# CS 341: ALGORITHMS

**Lecture 12: graph algorithms III – DAG testing, topsort, SCC**

Readings: see website

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## DFS APPLICATION: **TESTING** WHETHER A GRAPH IS A **DAG**

A directed graph $G$ is a **directed acyclic graph**, or **DAG**, if $G$ contains no directed cycle.



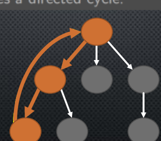
**Lemma 6.7**

*A directed graph is a DAG if and only if a depth-first search encounters no back edges.*

**Proof.**

($\Rightarrow$): Any back edge creates a directed cycle.

Back edge: **points to an ancestor** in the DFS forest

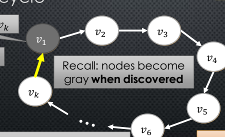




- Case ($\Leftarrow$): Suppose $\exists$ directed cycle. Show $\exists$ back edge.
  - Let $v_1, v_2, \ldots, v_k, v_1$ be a directed cycle
  - WLOG let $v_1$ be *earliest* discovered node in the cycle

$v_k \rightarrow v_1$

| edge type | discovery/finish times |
|---|---|
| tree | $d[v_k] < d[v_1] < f[v_1] < f[v_k]$ |
| forward | $d[v_k] < d[v_1] < f[v_1] < f[v_k]$ |
| back | $d[v_1] < d[v_k] < f[v_k] < f[v_1]$ |
| cross | $d[v_1] < f[v_1] < d[v_k] < f[v_k]$ |




Discovered **before** $v_2, \ldots, v_k$

Consider edge $\{v_k, v_1\}$

Since $d[v_1] < d[v_k]$, $\{v_k, v_1\}$ must be a **back** or **cross** edge. Why?

Recall: nodes become gray **when discovered**

So **when $v_1$ is discovered**, $v_2, \ldots, v_k$ are **all white**

**Recall: every** node $v_i$ that is **white-reachable** from $v_1$ when we discover $v_1$ (call $DFSVisit(v_1)$) turns **black before $v_1$** ($f[v_i] < f[v_1]$)

So $v_k$ must turn black **before** $v_1$, and we have $f[v_k] < f[v_1]$.

Thus, $\{v_k, v_1\}$ must be a **back edge**. QED



## TURNING THE **LEMMA** INTO AN **ALGORITHM**

**Lemma 6.7**

*A directed graph is a DAG if and only if a depth-first search encounters no back edges.*

- Search for back edges
- How to identify a back-edge?

When we observe an edge from $u$ to $v$, check if **$v$ is gray**

| edge type | colour of $v$ | discovery/finish times |
|---|---|---|
| tree | white | $d[u] < d[v] < f[v] < f[u]$ |
| forward | black | $d[u] < d[v] < f[v] < f[u]$ |
| back | gray | $d[v] < d[u] < f[u] < f[v]$ |
| cross | black | $d[v] < f[v] < d[u] < f[u]$ |

