: Motivation

*binary-search*$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r-\ell) \rfloor$



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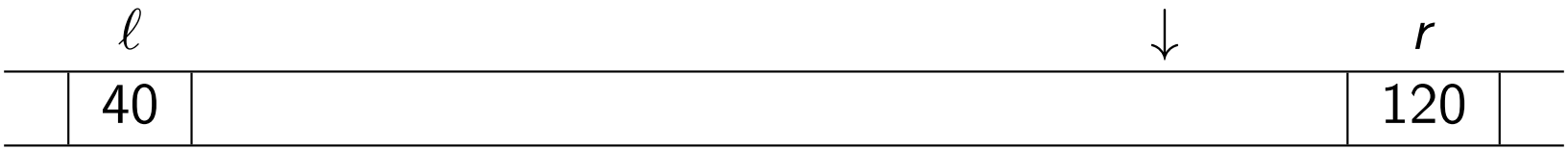


**Question**: If keys are *numbers*, where would you expect key $k = 100$?

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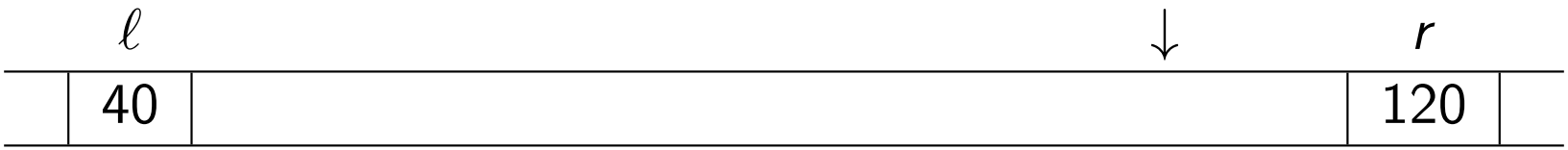



**Question**: If keys are *numbers*, where would you expect key $k = 100$?

*interpolation-search*$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k-A[\ell]}{A[r]-A[\ell]}(r-\ell) \right\rfloor$

# Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash during computation of $m$.

*interpolation-search*$(A, n, k)$
$A$: Sorted array of size $n$, $k$: key

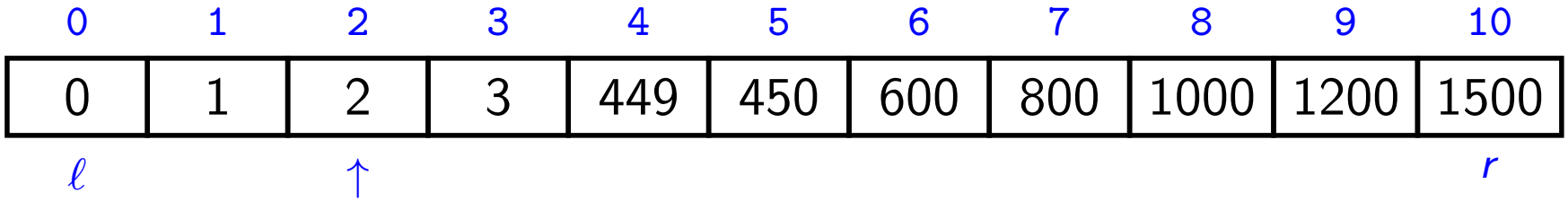
1. $\ell \leftarrow 0$, $r \leftarrow n-1$
2. **while** $(\ell \leq r)$
3. **if** $(k < A[\ell]$ or $k > A[r])$ **return** "not found"
4. **if** $(k = A[r])$ **then return** "found at $A[r]$"
5. $m \leftarrow \ell + \lfloor \frac{k - A[\ell])}{A[r] - A[\ell]} \cdot (r - \ell) \rfloor$
6. **if** $(A[m] == k)$ **then return** "found at $A[m]$"
7. **else if** $(A[m] < k)$ **then** $\ell \leftarrow m + 1$
8. **else** $r \leftarrow m - 1$
9. // We always return from somewhere within while-loop

# Interpolation Search Example

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 449 | 450 | 600 | 800 | 1000 | 1200 | 1500 |

*interpolation-search*(A[0..10],449):

# Interpolation Search Example




*interpolation-search*(A[0..10],449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$

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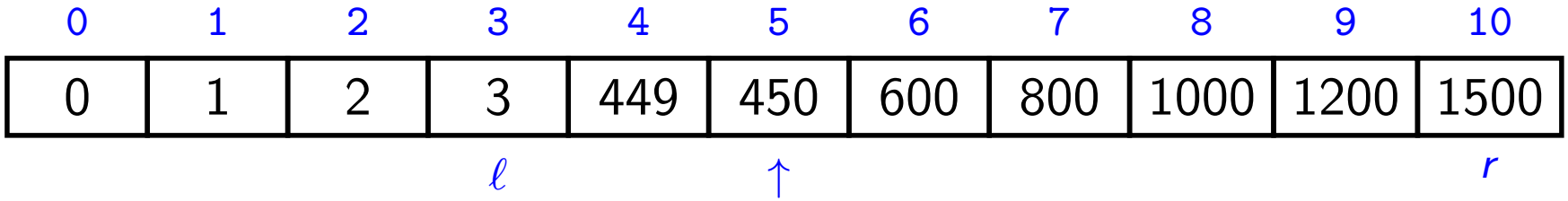


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- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$

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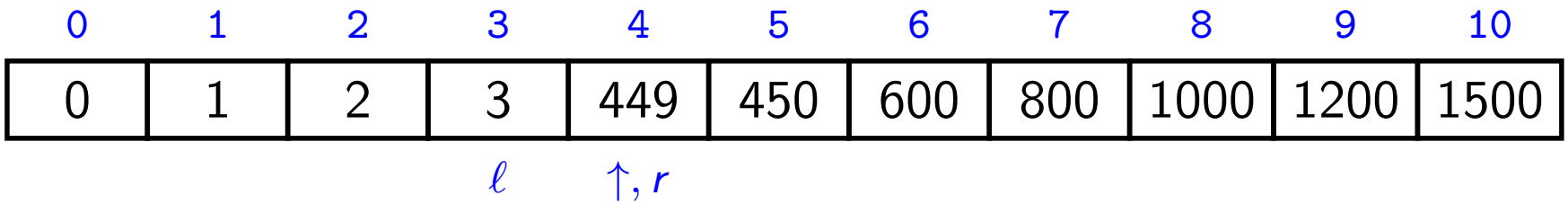


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- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, found at $A[4]$

# Interpolation Search Example

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Works well if keys are *uniformly* distributed:

- Can show: Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

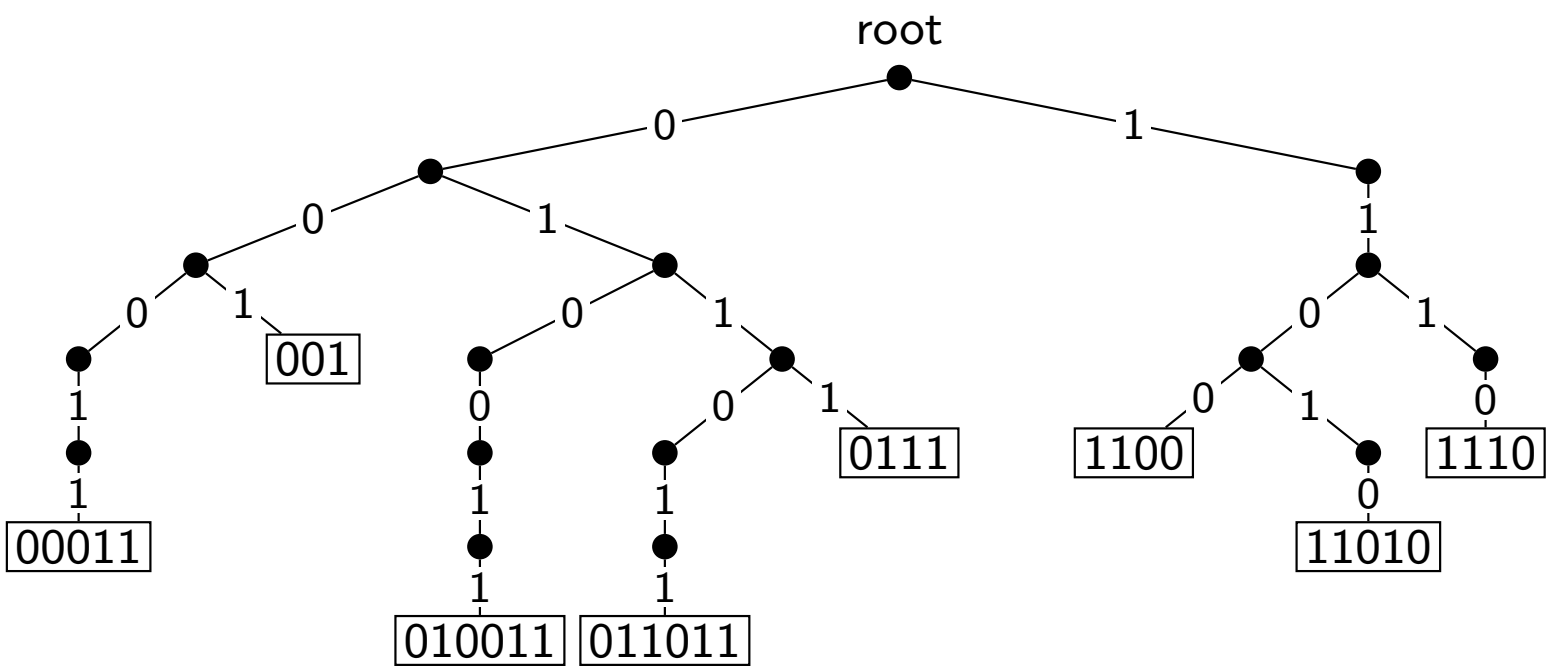
But: Worst case performance $\Theta(n)$

# Outline

# Tries: Introduction

**Trie** (also know as **radix tree**): A dictionary for bitstrings.
(Should know: string, word, $|w|$, alphabet, prefix, suffix, comparing words,....)

