CS 341: ALGORITHMS

Lecture 12: graph algorithms III – DAG testing, topsort, SCC

Readings: see website

Trevor Brown
https://student.cs.uwaterloo.ca/~cs341
trevor.brown@uwaterloo.ca

DFS APPLICATION: **TESTING** WHETHER A GRAPH IS A **DAG**

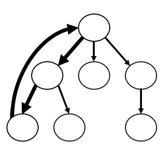
A directed graph G is a **directed acyclic graph**, or **DAG**, if G contains no directed cycle.

Lemma 6.7

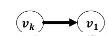
A directed graph is a DAG if and only if a depth-first search encounters no back edges.

back edges.

Back edge: **points to an ancestor** in the DFS forest



- Case (⇐): Suppose ∃ directed cycle. Show ∃ back edge.
  - Let $v_1, v_2, \ldots, v_k, v_1$ be a directed cycle
  - WLOG let v_1 be *earliest* discovered node in the cycle

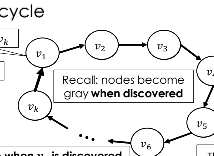


edge type	discovery/finish times
tree	$d[v_k] < d[v_1] < f[v_1] < f[v_k]$
forward	$d[v_k] < d[v_1] < f[v_1] < f[v_k]$
back	$d[v_1] < d[v_k] < f[v_k] < f[v_1]$
cross	$d[v_1] < f[v_1] < d[v_k] < f[v_k]$

Discovered **before** $v_2, \ldots, v_k$

Consider edge $\{v_k, v_1\}$

Since $d[v_1] < d[v_k]$, $\{v_k, v_1\}$ must be a **back** or **cross** edge. Why?



So **when v_1 is discovered**, $v_2, \ldots, v_k$ are **all white**

Recall: every node v_i that is **white-reachable** from v_1 when we discover v_1 (call $DFSVisit(v_1)$) turns **black before v_1** ($f[v_i] < f[v_1]$)

So v_k must turn black **before** v_1, and we have $f[v_k] < f[v_1]$.

Thus, $\{v_k, v_1\}$ must be a **back edge**. QED

TURNING THE **LEMMA** INTO AN **ALGORITHM**

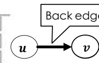
Lemma 6.7

A directed graph is a DAG if and only if a depth-first search encounters no back edges.

- Search for back edges
- How to identify a back-edge?

When we observe an edge from u to v, check if v **is gray**

edge type	colour of v	discovery/finish times
tree	**white**	$d[u] < d[v] < f[v] < f[u]$
forward	**black**	$d[u] < d[v] < f[v] < f[u]$
back	**gray**	$d[v] < d[u] < f[u] < f[v]$
cross	**black**	$d[v] < f[v] < d[u] < f[u]$

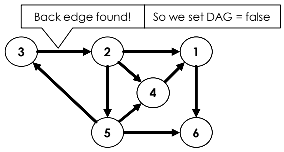


DFS: TESTING WHETHER A GRAPH IS A DAG

```
global variables:
    pred[1..n] = [null, null, ..., null]
    colour[1..n] = [white, white, ..., white]
    d[1..n] = [0, 0, ..., 0] // discovery times
    f[1..n] = [0, 0, ..., 0] // finish times
    time = 0
    DAG = true

IsDAG(adj[1..n])
    for v = 1..n
        if colour[v] == white
            DFSVisit(adj, v)
    return DAG
```

```
DFSVisit(adj[1..n], v)
    colour[v] = gray
    time = time + 1
    d[v] = time

    for each w in adj[v]
        if colour[w] == white
            pred[w] = v
            DFSVisit(w)
        if color[w] == gray
            DAG = false

    colour[v] = black
    time = time + 1
    f[v] = time
```

