You are allowed to discuss with others but are not allowed to use any references other than the coursenotes and the three reference books. Please list your collaborators for each question. You must write your own solutions. See the course outline for the homework policy.

This homework accounts for 8% of your total grade.

Problem 1 (25 Points) - Asymptotics

Determine the tightest asymptotic complexity $(O, \Omega, \Theta, o, \omega)$ of each pair of functions. Please show work.

For example: when given $f(n) = 5n \log n + 3n$ vs $g(n) = 2n \log n + 7n$, the correct answer should be $f(n) = \Theta(g(n))$, as $f(n) = \Theta(n \log n) = g(n)$.

1. $f(n) = 3n^2 + 4n + 5$ vs $g(n) = 2n^2 + 6n \log n$
2. $f(n) = 4^n$ vs $g(n) = n!$
3. $f(n) = 2^n$ vs $g(n) = n^3$
4. $f(n) = n \log n + n$ vs $g(n) = n^2$
5. $f(n) = 1000$ vs $g(n) = \log n$
6. $f(n) = n((-1)^n + 2)$ vs $g(n) = 10n$
7. $f(n) = 2n^3 + n^2$ vs $g(n) = 5n^3$
8. $f(n) = \log^\gamma n$ vs $g(n) = 2n$, where $\gamma > 0$ is a constant

Problem 2 (25 Points) - Recurrences

Solve the following recurrences (please show proof of correctness of your solution):

1. $T(n) = T(n - 10) + n$
2. $T(n) = 9 \cdot T(n/3) + n^2$
3. $T(n) = T(n/2) + T(n/4) + n$
4. $T(n) = 5 \cdot T(n/3) + \log^2 n$
5. $T(n) = 2 \cdot T(n - 2) + 1$

PROBLEM 3 (25 Points) - Maximum Ad Space

Computational model: For this question, you are required to work on the word RAM model, with word size $O(\log n)$ for input of size n, and you have enough memory. In your algorithm design and analysis, you are only required to make sure that the word size is always $O(\log n)$.

(a) Imagine an advertising company would like to post a huge poster in Toronto harbour front. There are n consecutive buildings there, with positive integer height $h_1, \ldots, h_n \in [1, n^{100}]$ and unit width as shown in the figure.

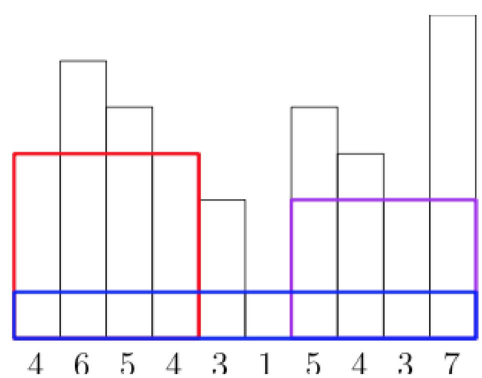


Figure 1: These are three possible solutions. The largest one has area $4 \times 4 = 16$.

Design a divide and conquer algorithm to find the maximum rectangular space to post their poster. Stated mathematically, find $1 \leq i \leq j \leq n$ to maximize

$$(j - i + 1) \cdot \min_{i \leq k \leq j} h_k$$

You will get full marks if your algorithm is correct, you present a (correct) proof of correctness, and the runtime is $O(n \log n)$.

For fun (i.e. no marks): try to get an algorithm which runs in time $O(n)$

(b) Imagine the government would like to build a huge park in the city. The city is an $n \times n$ grid, with some units occupied. Use (a) or otherwise, design an algorithm to find the maximum (axis-aligned) rectangular unoccupied space to build the park.

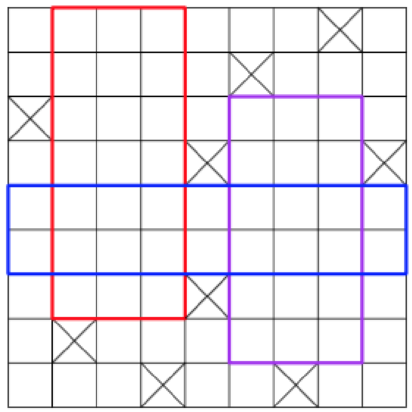


Figure 2: These are three possible solutions. The largest one has area $7 \times 3 = 21$.

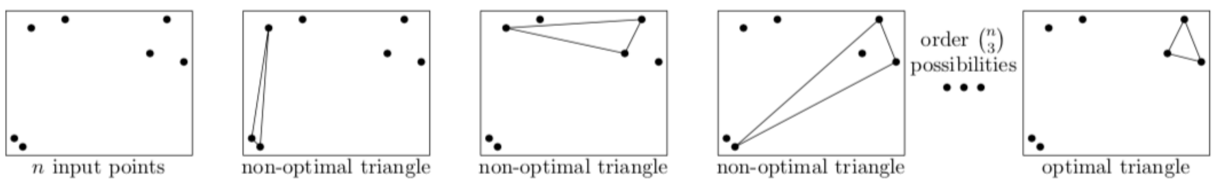
You will get full marks if the time complexity is $O(n^2)$ and the proofs are correct. You can assume that there is an $O(n)$ time algorithm for part (a).

Problem 4 (25 Points) - Smallest Perimeter

Computational model: you can work on the unit-cost model.

Suppose we are given n input points with positive integer coordinates, and we want to find the **optimal triangle** that can be formed by choosing 3 of these input points.

The optimal triangle is the one with the **smallest perimeter**.



Give an efficient divide & conquer algorithm to solve this problem, briefly justify its correctness (can be done as part of the algorithm explanation), and analyze its complexity using either Θ or big-O notation (doesn't matter which). You will get full marks if your algorithm is correct (with a proof of correctness) and has runtime $O(n \log n)$ (also a proof of this runtime must be provided).

In your recurrence relation you may ignore floors/ceilings/off-by-1-or-2.

You may assume you are given a constant time function $perimeter(a, b, c)$ that computes the perimeter of a triangle, given three points a, b, c.

Hint: try to modify the algorithm for closest pair.

Hint for the analysis: Suppose a triangle cannot have perimeter smaller than some δ. Then, how many *triangles* can be in a square of side length $\delta/4$?