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key

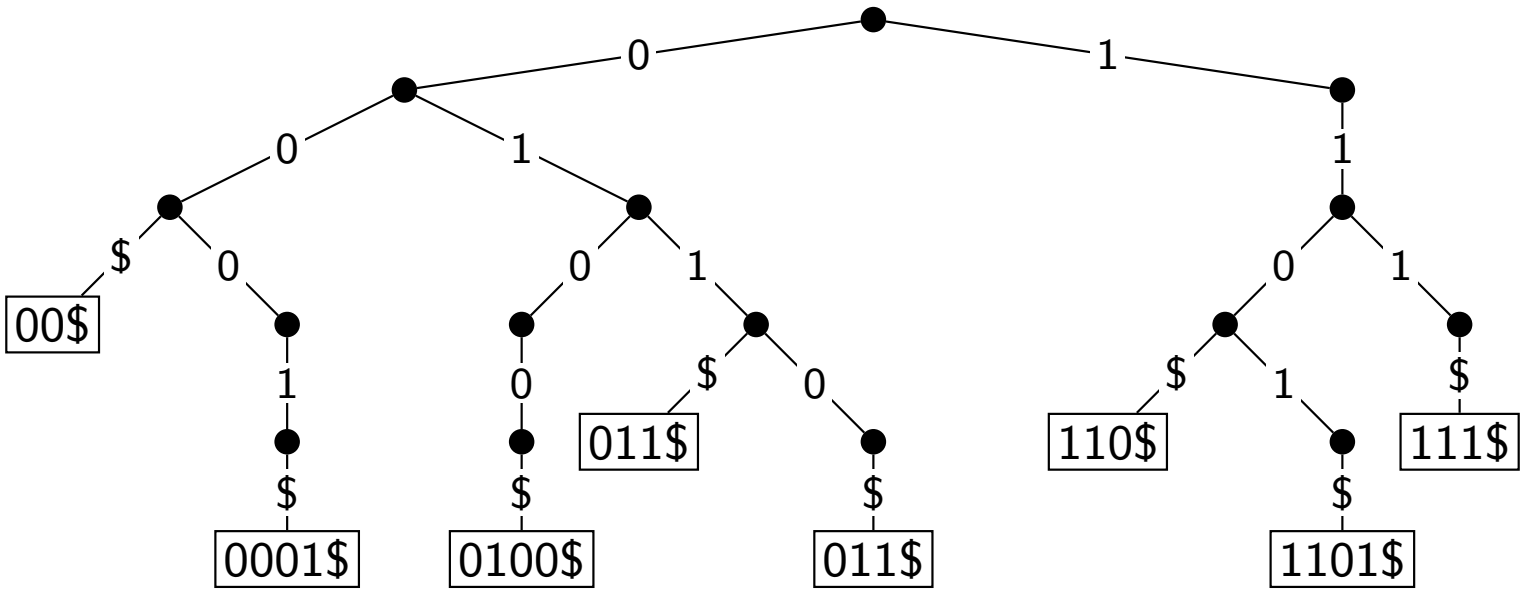


# More on tries

**Assumption:** Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with 'end-of-word' character \$.

**Example**: A trie for $\{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}$



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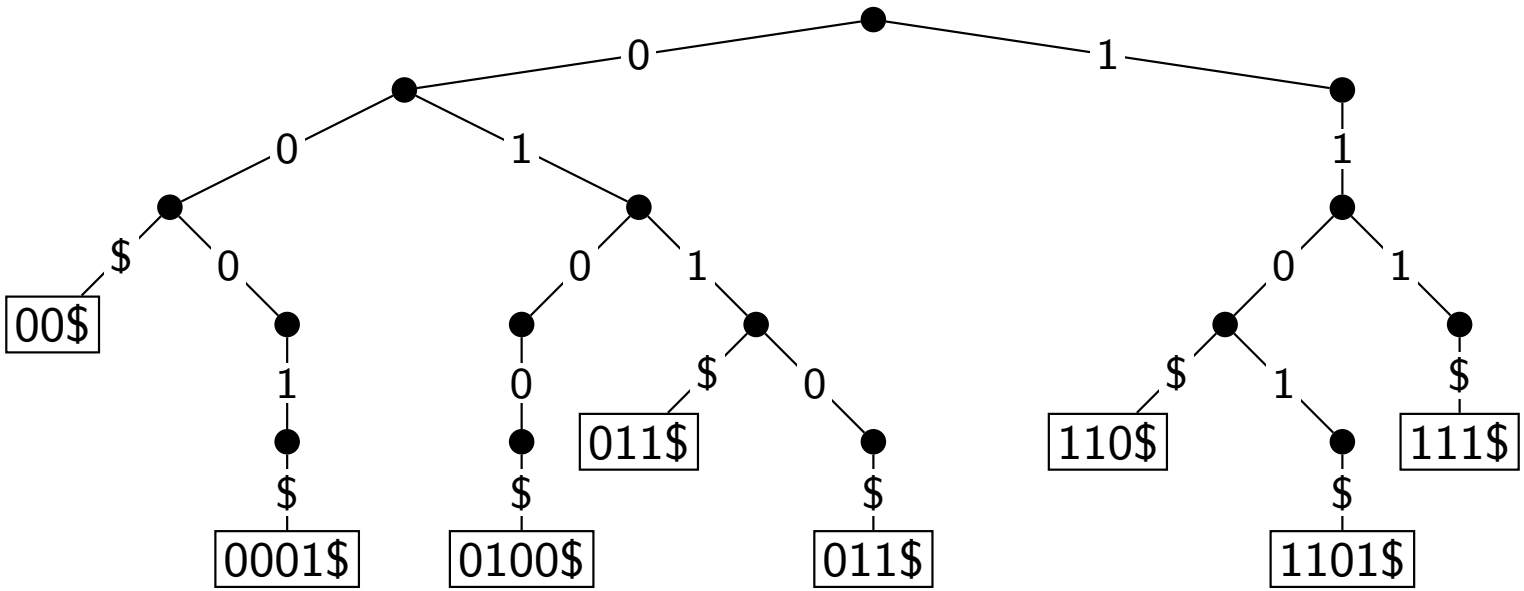



Then items (keys) are stored *only* in the leaf nodes

# Tries: Search

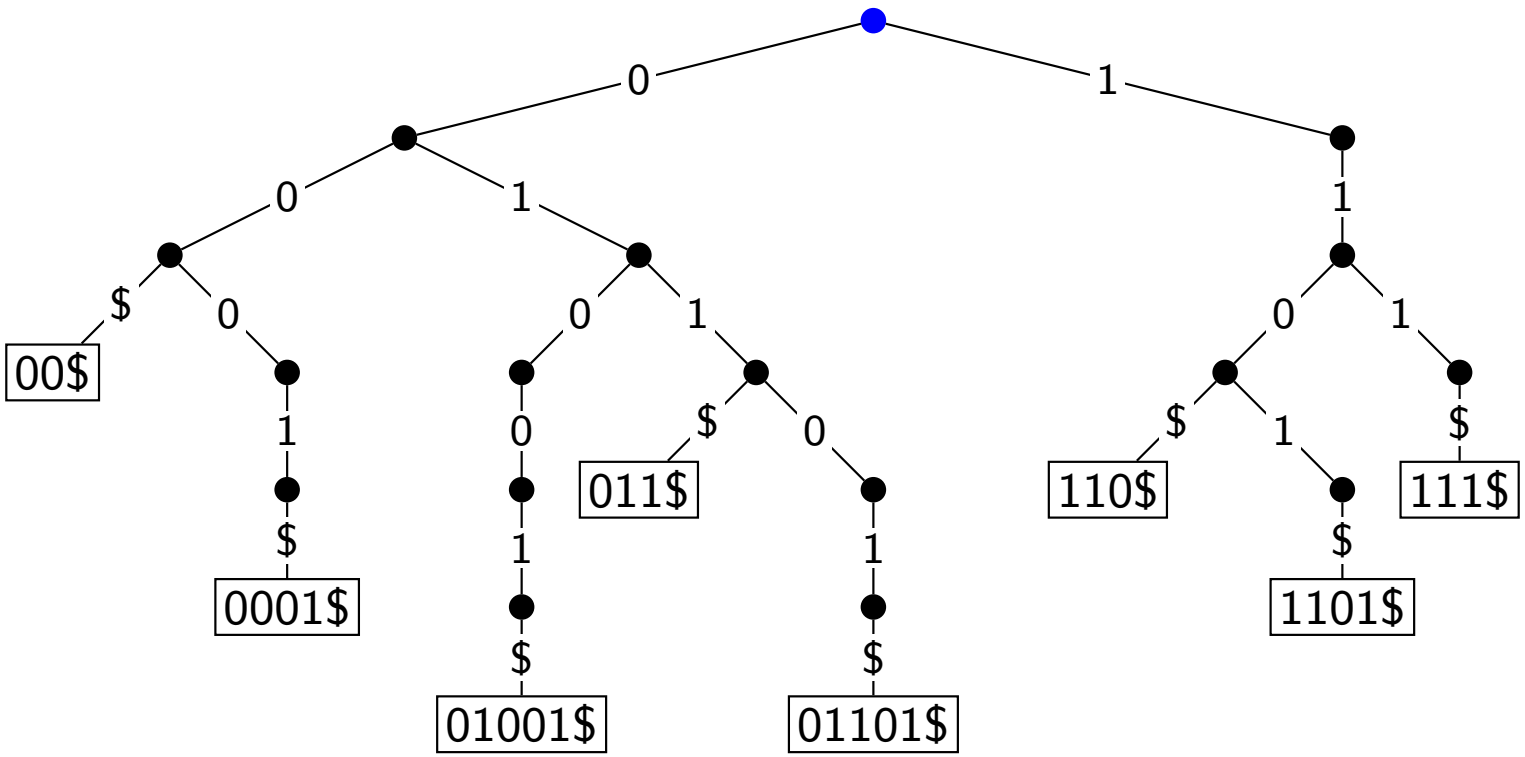
- start from the root and the most significant bit of $x$
- follow the link that corresponds to the current bit in $x$; return failure if the link is missing
- return success if we reach a leaf (it must store $x$)
- else recurse on the new node and the next bit of $x$

*Trie::search*$(v \leftarrow \text{root}, d \leftarrow 0, x)$
$v$: node of trie; $d$: level of $v$, $x$: word stored as array of chars

```
1.    if v is a leaf
2.        return v
3.    else
4.        let v' be child of v labelled with x[d]
5.        if there is no such child
6.            return "not found"
7.        else Trie::search(v', d + 1, x)
```

# Tries: Search Example

Example: Trie::search(011$)



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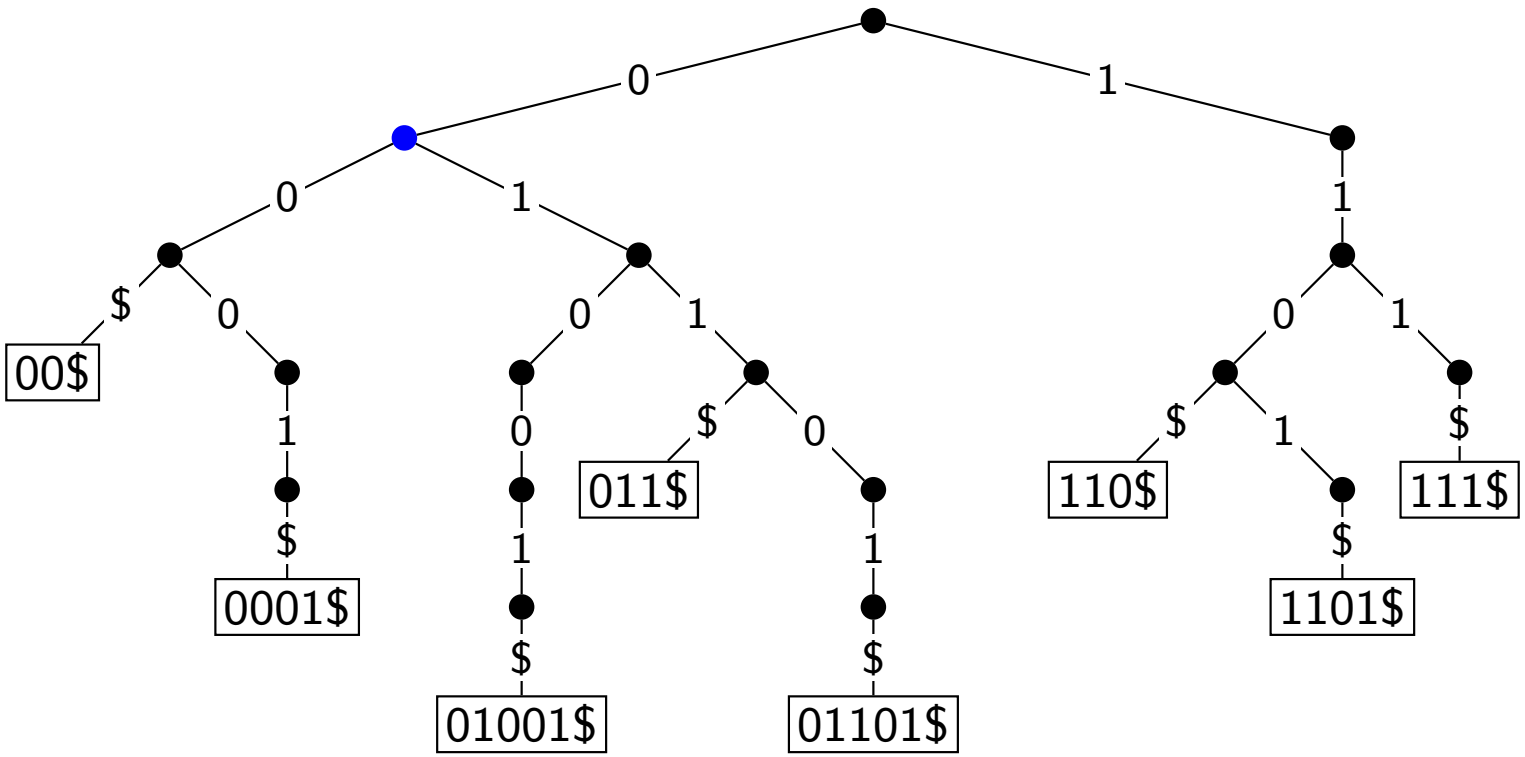