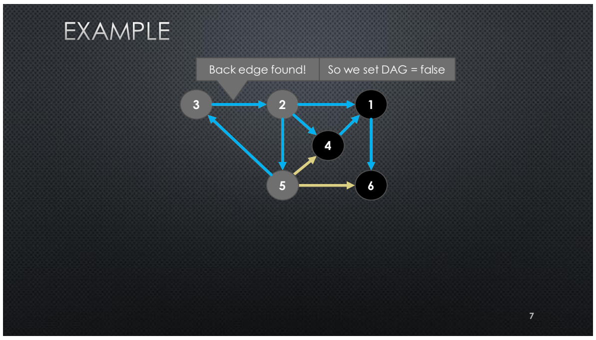
Back edge

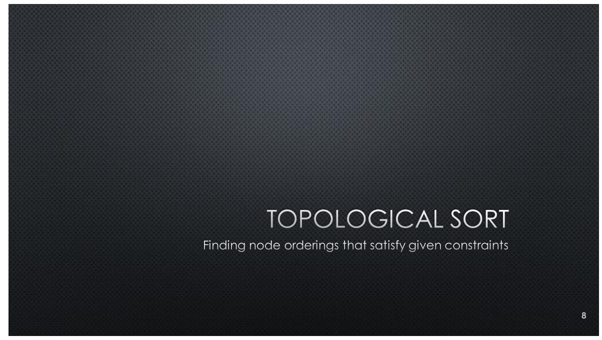


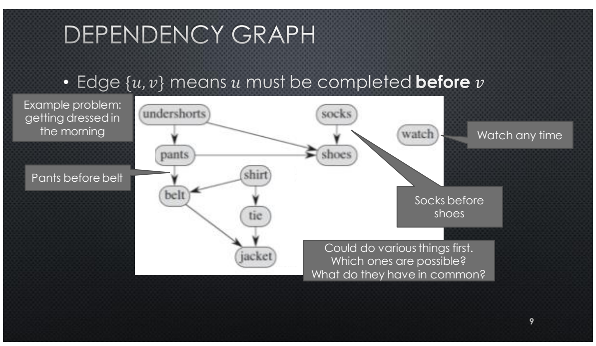

## DFS: TESTING WHETHER A GRAPH IS A DAG

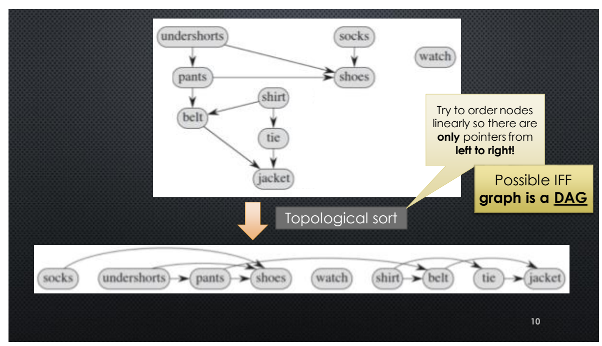
```
global variables:
    pred[1..n] = [null, null, ..., null]
    colour[1..n] = [white, white, ..., white]
    d[1..n] = [0, 0, ..., 0] // discovery times
    f[1..n] = [0, 0, ..., 0] // finish times
    time = 0
    DAG = true

IsDAG(adj[1..n])
    for v = 1..n
        if colour[v] == white
            DFSVisit(adj, v)
    return DAG
```

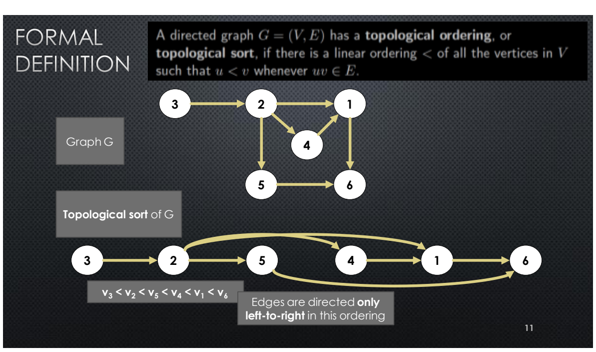
```
DFSVisit(adj[1..n], v)
    colour[v] = gray
    time = time + 1
    d[v] = time

    for each w in adj[v]
        if colour[w] == white
            pred[w] = v
            DFSVisit(w)
        if color[w] == gray
            DAG = false

    colour[v] = black
    time = time + 1
    f[v] = time
```

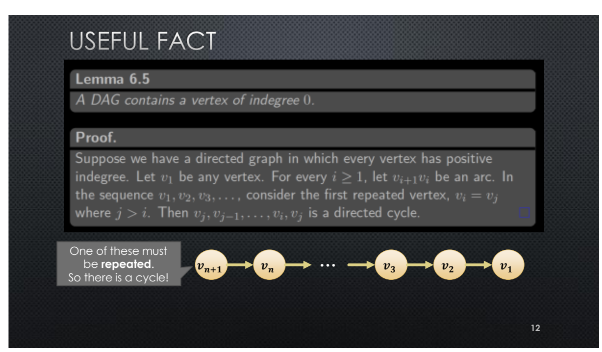

EXAMPLE
Back edge found!
So we set DAG = false
3
2
1
4
5
6
7


TOPOLOGICAL SORT
Finding node orderings that satisfy given constraints
8


DEPENDENCY GRAPH
• Edge $\{u, v\}$ means $u$ must be completed **before** $v$
Example problem: getting dressed in the morning
Pants before belt
Watch any time
Socks before shoes
Could do various things first. Which ones are possible? What do they have in common?
undershorts
socks
watch
pants
shoes
shirt
belt
tie
jacket
9


undershorts
socks
watch
pants
shoes
shirt
belt
tie
jacket
Try to order nodes linearly so there are **only** pointers from **left to right!**
Possible IFF **graph is a DAG**
Topological sort
socks
undershorts
pants
shoes
watch
shirt
belt
tie
jacket
10


