$\int \sqrt{1+t^4}\,dt$

Brian Bi

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1 Introduction

We wish to evaluate the indefinite integral: $\int \sqrt{1+t^4}\,dt$

Mathematica returns the result:

$$\int \sqrt{1+t^4}\,dt = \frac{t^5 - 2\sqrt[4]{-1}\sqrt{t^4+1}\,F(i\sinh^{-1}(\sqrt[4]{-1}\,t)\big| -1) + t}{3\sqrt{t^4+1}} + C \tag{1}$$

The antiderivative of $f(t) = \sqrt{1+t^4}$ cannot be expressed in terms of elementary functions. However, Mathematica gives the result in terms of the special function F, a nonelementary function known as the *incomplete elliptic integral of the first kind*. In the next section this function and its cousin, the *incomplete elliptic integral of the second kind*, are introduced. In the following sections we transform the integrand step by step into a more tractable form.

2 The E and F functions

The elliptic integral functions are so named because the context in which they originally arose was that of finding the arc length of an ellipse. Consider an ellipse with semimajor axis of length a oriented along the x-axis and semiminor axis of length 1 oriented along the y-axis. This ellipse has the parametric equations:

$$x = a\cos t \tag{2}$$

$$y = \sin t \tag{3}$$

(That is, the ellipse is an elongated circle.) As we know, the arc length of the part of this ellipse defined by $0 \le t \le \theta$ is given by

$$\begin{aligned}
s(\theta, a) &= \int_0^\theta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt && (4)\\
&= \int_0^\theta \sqrt{a^2\sin^2 t + \cos^2 t}\,dt && (5)\\
&= \int_0^\theta \sqrt{(a^2-1)\sin^2 t + \sin^2 t + \cos^2 t}\,dt && (6)\\
&= \int_0^\theta \sqrt{1 + (a^2-1)\sin^2 t}\,dt && (7)
\end{aligned}$$

Finally, define

$$E(x|k) = \int_0^x \sqrt{1 - k\sin^2 t}\,dt \tag{8}$$

Then we can write $s(\theta, a) = E(\theta|1-a^2)$, thus expressing the arc length of the ellipse in terms of this function E, which is known as the *incomplete elliptic integral of the second kind*. It is not possible to express the function E in terms of elementary functions (exponential, log, trig functions) of its arguments; if asked to integrate the function $f(x) = \sqrt{1 - k\sin^2 x}$, you have no choice but to reply $E(x|k) + C$.

In some cases we may similarly wish to integrate the function $g(x) = \frac{1}{\sqrt{1-k\sin^2 x}}$. This function *also* has no elementary antiderivative; so again mathematicians have given a special name to its antiderivative, the *incomplete elliptic integral of the first kind*, F. In particular, $F(x|k) + C$ is the antiderivative of $g(x)$.

3 A hyperbolic substitution

We wish to integrate $\sqrt{1+t^4}$. It is not at all obvious that the result can be expressed in terms of incomplete elliptic integrals; we shall have to proceed through several steps first. The first step is to make the substitution $t = (-1)^{1/4}\sinh u$, for which $dt = (-1)^{1/4}\cosh u\, du$. So

$$\begin{aligned}
\int \sqrt{1+t^4}\,dt &= \int \sqrt{1+((-1)^{1/4}\sinh u)^4}\,(-1)^{1/4}\cosh u\,du && (9)\\
&= (-1)^{1/4}\int \cosh u\sqrt{1-\sinh^4 u}\,du && (10)\\
&= (-1)^{1/4}\int \cosh u\sqrt{1+\sinh^2 u}\sqrt{1-\sinh^2 u}\,du && (11)\\
&= (-1)^{1/4}\int \cosh^2 u\sqrt{1-\sinh^2 u}\,du && (12)
\end{aligned}$$

4 Recast in terms of circular trig functions

By the definition of the cosh function, and de Moivre's equation:

$$\begin{aligned}
\cosh(ix) &= \frac{e^{ix}+e^{-ix}}{2} && (13)\\
&= \frac{\cos x + i\sin x + \cos(-x) + i\sin(-x)}{2} && (14)\\
&= \frac{2\cos x}{2} && (15)\\
&= \cos x && (16)
\end{aligned}$$

Likewise

$$\begin{aligned}
\sinh(ix) &= \frac{e^{ix}-e^{-ix}}{2} && (17)\\
&= \frac{\cos x + i\sin x - \cos(-x) - i\sin(-x)}{2} && (18)\\
&= \frac{2i\sin x}{2} && (19)\\
&= i\sin x && (20)
\end{aligned}$$