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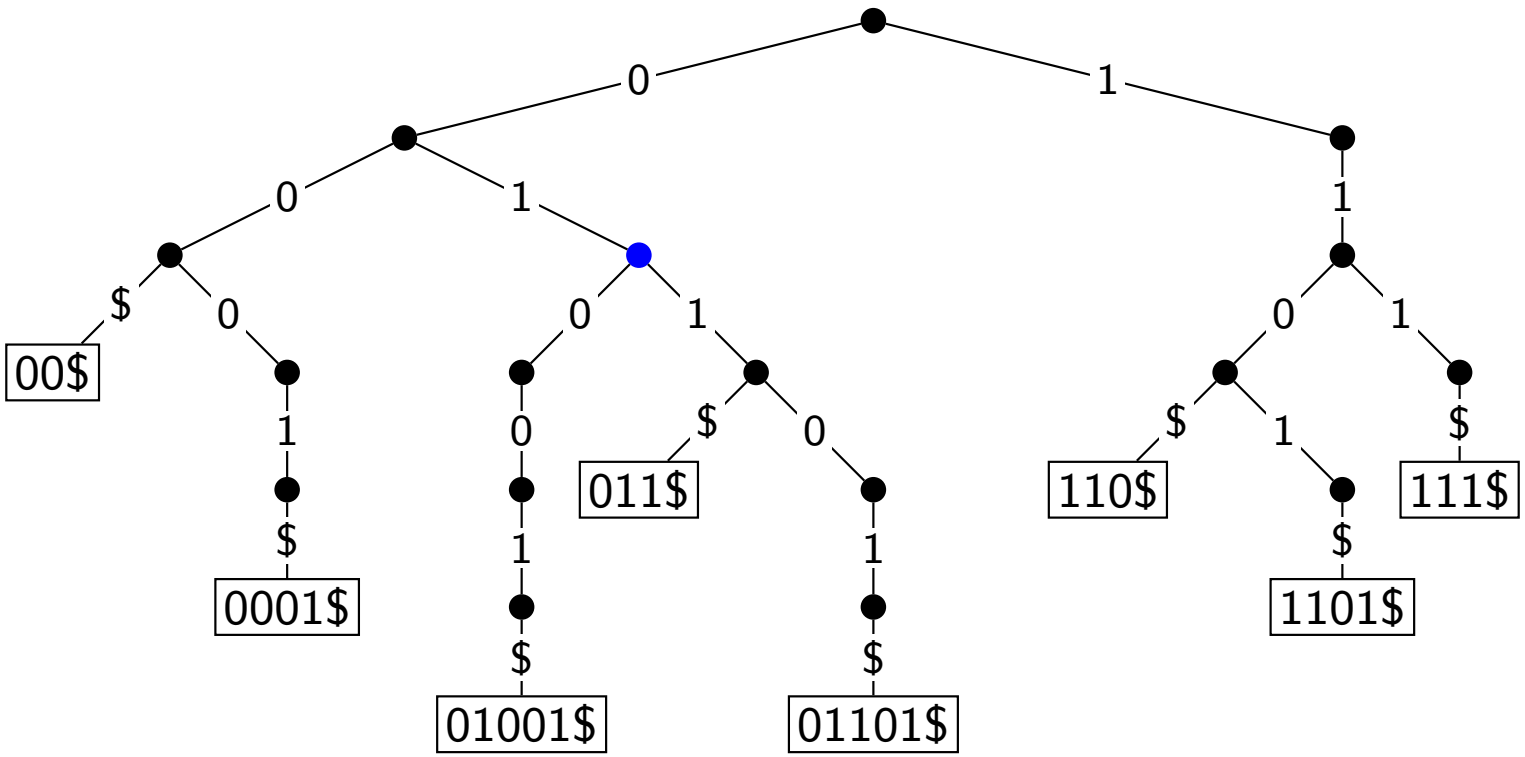

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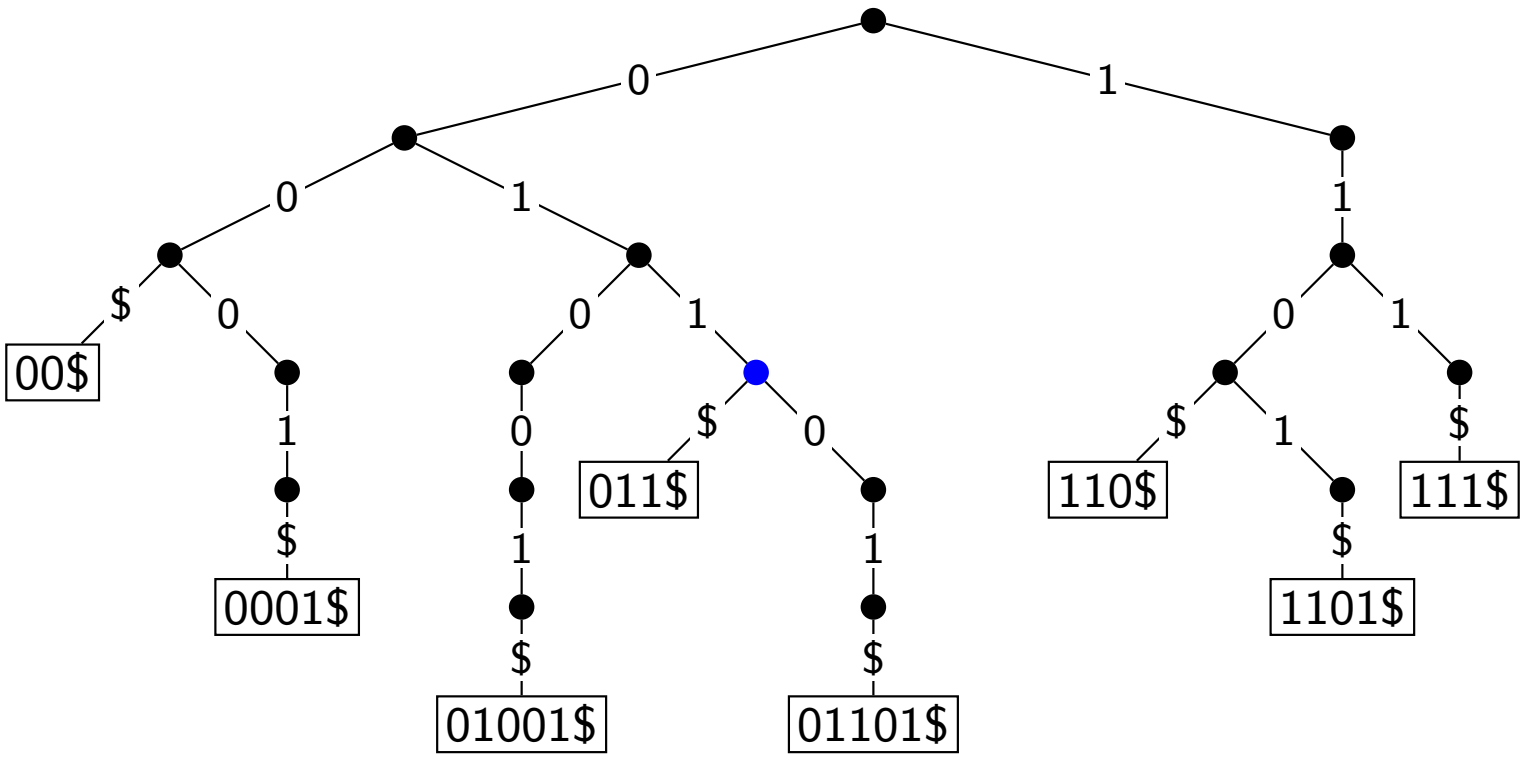
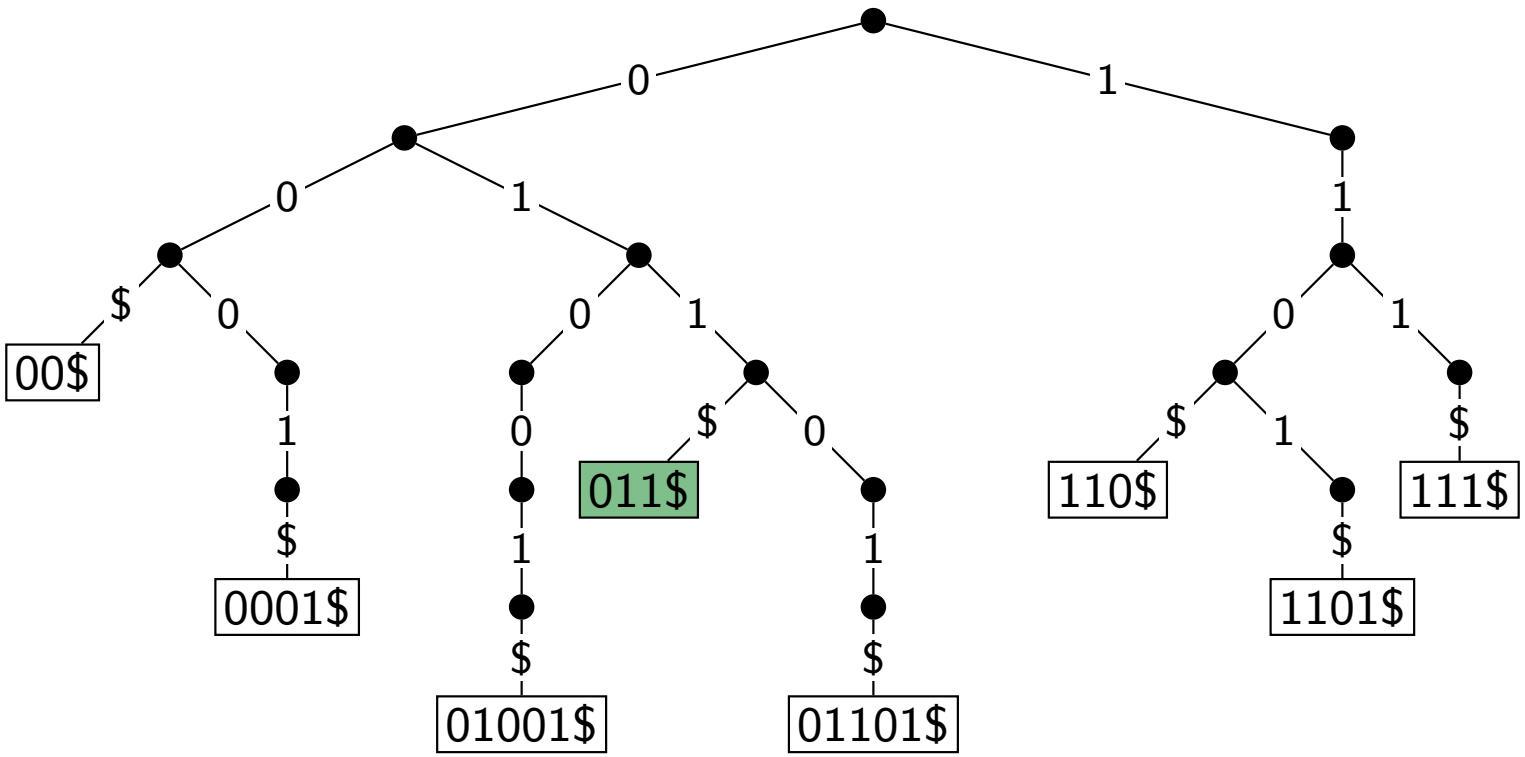