FORMAL DEFINITION
A directed graph $G = (V, E)$ has a **topological ordering**, or **topological sort**, if there is a linear ordering $<$ of all the vertices in $V$ such that $u < v$ whenever $uv \in E$.
Graph G
**Topological sort** of G
3
2
1
4
5
6
$v_3 < v_2 < v_5 < v_4 < v_1 < v_6$
Edges are directed **only left-to-right** in this ordering
11


USEFUL FACT
**Lemma 6.5**
*A DAG contains a vertex of indegree 0.*
**Proof.**
Suppose we have a directed graph in which every vertex has positive indegree. Let $v_1$ be any vertex. For every $i \geq 1$, let $v_{i+1}v_i$ be an arc. In the sequence $v_1, v_2, v_3, \ldots$, consider the first repeated vertex, $v_i = v_j$ where $j > i$. Then $v_j, v_{j-1}, \ldots, v_i, v_j$ is a directed cycle.
One of these must be **repeated**. So there is a cycle!
$v_{n+1}$
$v_n$
$\cdots$
$v_3$
$v_2$
$v_1$
12

# TOPOLOGICAL SORT VIA DFS

- We can implement topological sort by using **DFS**!
- The **finishing times** of nodes help us
- Understanding this algo will be **key** for understanding **strongly connected components**



**Lemma 6.8**

*Suppose $D$ is a DAG. Then $f[v] < f[u]$ for every arc $uv$.*

Recall from DAG-testing: there are **no back edges** in a DAG

| edge type | colour of $v$ | discovery/finish times |
|---|---|---|
| tree | white | $d[u] < d[v] < \boxed{f[v] < f[u]}$ |
| forward | black | $d[u] < d[v] < \boxed{f[v] < f[u]}$ |
| ~~back~~ | ~~gray~~ | $\cancel{d[v] < d[u] < f[u] < f[v]}$ |
| cross | black | $d[v] < \boxed{f[v] < d[u] < f[u]}$ |




**Theorem:** if D is a DAG, and we order vertices in **reverse order of finishing time**, (i.e., by largest to smallest finish time) then we get a topological ordering!

To see **why**, suppose D is a DAG and we order nodes in this way, so $f_{v_1} > f_{v_2} > \cdots > f_{v_{n-1}} > f_{v_n}$


For contradiction, suppose a **right-to-left** edge $\{u, v\}$ exists

By our node ordering, $f_v > f_u$

But the lemma says for every edge $\{u, v\}$, we must have $f_v < f_u$

**Lemma 6.8**

*Suppose $D$ is a DAG. Then $f[v] < f[u]$ for every arc $uv$.*

**Contradiction!** Right-to-left edge cannot exist. So is is a topological ordering.



# TOPOLOGICAL ORDERING VIA DFS

$O(n + m)$ w/adj. lists

```
global variables:
    pred[1..n] = [null, null, ..., null]
    colour[1..n] = [white, white, ..., white]
    d[1..n] = [0, 0, ..., 0] // discovery times
    f[1..n] = [0, 0, ..., 0] // finish times
    time = 0
    DAG = true

TopologicalSort(adj[1..n])
    S = new stack
    for v = 1..n
        if colour[v] == white
            DFSVisit(adj, v, S)
    if DAG then return S
    return null
```

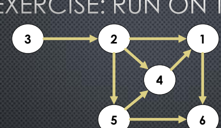
```
DFSVisit(adj[1..n], v, S)
    colour[v] = gray
    time = time + 1
    d[v] = time

    for each w in adj[v]
        if colour[w] == white
            pred[w] = v
            DFSVisit(w)
        if color[w] == gray
            DAG = false

    colour[v] = black
    S.push(v)
    time = time + 1
    f[v] = time
```