To proceed, we employ the substitution $u = ix$, for which $du = i\,dx$. We then obtain

$$\begin{aligned}
\int \cosh^2 u\sqrt{1-\sinh^2 u}\,du &= \int \cosh^2(ix)\sqrt{1-\sinh^2(ix)}\,i\,dx && (21)\\
&= i\int \cos^2 x\sqrt{1-(i\sin x)^2}\,dx && (22)\\
&= i\int \cos^2 x\sqrt{1+\sin^2 x}\,dx && (23)
\end{aligned}$$

5 Parts

Now, use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\begin{aligned} J &= \int \cos^2 x\sqrt{1+\sin^2 x}\,dx && (24)\\ &= \int \frac{1}{2}(1+\cos 2x)\sqrt{1+\sin^2 x}\,dx && (25)\\ &= \frac{1}{2}\int \cos 2x\sqrt{1+\sin^2 x}\,dx + \frac{1}{2}\int \sqrt{1+\sin^2 x}\,dx && (26)\\ &= \frac{1}{4}\int 2\cos 2x\sqrt{1+\sin^2 x}\,dx + \frac{1}{2}E(x|-1) && (27) \end{aligned}$$

Note that we have used the function E; keep watching and this will later cancel out of the answer, although F will shortly enter the picture.

We tackle the first term by parts:

$$\begin{aligned} \int \sqrt{1+\sin^2 x}\,(2\cos 2x\,dx) &= \sqrt{1+\sin^2 x}\int 2\cos 2x\,dx - \int\left(\int 2\cos 2x\,dx\right)\left(\frac{d}{dx}\sqrt{1+\sin^2 x}\right)dx && (28)\\ &= \sin 2x\sqrt{1+\sin^2 x} - \int \sin 2x\frac{\sin 2x}{2\sqrt{1+\sin^2 x}}\,dx && (29)\\ &= \sin 2x\sqrt{1+\sin^2 x} - \int \frac{\sin^2 2x}{2\sqrt{1+\sin^2 x}}\,dx && (30)\\ & && (31) \end{aligned}$$

Now

$$\begin{aligned} \int \frac{\sin^2 2x}{2\sqrt{1+\sin^2 x}}\,dx &= \int \frac{(2\sin x\cos x)^2}{2\sqrt{1+\sin^2 x}}\,dx && (32)\\ &= \int \frac{2\sin^2 x\cos^2 x}{\sqrt{1+\sin^2 x}}\,dx && (33)\\ &= \int \frac{2\sin^2 x(1-\sin^2 x)}{\sqrt{1+\sin^2 x}}\,dx && (34)\\ &= \int \frac{2\sin^2 x - 2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx && (35) \end{aligned}$$

Putting it all together we have so far, from (27), (31), and (35):

$$J = \frac{1}{4}\left(\sin 2x\sqrt{1+\sin^2 x} - \int \frac{2\sin^2 x - 2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx\right) + \frac{1}{2}E(x|-1) \tag{36}$$

6 Some *kombinowanie*

We cast (24) into an alternative form as follows:

$$\begin{aligned} J/2 &= \frac{1}{2}\int \cos^2 x\sqrt{1+\sin^2 x}\,dx && (37)\\ &= \frac{1}{4}\int \frac{2\cos^2 x(1+\sin^2 x)}{\sqrt{1+\sin^2 x}}\,dx && (38)\\ &= \frac{1}{4}\int \frac{2(1-\sin^2 x)(1+\sin^2 x)}{\sqrt{1+\sin^2 x}}\,dx && (39)\\ &= \frac{1}{4}\int \frac{2-2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx && (40) \end{aligned}$$