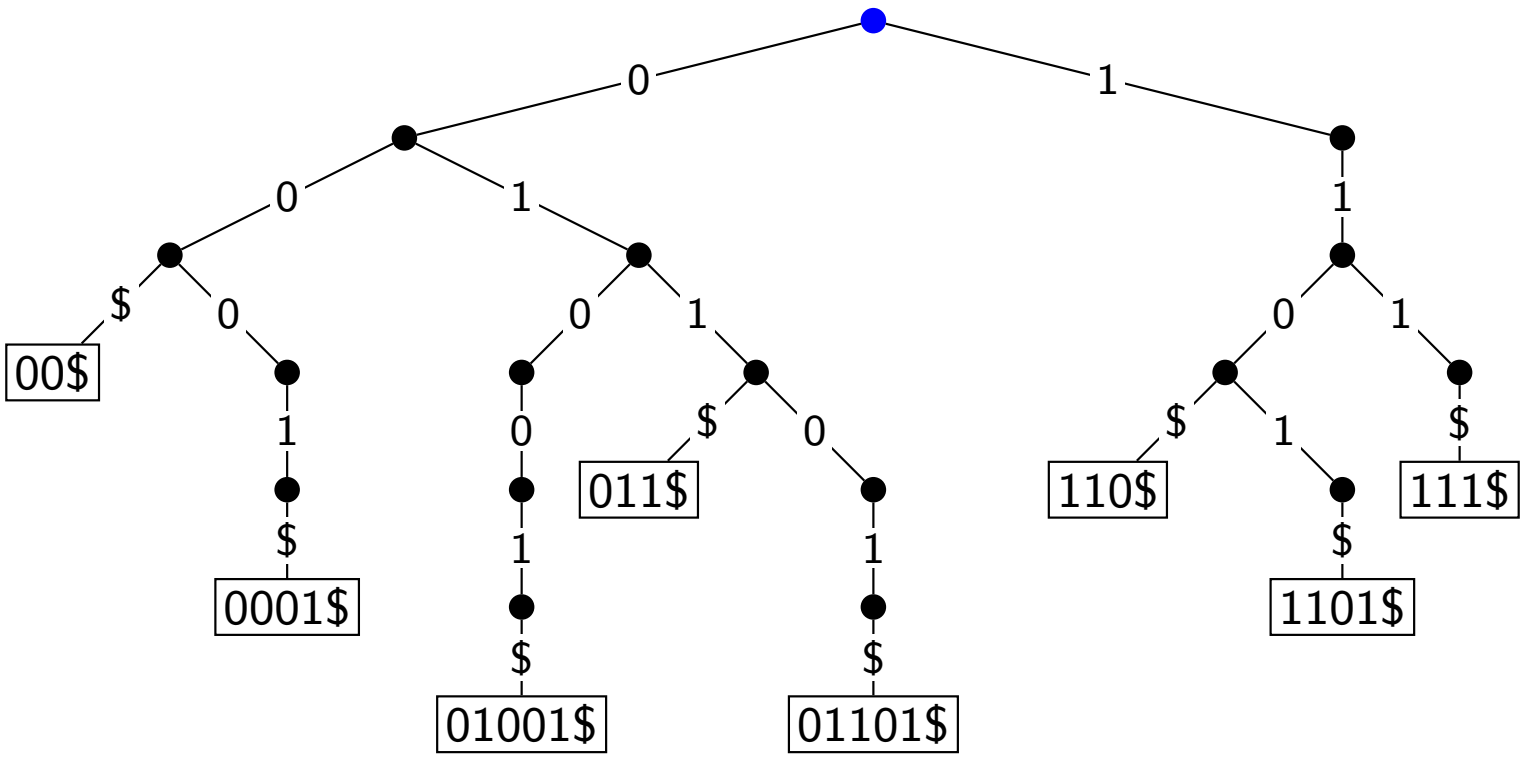
# Tries: Search Example

Example: Trie::search(011$) successful



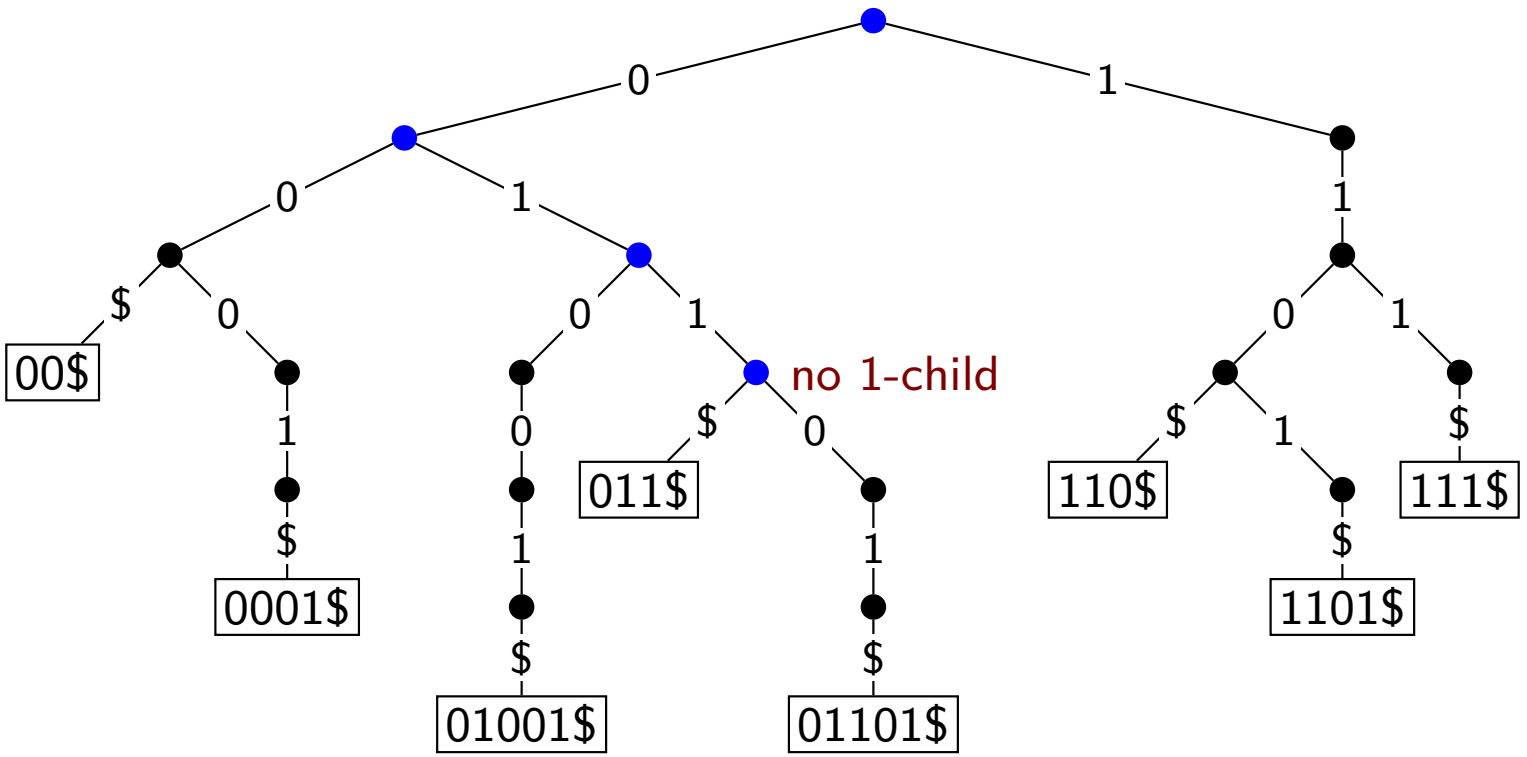
# Tries: Search Example

Example: Trie::search(0111$)