Save each node when it **finishes**

**Push smallest** finishing time first → **pop largest** first



# HOME EXERCISE: RUN ON THIS GRAPH



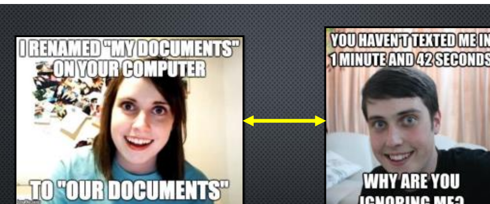

The initial calls are $DFSvisit(1)$, $DFSvisit(2)$ and $DFSvisit(3)$.

The discovery/finish times are as follows:

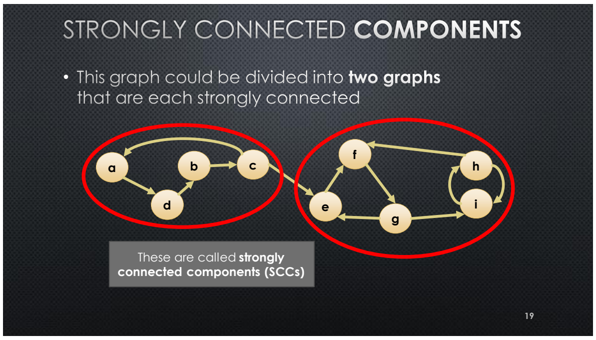
| $v$ | $d[v]$ | $f[v]$ | $v$ | $d[v]$ | $f[v]$ |
|---|---|---|---|---|---|
| 1 | 1 | 4 | 4 | 6 | 7 |
| 2 | 5 | 10 | 5 | 8 | 9 |
| 3 | 11 | 12 | 6 | 2 | 3 |

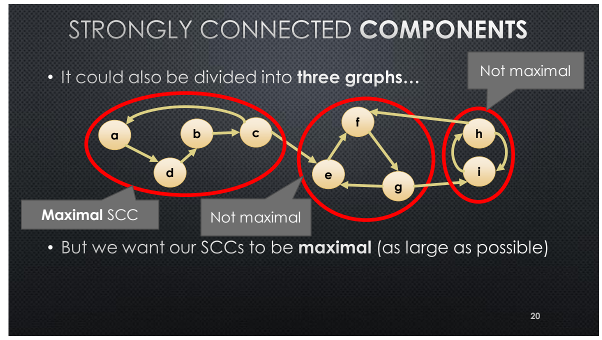
The topological ordering is 3, 2, 5, 4, 1, 6 (reverse order of finishing time).