EXAMPLE



7

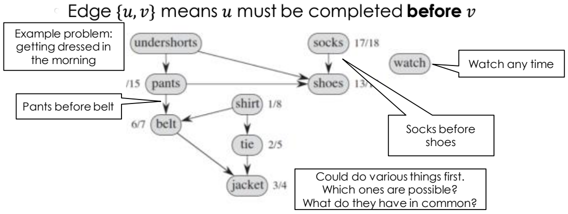
TOPOLOGICAL SORT

Finding node orderings that satisfy given constraints

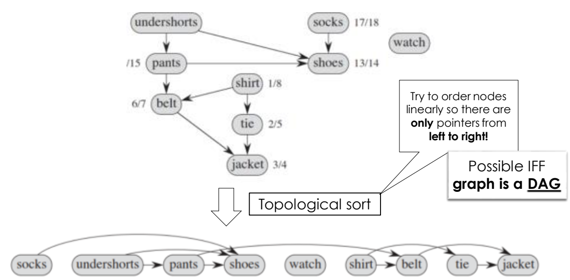
8

DEPENDENCY GRAPH

- Edge $\{u, v\}$ means u must be completed **before** v



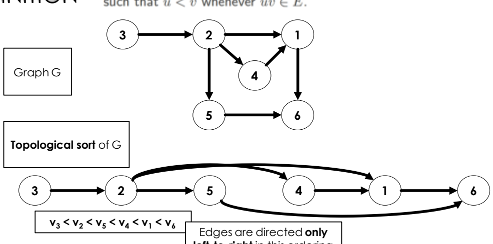
9



10

FORMAL DEFINITION

A directed graph $G = (V, E)$ has a **topological ordering**, or **topological sort**, if there is a linear ordering $<$ of all the vertices in V such that $u < v$ whenever $uv \in E$.



11

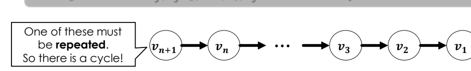
USEFUL FACT

Lemma 6.5

A DAG contains a vertex of indegree 0.

Proof.

Suppose we have a directed graph in which every vertex has positive indegree. Let v_1 be any vertex. For every $i \geq 1$, let $v_{i+1}v_i$ be an arc. In the sequence $v_1, v_2, v_3, \ldots$, consider the first repeated vertex, $v_i = v_j$ where $j > i$. Then $v_j, v_{j-1}, \ldots, v_i, v_j$ is a directed cycle.



12

TOPOLOGICAL SORT VIA DFS

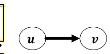
- We can implement topological sort by using **DFS**!
- The **finishing times** of nodes help us
- Understanding this algo will be **key** for understanding **strongly connected components**

Lemma 6.8

Suppose D is a DAG. Then $f[v] < f[u]$ for every arc uv.

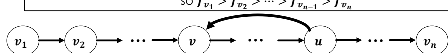
edge type	colour of v	discovery/finish times
tree	**white**	$d[u] < d[v] < f[v] < f[u]$
forward	**black**	$d[u] < d[v] < f[v] < f[u]$
~~back~~	~~**gray**~~	~~$d[v] < d[u] < f[u] < f[v]$~~
cross	**black**	$d[v] < f[v] < d[u] < f[u]$

Recall from DAG-testing: there are **no back edges** in a DAG



Theorem: if D is a DAG, and we order vertices in **reverse order of finishing time**, (i.e., by largest to smallest finish time) then we get a topological ordering!

To see **why**, suppose D is a DAG and we order nodes in this way, so $f_{v_1} > f_{v_2} > \cdots > f_{v_{n-1}} > f_{v_n}$



For contradiction, suppose a **right-to-left** edge $\{u, v\}$ exists

By our node ordering, $f_v > f_u$

But the lemma says for every edge $\{u, v\}$, we must have $f_v < f_u$

Lemma 6.8

Suppose D is a DAG. Then $f[v] < f[u]$ for every arc uv.

Contradiction! Right-to-left edge cannot exist. So is is a topological ordering.

TOPOLOGICAL ORDERING VIA DFS

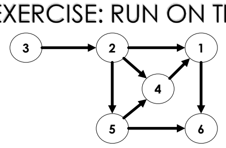
$O(n + m)$ w/adj. lists

```
global variables:
    pred[1..n] = [null, null, ..., null]
    colour[1..n] = [white, white, ..., white]
    d[1..n] = [0, 0, ..., 0] // discovery times
    f[1..n] = [0, 0, ..., 0] // finish times
    time = 0
    DAG = true

TopologicalSort(adj[1..n])
    S = new stack
    for v = 1..n
        if colour[v] == white
            DFSVisit(adj, v, S)
    if DAG then return S
    return null

DFSVisit(adj[1..n], v, S)
    colour[v] = gray
    time = time + 1
    d[v] = time

    for each w in adj[v]
        if colour[w] == white
            pred[w] = v
            DFSVisit(w)
        if color[w] == gray
            DAG = false

    colour[v] = black
    S.push(v)
    time = time + 1
    f[v] = time
```

Save each node when it **finishes**

Push smallest finishing time first → **pop largest** first

HOME EXERCISE: RUN ON THIS GRAPH

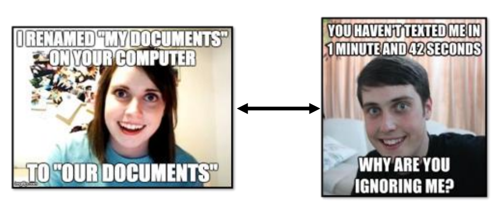


The initial calls are $DFSvisit(1)$, $DFSvisit(2)$ and $DFSvisit(3)$.