# Tries: Search Example

Example: Trie::search(0111$) unsuccessful

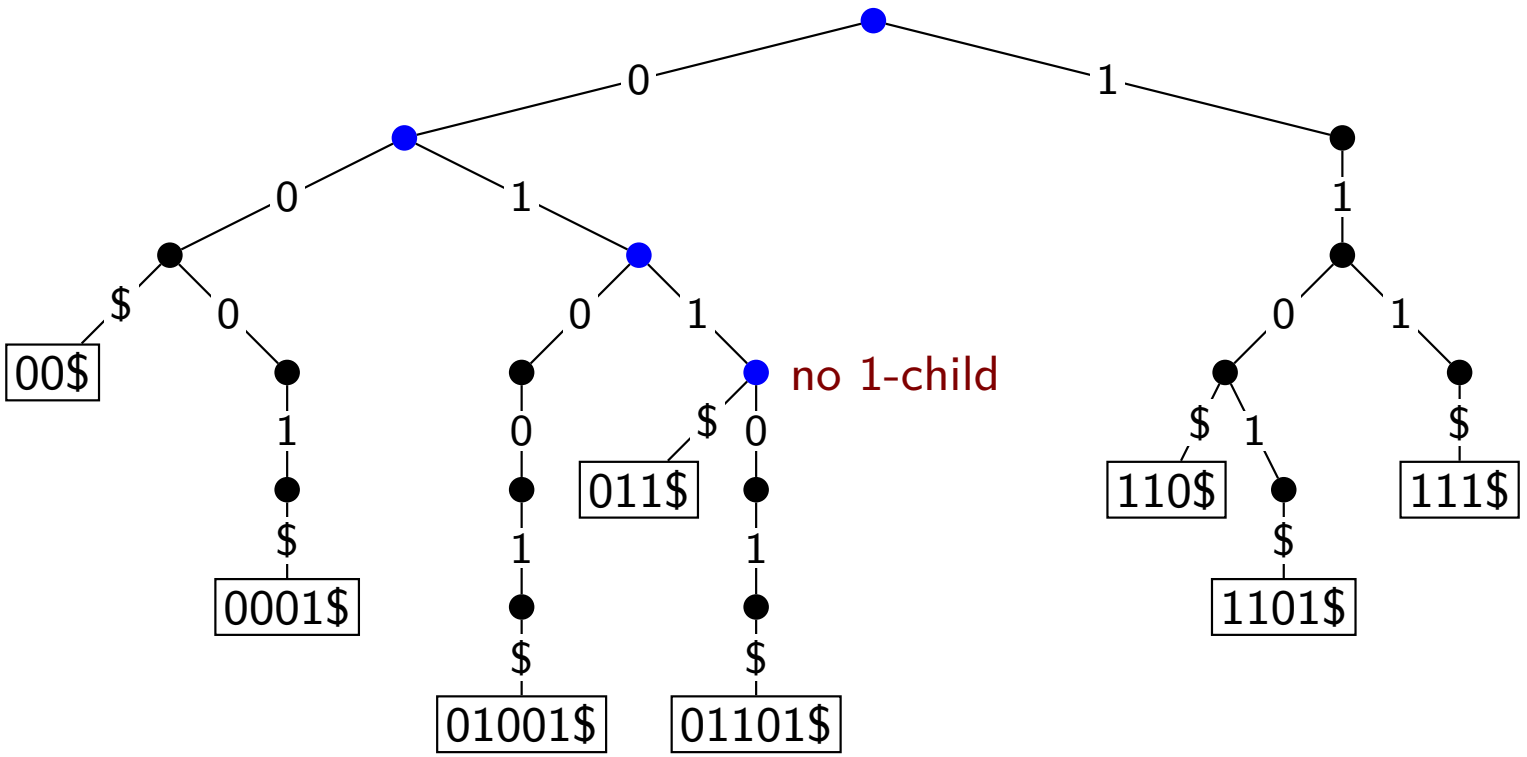


# Tries: Insert & Delete

- *Trie::insert(x)*
  - Search for $x$, this should be unsuccessful
  - Suppose we finish at a node $v$ that is missing a suitable child. Note: $x$ has extra bits left.
  - Expand the trie from the node $v$ by adding necessary nodes that correspond to extra bits of $x$.
- *Trie::delete*$(x)$
  - Search for $x$
  - let $v$ be the leaf where $x$ is found
  - delete $v$ and all ancestors of $v$ until we reach an ancestor that has two children.
- **Time Complexity** of all operations: $\Theta(|x|)$
  $|x|$: length of binary string $x$, i.e., the number of bits in $x$

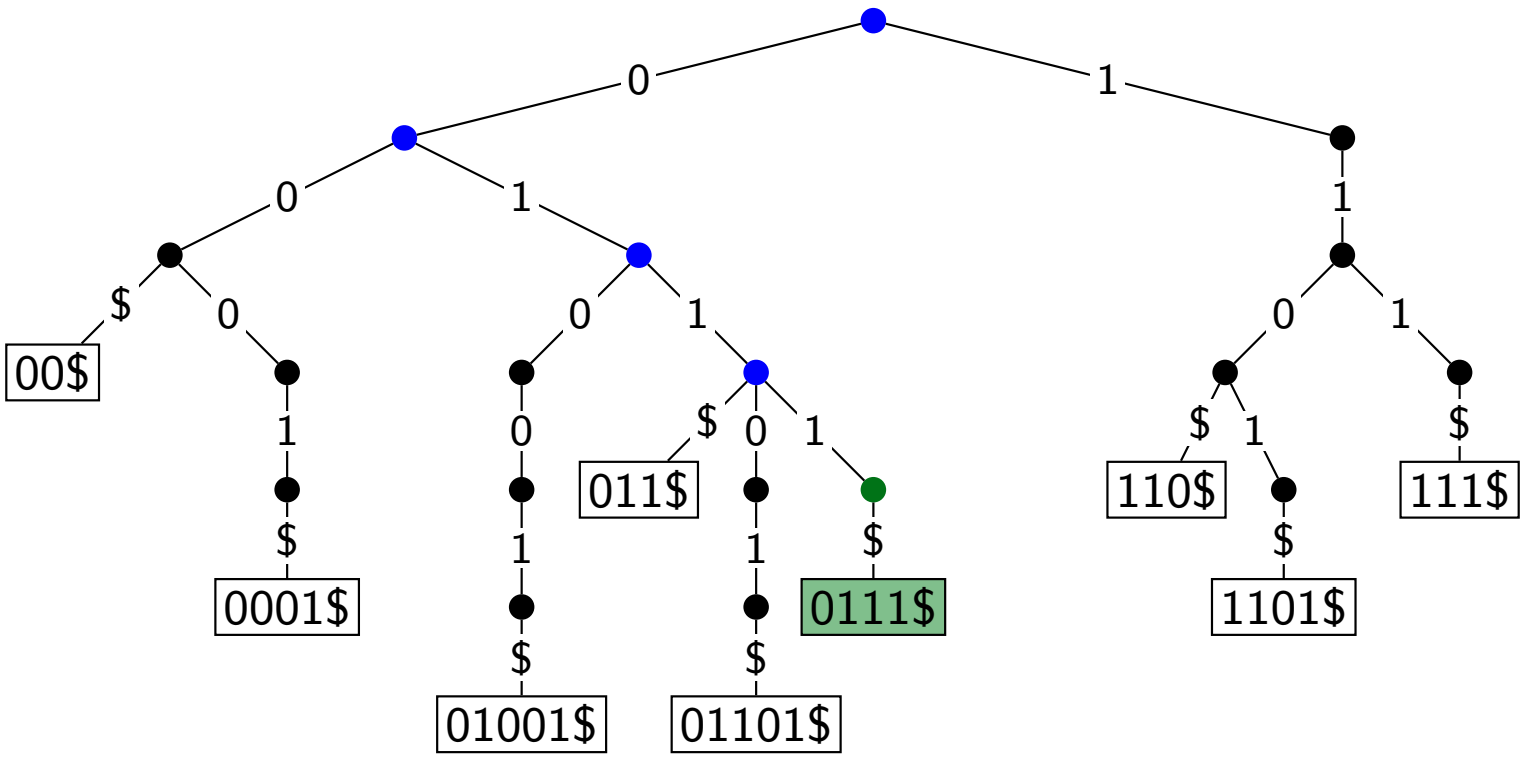
# Tries: Insert Example

Example: *Trie::insert*(0111$)



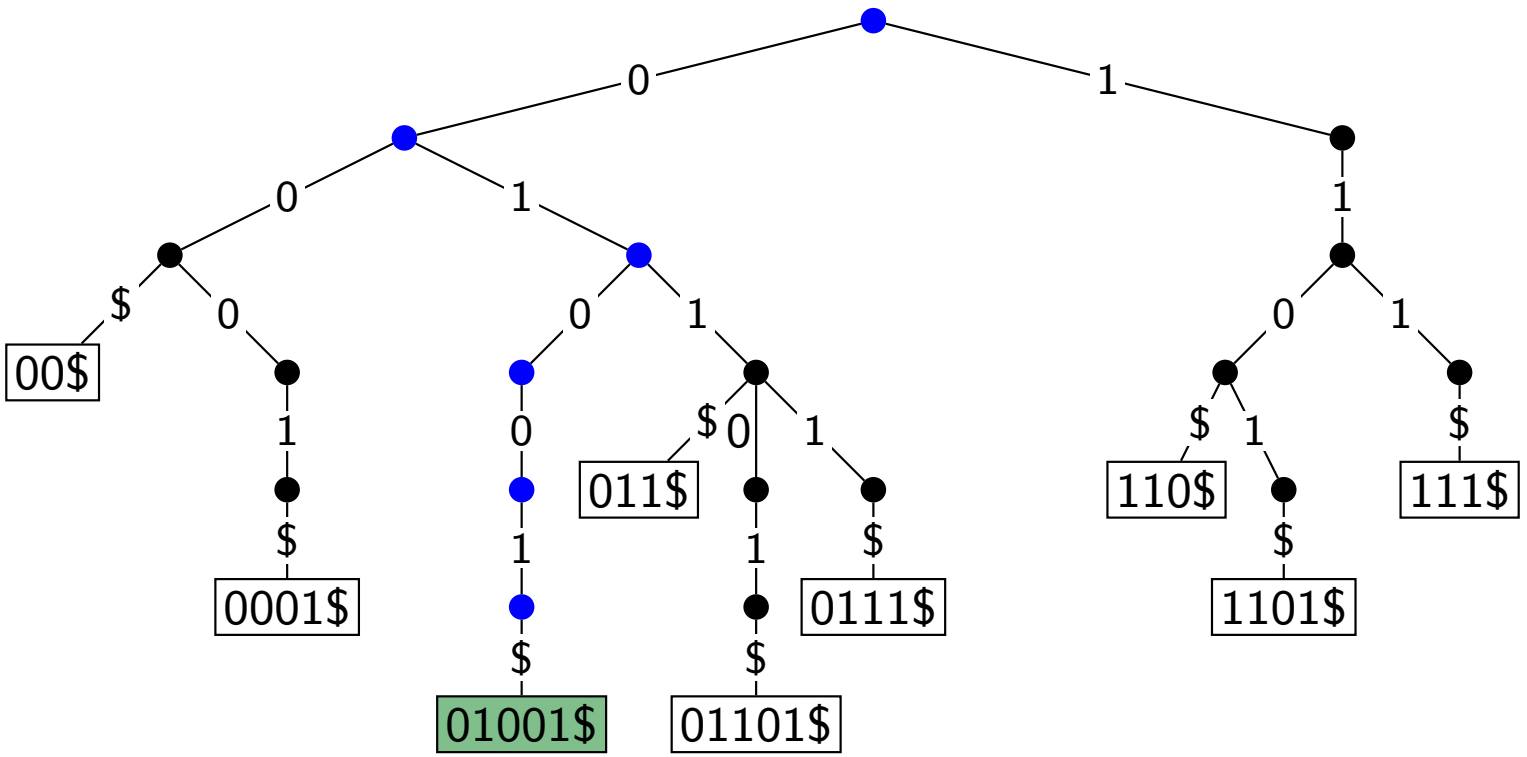
# Tries: Insert Example

Example: *Trie::insert*(0111$)