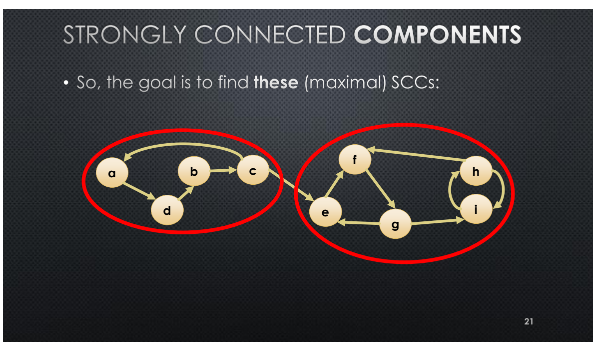


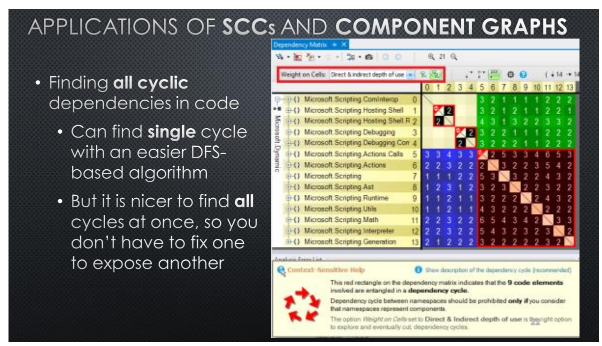


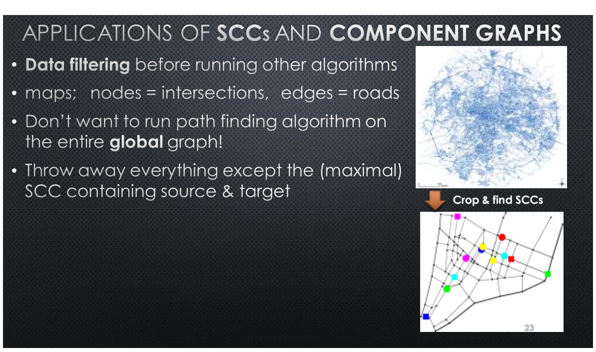

# STRONGLY CONNECTED COMPONENTS

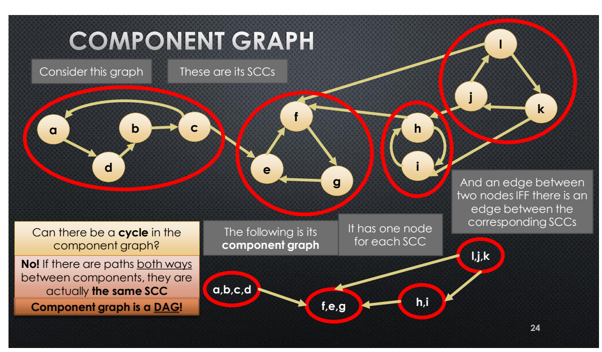

STRONGLY CONNECTED COMPONENTS
• This graph could be divided into two graphs that are each strongly connected
a
b
c
d
e
f
g
h
i
These are called strongly connected components (SCCs)
19

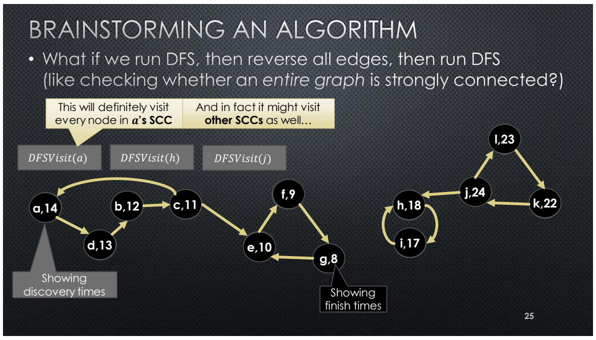

STRONGLY CONNECTED COMPONENTS
• It could also be divided into three graphs…
Not maximal
a
b
c
d
e
f
g
h
i
Maximal SCC
Not maximal
• But we want our SCCs to be maximal (as large as possible)
20

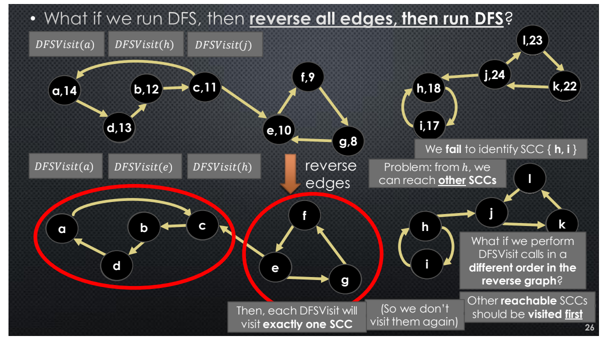

STRONGLY CONNECTED COMPONENTS
• So, the goal is to find these (maximal) SCCs:
a
b
c
d
e
f
g
h
i
21

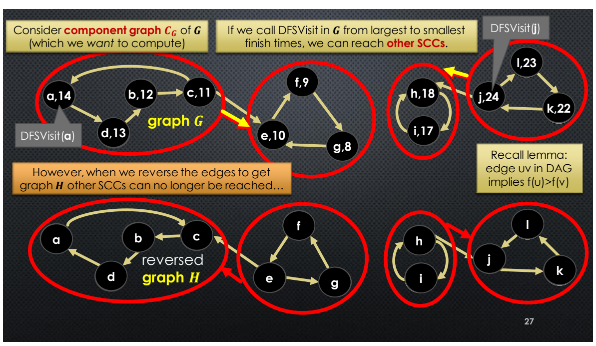

APPLICATIONS OF SCCs AND COMPONENT GRAPHS
• Finding all cyclic dependencies in code
• Can find single cycle with an easier DFS-based algorithm
• But it is nicer to find all cycles at once, so you don't have to fix one to expose another
22