The discovery/finish times are as follows:

v	$d[v]$	$f[v]$	v	$d[v]$	$f[v]$
1	1	4	4	6	7
2	5	10	5	8	9
3	11	12	6	2	3

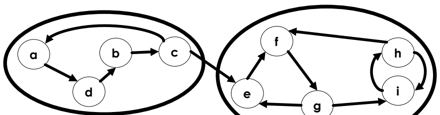
The topological ordering is $3, 2, 5, 4, 1, 6$ (reverse order of finishing time).



STRONGLY CONNECTED COMPONENTS

STRONGLY CONNECTED **COMPONENTS**

- This graph could be divided into **two graphs** that are each strongly connected

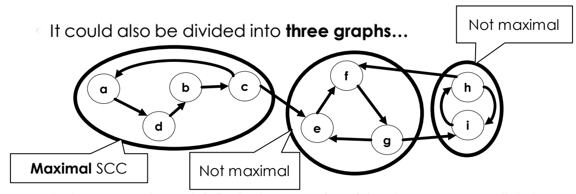


These are called **strongly connected components (SCCs)**

19

STRONGLY CONNECTED **COMPONENTS**

- It could also be divided into **three graphs…**

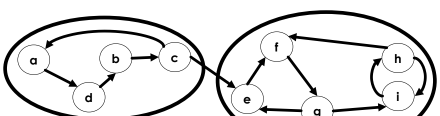


- But we want our SCCs to be **maximal** (as large as possible)

20

STRONGLY CONNECTED **COMPONENTS**

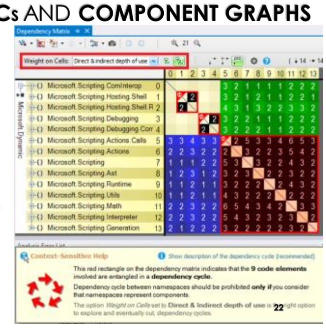
- So, the goal is to find **these** (maximal) SCCs:



21

APPLICATIONS OF **SCCs** AND **COMPONENT GRAPHS**

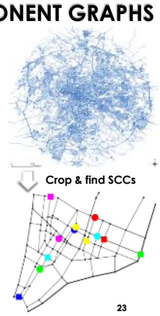
- Finding **all cyclic** dependencies in code
  - Can find **single** cycle with an easier DFS-based algorithm
  - But it is nicer to find **all** cycles at once, so you don't have to fix one to expose another



22

APPLICATIONS OF **SCCs** AND **COMPONENT GRAPHS**

- **Data filtering** before running other algorithms
- maps; nodes = intersections, edges = roads
- Don't want to run path finding algorithm on the entire **global** graph!
- Throw away everything except the (maximal) SCC containing source & target

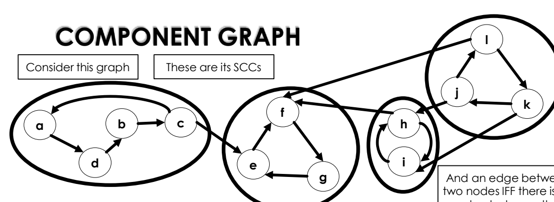


23

COMPONENT GRAPH

Consider this graph

These are its SCCs



And an edge between two nodes IFF there is an edge between the corresponding SCCs

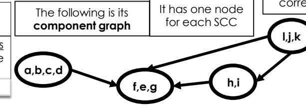
Can there be a **cycle** in the component graph?

No! If there are paths both ways between components, they are actually **the same SCC**

Component graph is a DAG!