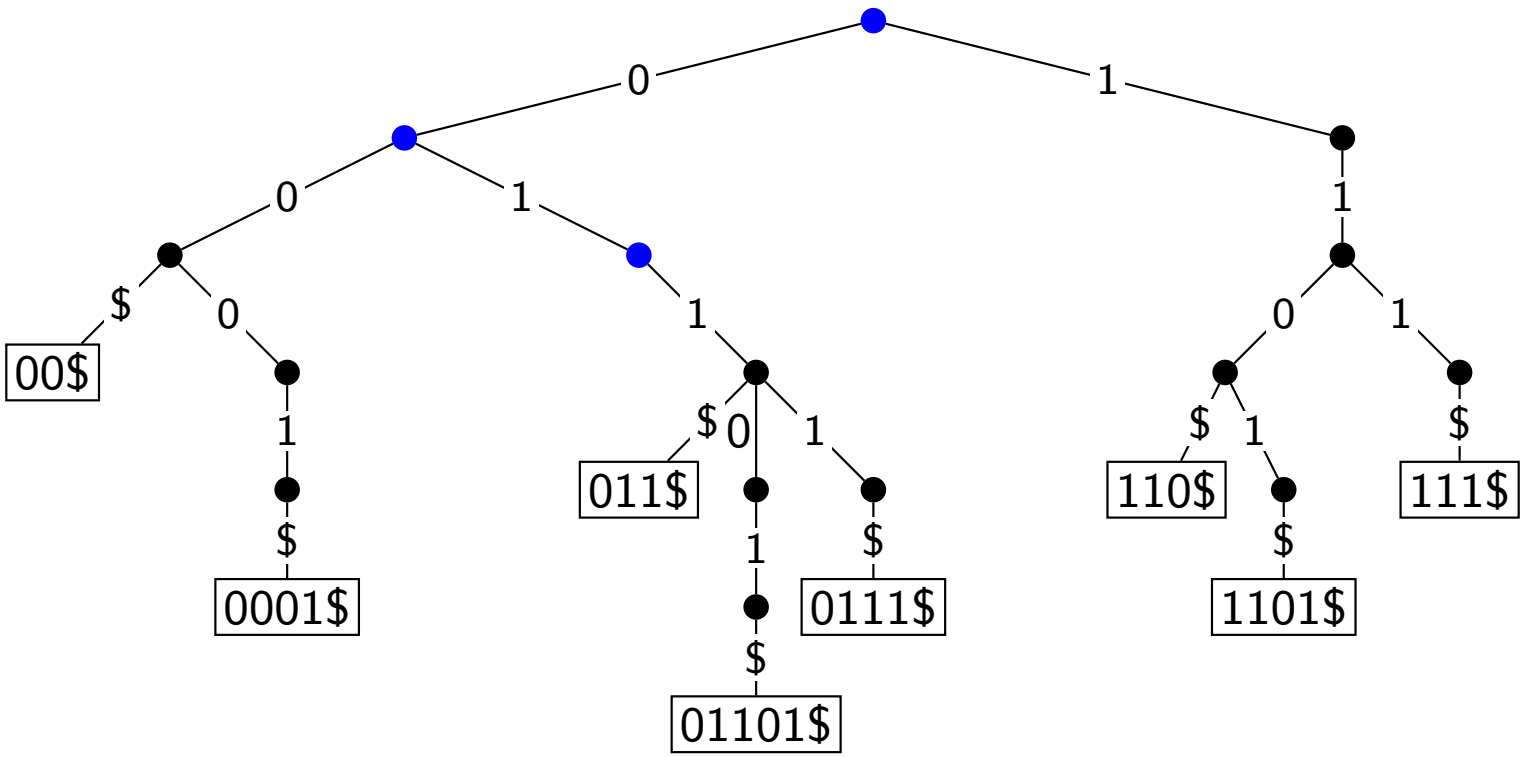
# Tries: Delete Example

Example: *Trie::delete*(01001$)



# Tries: Delete Example

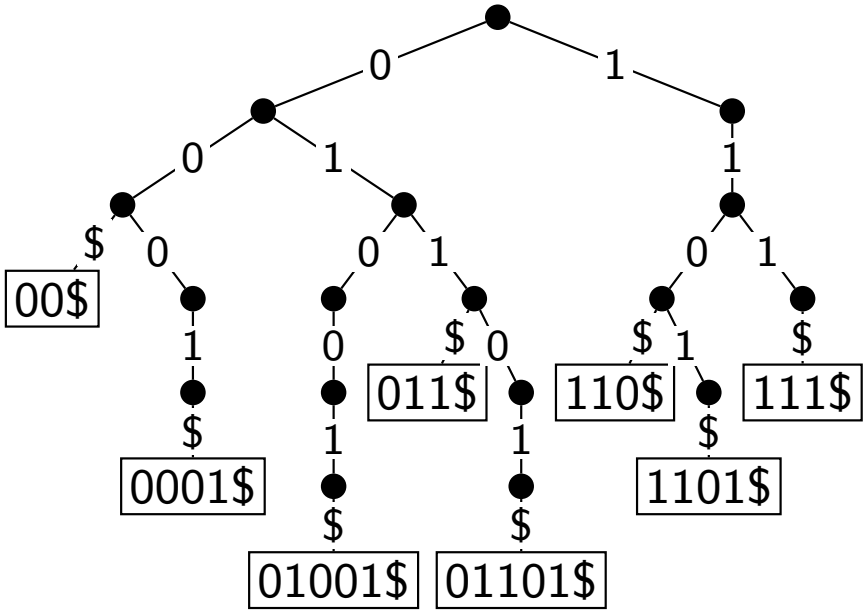
Example: *Trie::delete*(01001$)

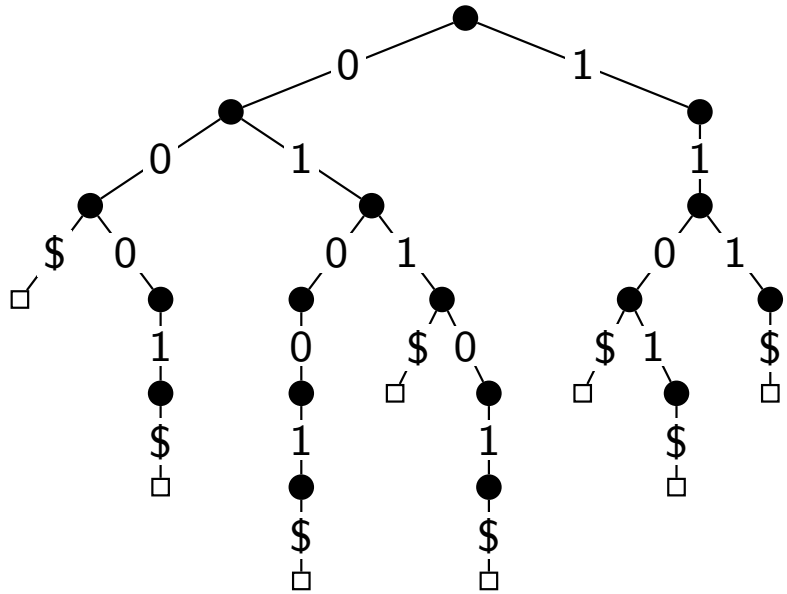


# Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.

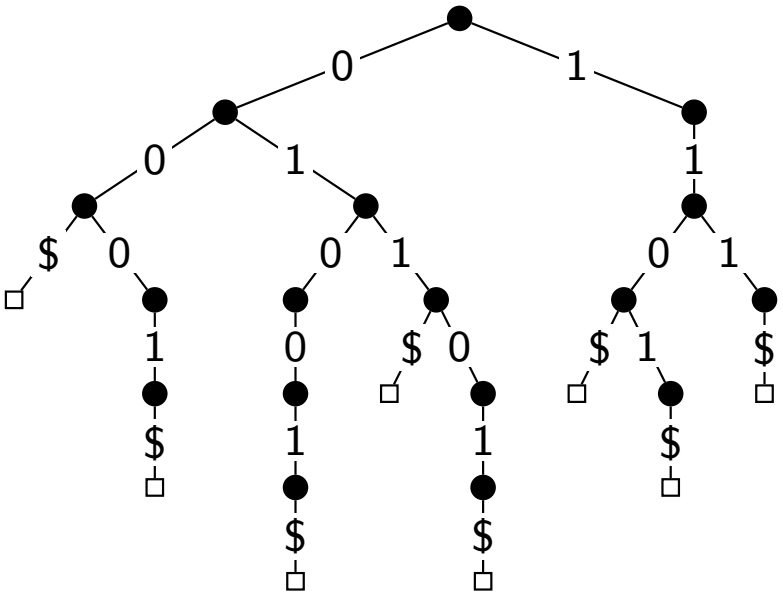


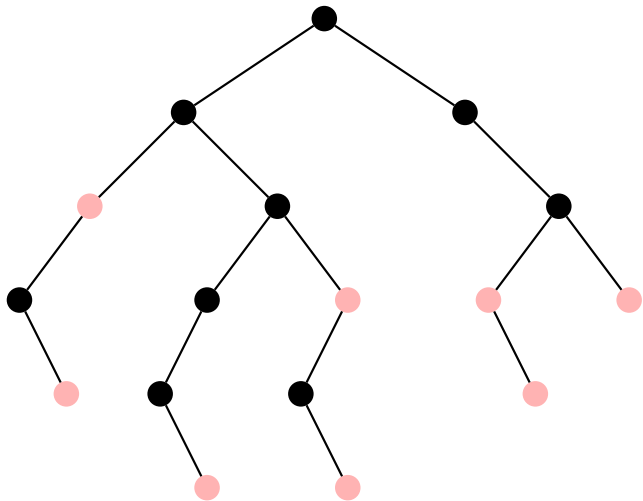



# Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys.
  Use a *flag* to indicate such nodes.
- No need for end-of-word character \$
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.