APPLICATIONS OF SCCs AND COMPONENT GRAPHS
• Data filtering before running other algorithms
• maps; nodes = intersections, edges = roads
• Don't want to run path finding algorithm on the entire global graph!
• Throw away everything except the (maximal) SCC containing source & target
Crop & find SCCs
23

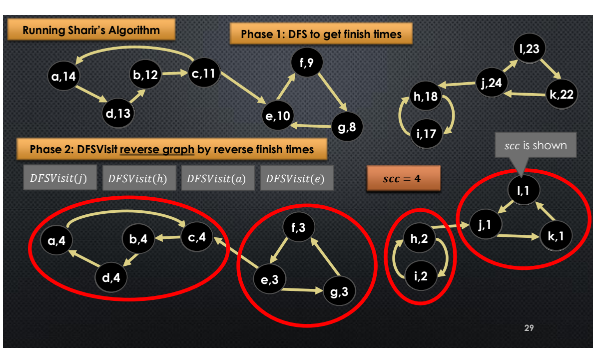

COMPONENT GRAPH
Consider this graph
These are its SCCs
a
b
c
d
e
f
g
h
i
j
k
l
And an edge between two nodes IFF there is an edge between the corresponding SCCs
Can there be a cycle in the component graph?
No! If there are paths both ways between components, they are actually the same SCC
Component graph is a DAG!
The following is its component graph
It has one node for each SCC
l,j,k
a,b,c,d
f,e,g
h,i
24

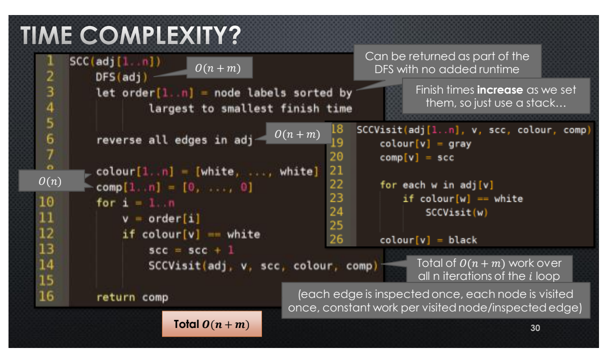

BRAINSTORMING AN ALGORITHM
What if we run DFS, then reverse all edges, then run DFS
(like checking whether an entire graph is strongly connected?)
This will definitely visit every node in a's SCC
And in fact it might visit other SCCs as well…
DFSVisit(a)
DFSVisit(h)
DFSVisit(j)
a,14
b,12
c,11
d,13
e,10
f,9
g,8
h,18
i,17
j,24
k,22
l,23
Showing discovery times
Showing finish times
25


What if we run DFS, then reverse all edges, then run DFS?
DFSVisit(a)
DFSVisit(h)
DFSVisit(j)
DFSVisit(a)
DFSVisit(e)
DFSVisit(h)
reverse edges
We fail to identify SCC { h, i }
Problem: from h, we can reach other SCCs
What if we perform DFSVisit calls in a different order in the reverse graph?
Then, each DFSVisit will visit exactly one SCC
(So we don't visit them again)
Other reachable SCCs should be visited first
26


Consider component graph $C_G$ of G (which we want to compute)
If we call DFSVisit in G from largest to smallest finish times, we can reach other SCCs.
DFSVisit(j)
DFSVisit(a)
graph G
However, when we reverse the edges to get graph H other SCCs can no longer be reached…
Recall lemma: edge uv in DAG implies f(u)>f(v)
reversed graph H
27


SCC ALGORITHM
SCC(adj[1..n])
DFS(adj)
let order[1..n] = node labels sorted by
largest to smallest finish time
reverse all edges in adj
colour[1..n] = [white, ..., white]
comp[1..n] = [0, ..., 0]
for i = 1..n
v = order[i]
if colour[v] == white
scc = scc + 1
SCCVisit(adj, v, scc, colour, comp)
return comp
SCCVisit(adj[1..n], v, scc, colour, comp)
colour[v] = gray
comp[v] = scc
for each w in adj[v]
if colour[w] == white
SCCVisit(w)
colour[v] = black
This is called Sharir's algorithm (sometimes Kosaraju's algorithm). This paper first introduced it.
28


