CS 370 Spring 2012: Assignment 1

Due Thursday May 24, 5:00 PM, in the Assignment Boxes, 3rd Floor MC.

1. **(8 marks)** The roots of a quadratic equation, $ax^2 + bx + c = 0$, are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

when $a \neq 0$.

(a) Simulating the four-digit accuracy of the floating point number system $F(10, 4, -10, 10)$ with rounding, carry out by hand the computation of the roots, using the above formulae, for the following quadratic equation

$$1.002x^2 + 11.01x + 0.01265 = 0. \tag{1}$$

The roots of the given quadratic, correct to five significant digits, are -0.0011491 and -10.987. What is the relative error of your computed roots?

(b) As we saw in class, the *cancellation problem* arises when applying the above formulae for any quadratic equation having the property that

$$|b| \approx \sqrt{b^2 - 4ac}.$$

If $b > 0$ then the quadratic formula for x_1 will exhibit cancellation, and if $b < 0$ then x_2 will exhibit cancellation. We saw that a mathematically equivalent formula for x_1 is

$$x_1 = \frac{2c}{-b - \sqrt{b^2 - 4ac}}.$$

Derive a formula for x_2 that avoids the cancellation problem when $b < 0$.

(c) Deduce an algorithm for calculating the roots of a quadratic equation that avoids cancellation errors. Give pseudocode for your algorithm.

(d) Redo the calculation of the roots of equation (1) by applying your improved algorithm using the same floating point number system. Compute the relative error of the computed roots and compare with your results from part (a).

2. **(8 marks)** Consider the recurrence sequence $\{p_n\}$ defined by

$$p_n = \frac{26}{5} p_{n-1} - p_{n-2}, \quad n \geq 2.$$

(a) What is the exact solution? Show your work.

(b) Show that if we use

$$p_0 = 1, \quad p_1 = \frac{1}{5},$$

with exact arithmetic then $p_n = \frac{1}{5^n}$ for all $n \geq 0$.

(c) Suppose we compute this sequence using the same starting values but now using floating point arithmetic. Carry out a stability analysis for this computation by considering the propagation of errors introduced in the initial values p_0 and p_1. Is this computation stable or unstable? Justify your answer mathematically.

3. **(10 marks)** Suppose we have a natural spline:

$$S(x) = \begin{cases} a_1 + 25x + 9x^2 + x^3 & x \in [-3, -1] \\ 26 + a_2 x + a_3 x^2 - x^3 & x \in [-1, 0] \\ 26 + 19x + a_4 x^2 + a_5 x^3 & x \in [0, 3] \\ -163 + 208x - 60x^2 + a_6 x^3 & x \in [3, 4] \end{cases}$$

   (a) Determine the values of a_1, a_2, $\cdots$, a_6.

   (b) Using the Lagrange basis, construct a cubic interpolation polynomial which interpolates the values of $S(x)$ at $-3, -1, 0, 3$. (If you have trouble determining the a_i, use symbolic values of $S(x)$ to complete this part.)

4. **(5 marks)** For general data points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$, where $x_1 < x_2 < \ldots < x_N$ and $N \geq 4$, a *not-a-knot* cubic spline has the end condition that the third-order derivative is continuous at x_2 and x_{N-1}. Assume that $S(x)$ is a cubic spline interpolant for four data points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and (x_4, y_4):

$$S(x) = \begin{cases} p_1(x) & x \in [x_1, x_2] \\ p_2(x) & x \in [x_2, x_3] \\ p_3(x) & x \in [x_3, x_4]. \end{cases}$$

   Suppose $P(x) = 2x^3 + 5x - 7$ is the cubic interpolant for the same four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) where $x_1 < x_2 < x_3 < x_4$ are knots. What is the not-a-knot spline interpolant $S(x)$? Prove your answer.

5. **(14 marks)** For this question, you will use MATLAB to create parametric curves representing your first and last initials separated by an ampersand, inside the outline of your hand. For example, John Smith would produce the writing

$$J \ \& \ S$$

   in handwriting, inside the outline of his hand.

   Follow the steps below:

   (1) The first part involves getting the input data for the handwriting. First, trace your hand on a piece of paper, and write your initials separated by & inside it. Using the following script, you can put the paper on the computer screen and create interpolation data points for the drawing by clicking with the mouse.

```
figure('position', get(0,'screensize'))
axes('position', [0 0 1 1])
[x,y] = ginput;
% click...
clf;
save('data.mat','x','y');
```

   Use the **help** command to familiarize yourself with any matlab command. Select a few dozen points outlining your drawing by clicking on the screen. Press enter to terminate the input sequence. You can load the data you created and saved using the **load** command.

(2) Using the data collected in the previous step, generate parametric curves based on smooth parametric curve interpolations as described in the course notes (pages 35-39). Show the output using piecewise linear splines and cubic splines.

Prepare three Matlab .m files, one for each of the following tasks:

(a) Create a plot of the interpolation data corresponding to the crude initial shape, plotted with the '*' symbol. Your plot should have a title and should show the axes and gridlines.

(b) Create a plot of your drawing/handwriting created by joining the original data points with straight lines. (The letters will not look very smooth.) Your plot should have a title and should show the axes and gridlines.

(c) Create two plots showing smoother spline representations of your drawing. Refine the partition by a factor of 5. The two plots should use different spline end conditions: use not-a-knot conditions for the first plot, and use natural conditions for the second plot. See matlab functions *csape* and *fnval* for cubic spline interpolation with end conditions. The plots should have titles, but do not include the axes or grid lines.

Submit a hard copy of all of your plots and code.