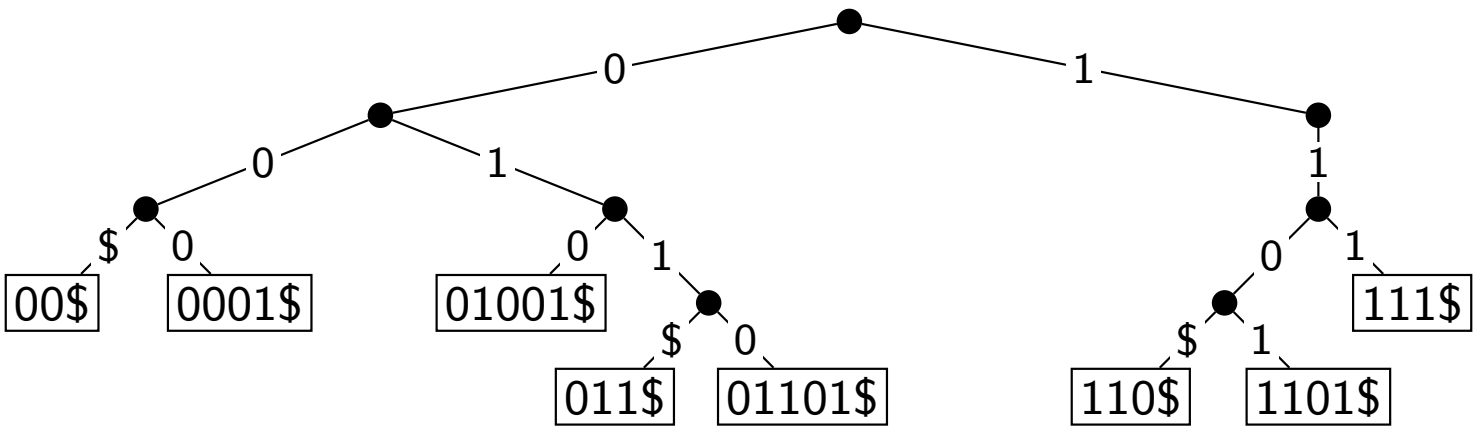


# Variations 3 of Tries

**Pruned Trie:** Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)

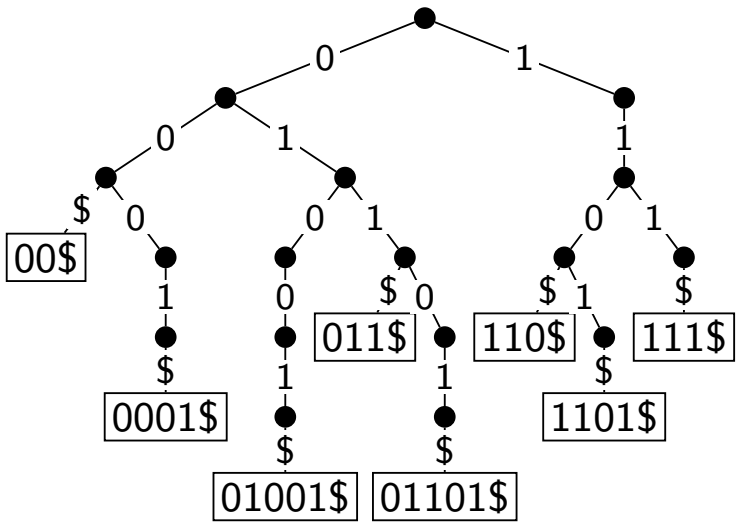


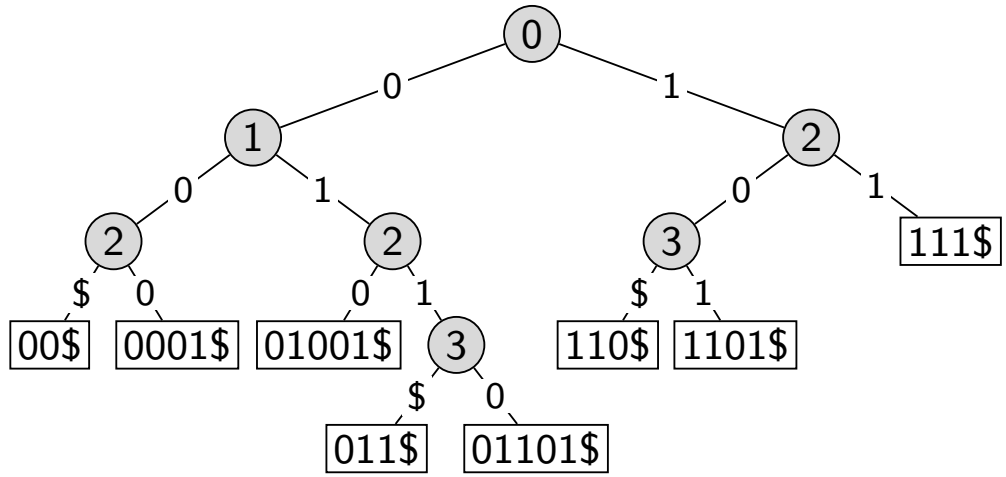

A more efficient version of tries, but the operations get a bit more complicated.

# Variation 4 of Tries

**Compressed Trie:** compress paths of nodes with only one child

- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with $n$ keys has at most $n - 1$ internal nodes




Also known as **Patricia-Tries**:
*Practical Algorithm to Retrieve Information Coded in Alphanumeric*

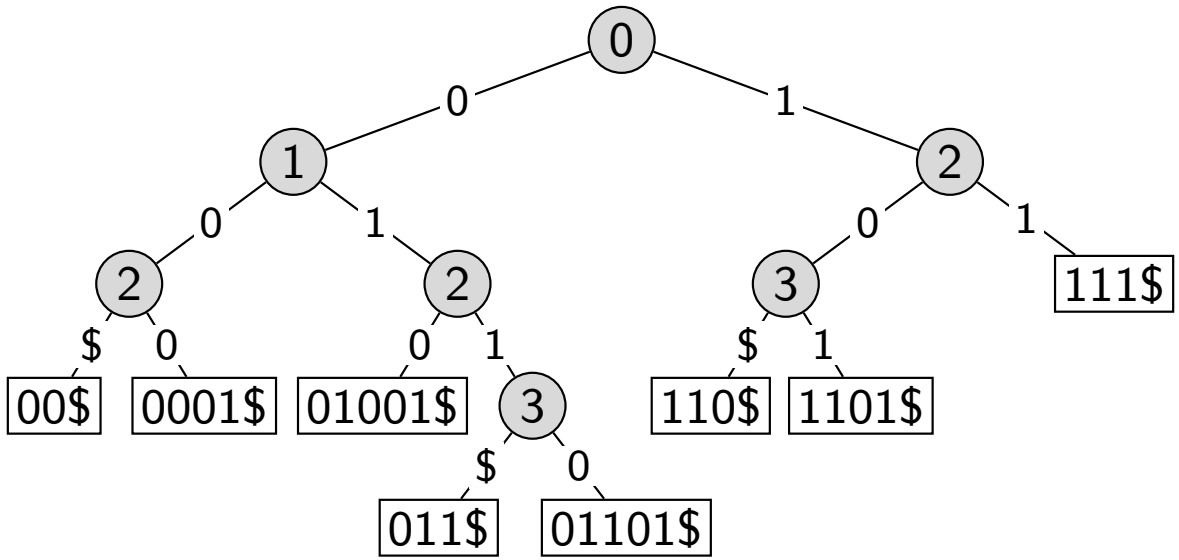
# Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in $x$;
  return failure if the link is missing
- if we reach a leaf, expicitly check whether word stored at leaf is $x$
- else recurse on the new node and the next bit of $x$

```
CompressedTrie::search(v ← root, x)
v: node of trie; x: word
1.    if v is a leaf
2.        return strcmp(x, v.key)
3.    d ← index stored at v
4.    if x has at most d bits
5.        return "not found"
6.    v' ← child of v labelled with x[d]
7.    if there is no such child
8.        return "not found"
9.    CompressedTrie::search(v', x)
```

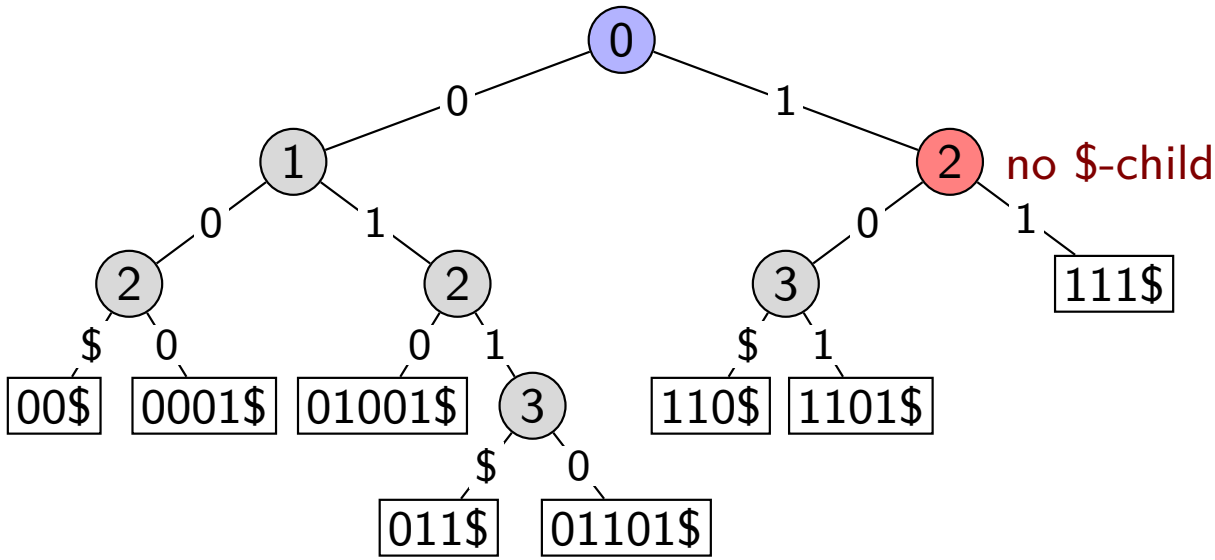
# Compressed Tries: Search Example

Example: CompressedTrie::search(10$)



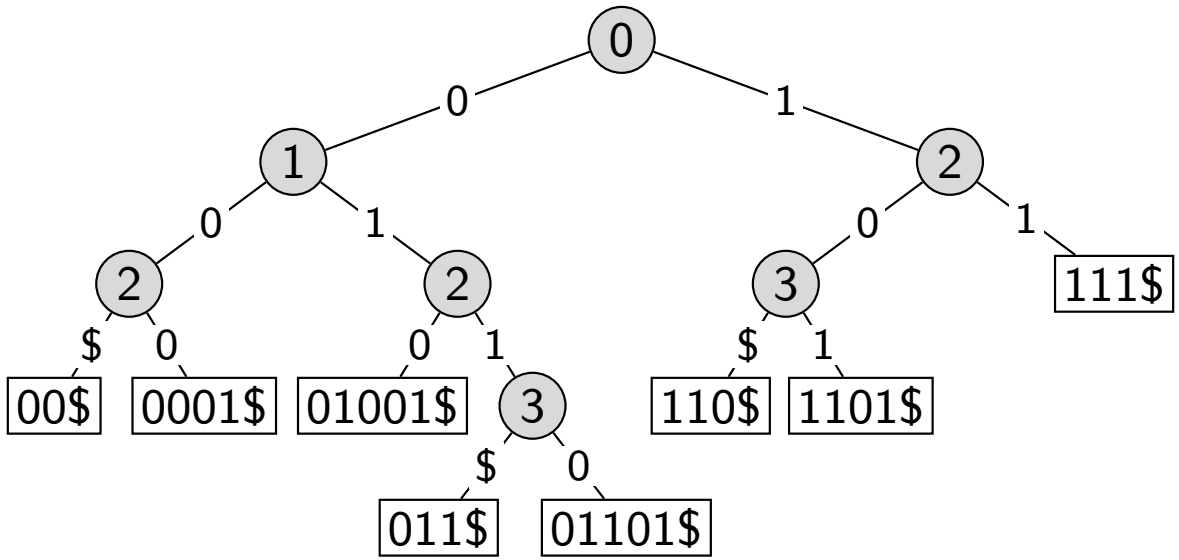
# Compressed Tries: Search Example

Example: CompressedTrie::search(10$) unsuccessful



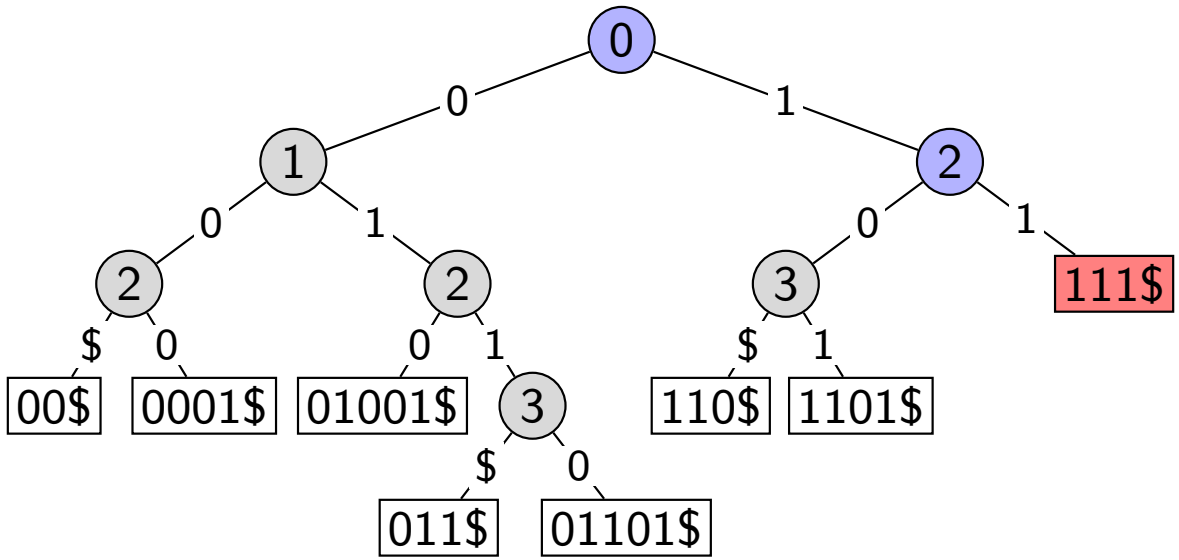
# Compressed Tries: Search Example

Example: CompressedTrie::search(101$)