Running Sharir's Algorithm
Phase 1: DFS to get finish times
Phase 2: DFSVisit reverse graph by reverse finish times
DFSVisit(j)
DFSVisit(h)
DFSVisit(a)
DFSVisit(e)
scc = 4
scc is shown
29


TIME COMPLEXITY?
O(n + m)
Can be returned as part of the DFS with no added runtime
Finish times increase as we set them, so just use a stack…
O(n + m)
O(n)
Total of O(n + m) work over all n iterations of the i loop
(each edge is inspected once, each node is visited once, constant work per visited node/inspected edge)
Total O(n + m)
30

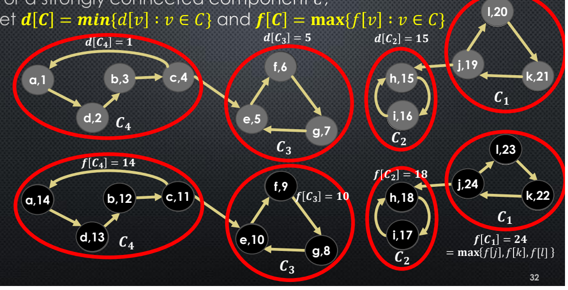
## CORRECTNESS

- Want to prove that each top-level call to SCCVisit explores **exactly** the nodes **in one SCC**
- Proof hinges on a key lemma that talks about the **finish times of SCCs** in the **component graph**
- To talk about finish times of **SCCs**, we need a definition…



## A KEY DEFINITION

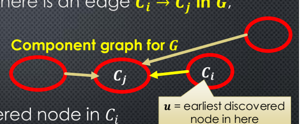
- For a strongly connected component $C$, let $\boldsymbol{d[C] = min\{d[v] : v \in C\}}$ and $\boldsymbol{f[C] = \max\{f[v] : v \in C\}}$






## A KEY LEMMA

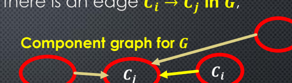
- **Lemma:** if $C_i, C_j$ are SCCs and there is an edge $\boldsymbol{C_i \to C_j}$ **in** $\boldsymbol{G}$, then $\boldsymbol{f[C_i] > f[C_j]}$
- **Proof.** Case 1 $(\boldsymbol{d[C_i] < d[C_j]})$:
  - Let $u$ be the earliest discovered node in $C_i$
  - All nodes in $C_i \cup C_j$ are white-reachable from $u$, so they are **descendants in the DFS forest** and **finish before** $\boldsymbol{u}$
  - So $f[C_i] = f[u] > f[C_j]$






## A KEY LEMMA

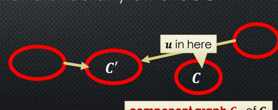
- **Lemma:** if $C_i, C_j$ are SCCs and there is an edge $\boldsymbol{C_i \to C_j}$ **in** $\boldsymbol{G}$, then $\boldsymbol{f[C_i] > f[C_j]}$
- **Proof. Case 2 $(\boldsymbol{d[C_j] < d[C_i]})$:**
  - Since component graph is a DAG, there is **no path** $\boldsymbol{C_j \to C_i}$
  - Thus, **no nodes** in $C_i$ are reachable from $C_j$
  - So we discover $C_j$ and finish $C_j$ **without** discovering $C_i$
  - Therefore $d[C_j] < \boldsymbol{f[C_j]} < d[C_i] < \boldsymbol{f[C_i]}$. **QED**






## COMPLETING THE PROOF

- Suppose we have performed DFS to get our finish times, and we are about to perform SCCVisits on the reverse graph
- **We prove each top-level SCCVisit call visits precisely one SCC**
- Consider the first top-level SCCVisit$(u)$
- Let $C$ be the SCC containing $u$ and $C'$ be any other SCC
- Since we call SCCVisit on nodes starting from the **largest finish time**,
  - We know $\boldsymbol{f(C) > f(C')}$