The following is its **component graph**

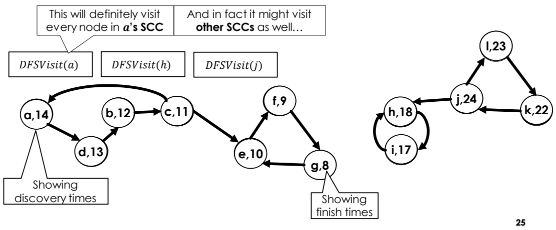
It has one node for each SCC



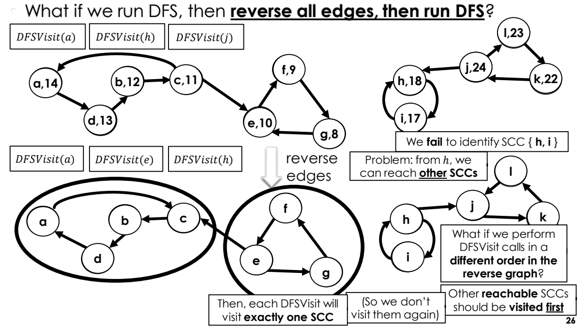
24

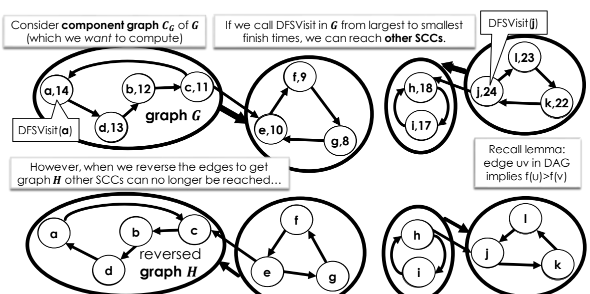
BRAINSTORMING AN ALGORITHM

- What if we run DFS, then reverse all edges, then run DFS (like checking whether an *entire graph* is strongly connected?)



What if we run DFS, then **reverse all edges, then run DFS**?



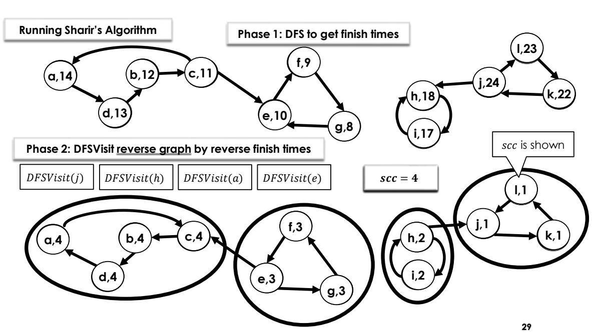


SCC ALGORITHM

This is called Sharir's algorithm (sometimes Kosaraju's algorithm). **This paper** first introduced it.

```
SCC(adj[1..n])
    DFS(adj)
    let order[1..n] = node labels sorted by
            largest to smallest finish time

    reverse all edges in adj

    colour[1..n] = [white, ..., white]
    comp[1..n] = [0, ..., 0]
    for i = 1..n
        v = order[i]
        if colour[v] == white
            scc = scc + 1
            SCCVisit(adj, v, scc, colour, comp)

    return comp
```

```
SCCVisit(adj[1..n], v, scc, colour, comp)
    colour[v] = gray
    comp[v] = scc

    for each w in adj[v]
        if colour[w] == white
            SCCVisit(w)

    colour[v] = black
```

TIME COMPLEXITY?

```
SCC(adj[1..n])
    DFS(adj)
    let order[1..n] = node labels sorted by
            largest to smallest finish time

    reverse all edges in adj

    colour[1..n] = [white, ..., white]
    comp[1..n] = [0, ..., 0]
    for i = 1..n
        v = order[i]
        if colour[v] == white
            scc = scc + 1
            SCCVisit(adj, v, scc, colour, comp)

    return comp
```

```
SCCVisit(adj[1..n], v, scc, colour, comp)
    colour[v] = gray
    comp[v] = scc

    for each w in adj[v]
        if colour[w] == white
            SCCVisit(w)

    colour[v] = black
```

- DFS(adj): $O(n+m)$
- Can be returned as part of the DFS with no added runtime
- Finish times **increase** as we set them, so just use a stack…
- reverse all edges in adj: $O(n+m)$
- colour/comp initialization: $O(n)$
- Total of $O(n+m)$ work over all n iterations of the i loop
- (each edge is inspected once, each node is visited once, constant work per visited node/inspected edge)

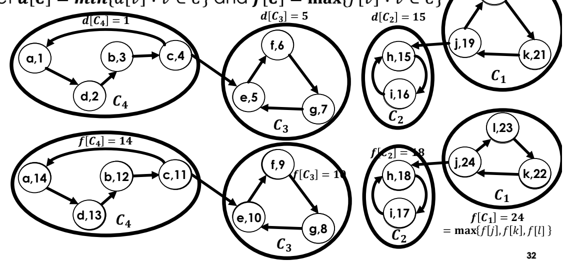
Total $O(n+m)$

CORRECTNESS

- Want to prove that each top-level call to SCCVisit explores **exactly** the nodes **in one SCC**
- Proof hinges on a key lemma that talks about the **finish times of SCCs** in the **component graph**
- To talk about finish times of **SCCs**, we need a definition…