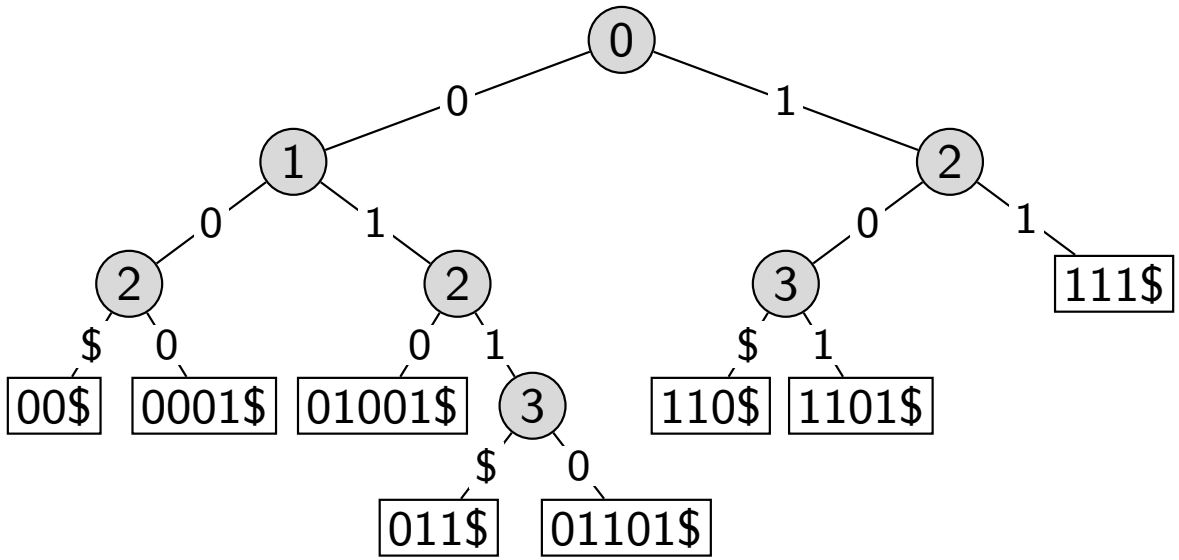
# Compressed Tries: Search Example

Example: CompressedTrie::search(101$) unsuccessful



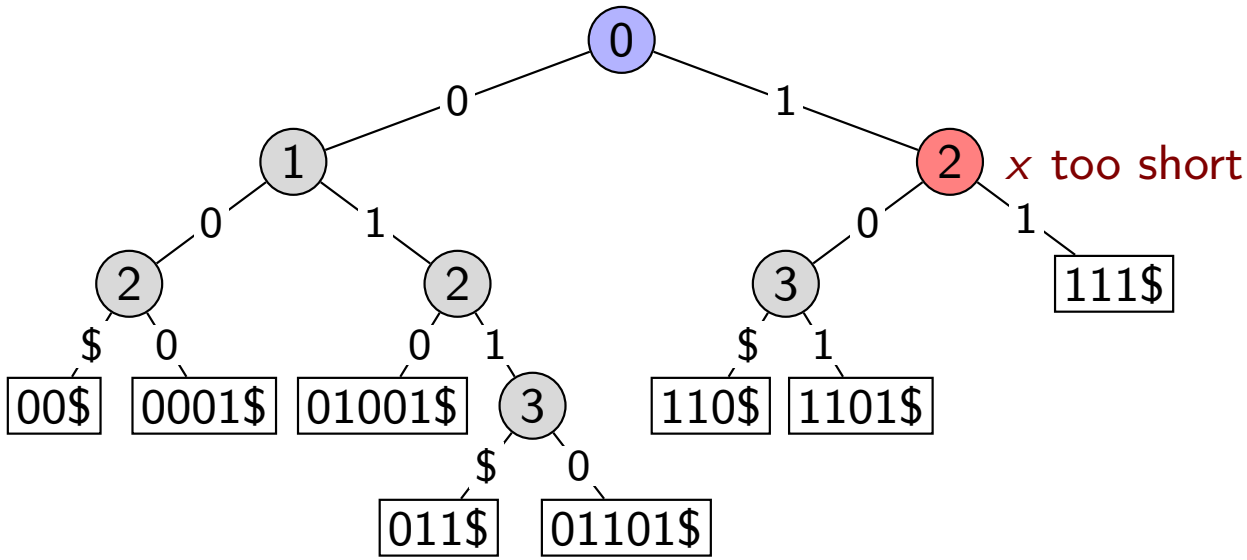
# Compressed Tries: Search Example

Example: CompressedTrie::search(1$)



# Compressed Tries: Search Example

Example: CompressedTrie::search(1$) unsuccessful



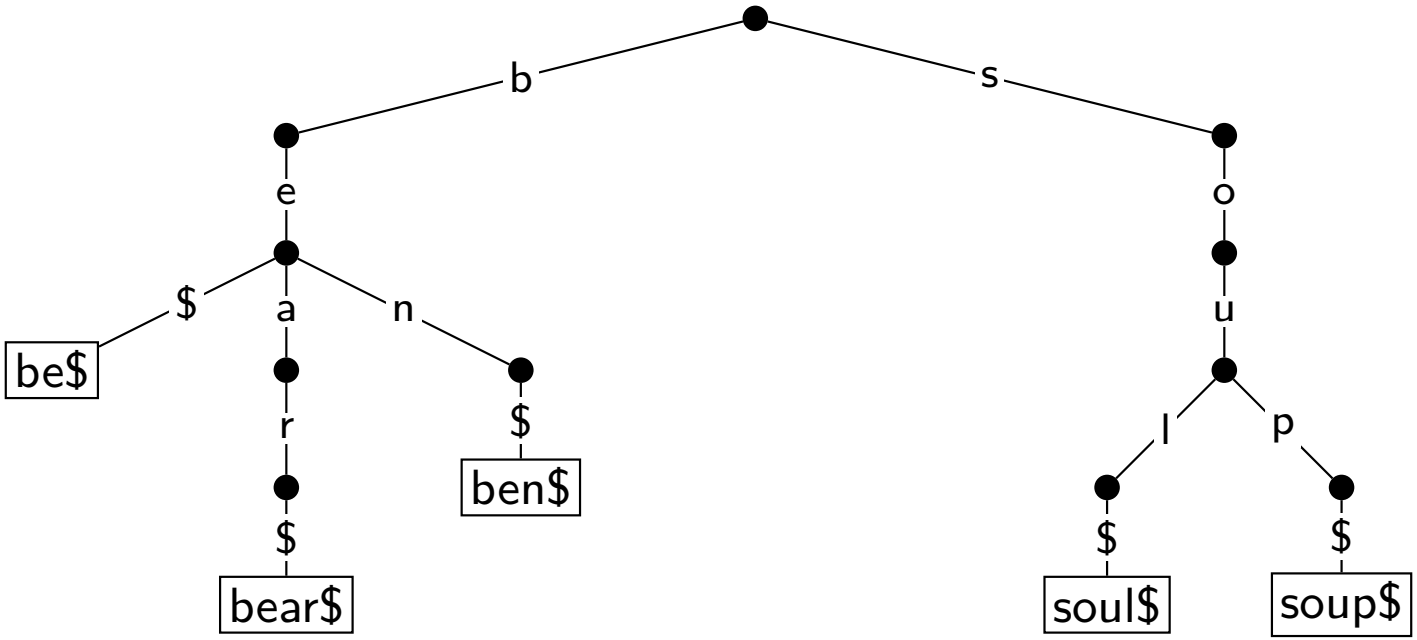
# Compressed Tries: Insert & Delete

- *CompressedTrie::delete*($x$):
  - Perform *search*($x$)
  - Remove the node $v$ that stored $x$
  - Compress along path to $v$ whenever possible.
- *CompressedTrie::insert*($x$):
  - Perform *search*($x$)
  - Let $v$ be the node where the search ended.
  - Conceptually simplest approach:
    - ★ Uncompress path from root to $v$.
    - ★ Insert $x$ as in an uncompressed trie.
    - ★ Compress paths from root to $v$ and from root to $x$.
  - But it can also be done by only adding those nodes that are needed.
  - Requires **leaf-links**: Every node stores a link to a leaf that is a descendant.
- All operations take $O(|x|)$ time.

Much more complicated, but space-savings are worth it if words are unevenly distributed.

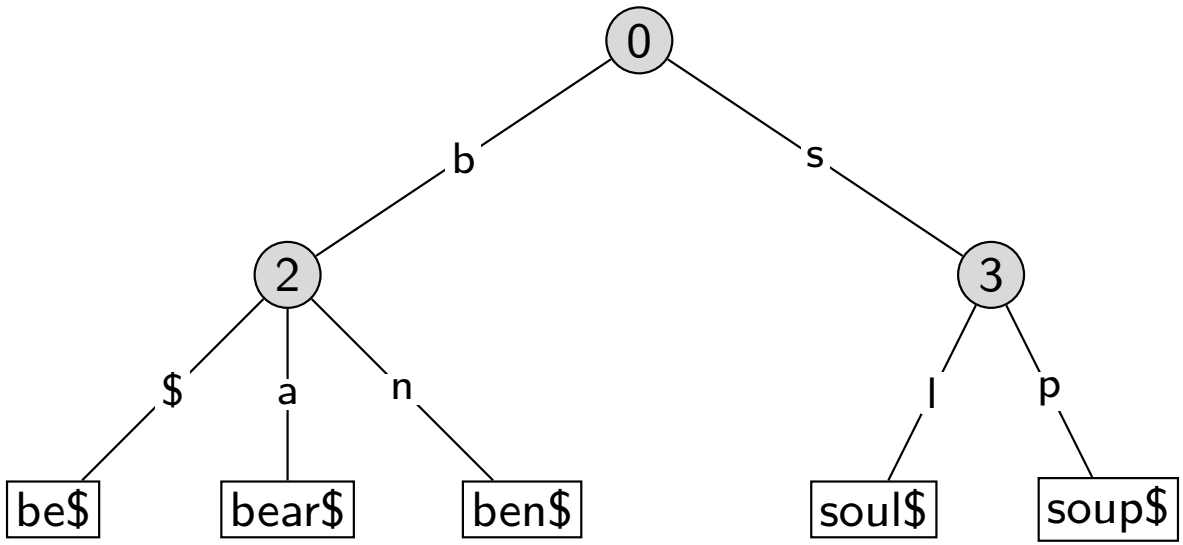
# Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* $\Sigma$
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



# Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



# Multiway Tries: Summary

- Operations *search*($x$), *insert*($x$) and *delete*($x$) are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot (\text{time to find the appropriate child}))$

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Each node now has up to $|\Sigma| + 1$ children. How should they be stored?

**Solution 1:** Array of size $|\Sigma| + 1$ for each node.
Complexity: $O(1)$ time to find child, $O(|\Sigma|)$ space per node.

**Solution 2:** List of children for each node.
Complexity: $O(|\Sigma|)$ time to find child, $O(\#\text{children})$ space per node.

**Solution 3:** Dictionary (AVL-tree?) of children for each node.
Complexity: $O(\log(\#\text{children}))$ time, $O(\#\text{children})$ space per node.

Best in theory, but not worth it in practice unless $|\Sigma|$ is huge.

In practice, use *hashing* (keys are in (typically small) range $\Sigma$).