Adding (40) and (36) gives

$$\begin{aligned}
3J/2 &= \frac{1}{4}\left(\sin 2x\sqrt{1+\sin^2 x} - \int \frac{2\sin^2 x - 2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx\right) + \frac{1}{2}E(x|-1) + \frac{1}{4}\int \frac{2-2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx && (41)\\
&= \frac{1}{4}\sin 2x\sqrt{1+\sin^2 x} + \frac{1}{2}E(x|-1) + \frac{1}{4}\left(\int \frac{2-2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx - \int \frac{2\sin^2 x - 2\sin^4 x}{\sqrt{1+\sin^2 x}}\,dx\right) && (42)\\
&= \frac{1}{4}\sin 2x\sqrt{1+\sin^2 x} + \frac{1}{2}E(x|-1) + \frac{1}{4}\left(\int \frac{2-2\sin^2 x}{\sqrt{1+\sin^2 x}}\,dx\right) && (43)\\
&= \frac{1}{4}\sin 2x\sqrt{1+\sin^2 x} + \frac{1}{2}E(x|-1) + \frac{1}{4}\left(\int \frac{4}{\sqrt{1+\sin^2 x}}\,dx - \int \frac{2+2\sin^2 x}{\sqrt{1+\sin^2 x}}\,dx\right) && (44)\\
&= \frac{1}{4}\sin 2x\sqrt{1+\sin^2 x} + \frac{1}{2}E(x|-1) + \frac{1}{4}\left(\int \frac{4}{\sqrt{1+\sin^2 x}}\,dx - \int 2\sqrt{1+\sin^2 x}\,dx\right) && (45)\\
&= \frac{1}{4}\sin 2x\sqrt{1+\sin^2 x} + \frac{1}{2}E(x|-1) + \frac{1}{4}(4F(x|-1) - 2E(x|-1)) && (46)\\
&= \frac{1}{4}\sin 2x\sqrt{1+\sin^2 x} + F(x|-1) && (47)\\
J &= \frac{1}{6}\sin 2x\sqrt{1+\sin^2 x} + \frac{2}{3}F(x|-1) && (48)
\end{aligned}$$

7 Home stretch

Recall that $u = ix$, or $x = -iu$. Also, we see that $\sin(ix) = (-i)i\sin(ix) = (-i)\sinh(i(ix)) = (-i)\sinh(-x) = i\sinh x$ by using (20). Now substitute (48) back into (23), to obtain:

$$\begin{aligned}
\int \cosh^2 u\sqrt{1-\sinh^2 u}\,du &= \frac{i}{6}\sin 2x\sqrt{1+\sin^2 x} + \frac{2i}{3}F(x|-1) && (49)\\
&= \frac{i}{6}\sin -2iu\sqrt{1+\sin^2 -iu} + \frac{2i}{3}F(-iu|-1) && (50)\\
&= \frac{i}{6}i\sinh -2u\sqrt{1+(i\sinh -u)^2} + \frac{2i}{3}F(-iu|-1) && (51)\\
&= \frac{1}{6}\sinh 2u\sqrt{1-\sinh^2 u} + \frac{2i}{3}F(-iu|-1) && (52)
\end{aligned}$$

Finally, recall that $t = (-1)^{1/4}\sinh u$, so that $u = \sinh^{-1}((-1)^{7/4}t)$. Substituting (52) back into (12) gives:

$$\int \sqrt{1+t^4}\,dt = (-1)^{1/4}\left(\frac{1}{6}\sinh 2u\sqrt{1-\sinh^2 u} + \frac{2i}{3}F(-iu|-1)\right) \tag{53}$$

$$\begin{aligned} &= (-1)^{1/4}\Bigg(\frac{1}{6}\sinh[2\sinh^{-1}((-1)^{7/4}t)]\sqrt{1-\sinh^2(\sinh^{-1}((-1)^{7/4}t))} \\ &+ \frac{2i}{3}F(-i[\sinh^{-1}((-1)^{7/4}t)]|-1)\Bigg) \end{aligned} \tag{54}$$

$$\begin{aligned} &= (-1)^{1/4}\Bigg(\frac{1}{6}\cdot 2\sinh[\sinh^{-1}((-1)^{7/4}t)]\cosh[\sinh^{-1}((-1)^{7/4}t)] \\ &\times \sqrt{1-((-1)^{7/4}t)^2} + \frac{2i}{3}F(-i[\sinh^{-1}((-1)^{7/4}t)]|-1)\Bigg) \end{aligned} \tag{55}$$

$$\begin{aligned} &= (-1)^{1/4}\Bigg(\frac{1}{3}((-1)^{7/4}t)\sqrt{1+((-1)^{7/4}t)^2}\sqrt{1-((-1)^{7/4})^2} \\ &+ \frac{2i}{3}F(-i[\sinh^{-1}((-1)^{7/4}t)]|-1)\Bigg) \end{aligned} \tag{56}$$

$$= (-1)^{1/4}\left(\frac{1}{3}((-1)^{7/4}t)\sqrt{1-((-1)^{7/4}t)^4} + \frac{2i}{3}F(-i[\sinh^{-1}((-1)^{7/4}t)]|-1)\right) \tag{57}$$

$$= \frac{1}{3}t\sqrt{1+t^4} + \frac{2}{3}(-1)^{3/4}F(-i[\sinh^{-1}((-1)^{7/4}t)]|-1) \tag{58}$$

$$= \frac{1}{3}t\sqrt{1+t^4} - \frac{2}{3}(-1)^{1/4}F(i\sinh^{-1}((-1)^{1/4}t)|-1) \tag{59}$$

which, when put over the denominator $3\sqrt{1+t^4}$, exactly matches Mathematica's output.