## COMPLETING THE PROOF

- We know $\boldsymbol{f(C) > f(C')}$
- By Lemma: if there were an edge $C' \to C$ in $G$, then we would have $f(C') > f(C)$
  - So there is no edge $C' \to C$ in $G$
  - and hence **no edge** $\boldsymbol{C \to C'}$ **in** $\boldsymbol{H}$
  - So, **SCCVisit**$\boldsymbol{(u)}$ **in** $\boldsymbol{H}$ **cannot visit** $\boldsymbol{C'}$

… and sets comp[v] = scc for all nodes in the SCC

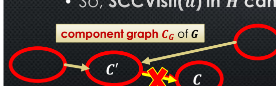
**So each top-level call explores one SCC…**

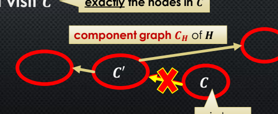
and **larger finish** time means **already explored**!

In $G$, edges go from larger to smaller finish times. **In $H$, edges go from smaller to larger.**

Similar argument for subsequent **top-level** calls to SCCVisit.

**So SCCVisit$(u)$ visits exactly the nodes in $C$**



## IF WE HAVE TIME

**topological sort** without relying on DFS



## EXISTENCE OF A TOPOLOGICAL SORT ORDER

**Theorem 6.6**

*A directed graph $D$ has a topological sort if and only if it is a DAG.*

**Proof.**

($\Rightarrow$): Suppose $D$ has a directed cycle $v_1, v_2, \ldots, v_j, v_1$. Then $v_1 < v_2 < \cdots < v_j < v_1$, so a topological ordering does not exist.

($\Leftarrow$): Suppose $D$ is a DAG. Then the algorithm below constructs a topological ordering. $\square$



```
Kahn(adj[1..n])
    indeg[1..n] = [0, ..., 0]
    for each edge (u,v) in adj
        indeg[v] = indeg[v] + 1

    order = new list
    q = new queue containing {v : indeg[v] == 0}
    for i = 1..n
        if q.empty() return null
        v = q.dequeue()
        order.append(v)

        for each w in adj[v]
            indeg[w] = indeg[w] - 1
            if indeg[w] == 0 then q.enqueue(w)

    return order
```

$indeg[v]$ = # of edges pointing **into** node $v$

= number of **unsatisfied constraints** on $v$

Nodes with $\boldsymbol{indeg}$ **0** have **no unsatisfied dependencies**

So this step is enqueuing nodes whose dependencies are already satisfied

$\boldsymbol{q}$ **always** contains nodes with no unsatisfied dependencies (indeg 0)

No such order!

Add $v$ to the topological order

Remove $v$'s out edges. If we have now satisfied all dependencies for some $w$, add $w$ to the queue also.



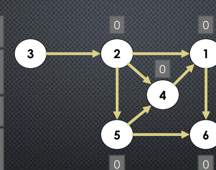
## EXAMPLE (KAHN'S ALGORITHM)

Compute **indegree** for all vertices

For each node u
For each w in adj(u)
w.deg = w.deg+1

vertices with indeg 0 go into the queue

Until Q is empty: pop, output that element, decrement its neighbours, enqueue new indeg 0's




Queue Q

3 2 5 4 1 6

Output



```
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    indeg[1..n] = [0, ..., 0]
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    return order
```

Running time with adjacency lists?

$O(n)$

$O(n+m)$ total work over all iterations

$O(n)$ iterations

$O(1)$ per check

$O(1)$

$O(\deg(v))$ per iteration $i$

$\sum_{v \in V} \deg(v) \in \boldsymbol{O(n+m)}$ **total work over all nodes** $v$

Total $O(n+m)$



## BONUS SLIDES

## SCC: HOW ABOUT A DIFFERENT ORDERING?

- Rather than doing DFS in the **reverse** graph in order of **decreasing** finish times
- Why not do DFS in the **original** graph in order of **increasing** finish times?
- Exercise: does this work?

43

## SCC: HOW ABOUT A DIFFERENT ORDERING?

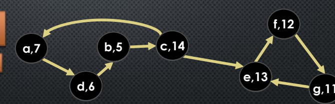
- Why not do DFS in the **original** graph in order of **increasing** finish times?

**Doesn't work!**

**Output depends where first DFS starts…**

**If first DFS starts at c, then…**

**DFSVisit(b) would reach <u>two</u> SCCs.**




44