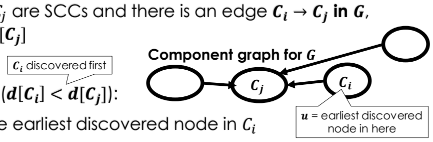
A KEY DEFINITION

- For a strongly connected component C, let $\boldsymbol{d[C]} = \boldsymbol{min}\{d[v] : v \in C\}$ and $\boldsymbol{f[C]} = \mathbf{max}\{f[v] : v \in C\}$



A KEY LEMMA

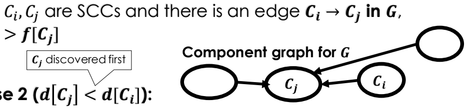
- **Lemma:** if C_i, C_j are SCCs and there is an edge $\boldsymbol{C_i \to C_j}$ **in** $\boldsymbol{G}$, then $\boldsymbol{f[C_i] > f[C_j]}$



- **Proof.** Case 1 ($\boldsymbol{d[C_i] < d[C_j]}$):
  - Let u be the earliest discovered node in C_i
  - All nodes in $C_i \cup C_j$ are white-reachable from u, so they are **descendants in the DFS forest** and **finish before** $\boldsymbol{u}$
  - So $f[C_i] = f[u] > f[C_j]$

A KEY LEMMA

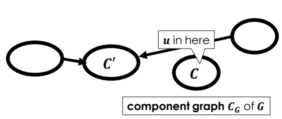
- **Lemma:** if C_i, C_j are SCCs and there is an edge $\boldsymbol{C_i \to C_j}$ **in** $\boldsymbol{G}$, then $\boldsymbol{f[C_i] > f[C_j]}$



- **Proof. Case 2 ($\boldsymbol{d[C_j] < d[C_i]}$):**
  - Since component graph is a DAG, there is **no path** $\boldsymbol{C_j \to C_i}$
  - Thus, **no nodes** in C_i are reachable from C_j
  - So we discover C_j and finish C_j **without** discovering C_i
  - Therefore $d[C_j] < \boldsymbol{f[C_j]} < d[C_i] < \boldsymbol{f[C_i]}$. **QED**

COMPLETING THE PROOF

- Suppose we have performed DFS to get our finish times, and we are about to perform SCCVisits on the reverse graph
- **We prove each top-level SCCVisit call visits precisely one SCC**
- Consider the first top-level SCCVisit(u)
- Let C be the SCC containing u and C' be any other SCC
- Since we call SCCVisit on nodes starting from the **largest finish time**,
  - We know $\boldsymbol{f(C) > f(C')}$



COMPLETING THE PROOF

- We know $\boldsymbol{f(C) > f(C')}$
- By Lemma: if there were an edge $C' \to C$ in G, then we would have $f(C') > f(C)$
  - So there is no edge $C' \to C$ in G
  - and hence **no edge** $\boldsymbol{C \to C'}$ **in** $\boldsymbol{H}$
  - So, **SCCVisit($\boldsymbol{u}$) in $\boldsymbol{H}$ cannot visit $\boldsymbol{C'}$**

… and sets comp[v] = scc for all nodes in the SCC

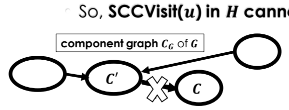
So each top-level call explores one SCC…

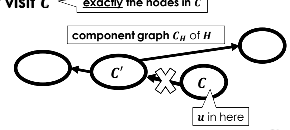
and **larger finish** time means **already explored**!

In G, edges go from larger to smaller finish times. **In $\boldsymbol{H}$, edges go from smaller to larger.**

Similar argument for subsequent **top-level** calls to SCCVisit.

So SCCVisit($\boldsymbol{u}$) visits exactly the nodes in $\boldsymbol{C}$





IF WE HAVE TIME

topological sort without relying on DFS

EXISTENCE OF A TOPOLOGICAL SORT ORDER

Theorem 6.6

A directed graph D has a topological sort if and only if it is a DAG.

Proof.

($\Rightarrow$): Suppose D has a directed cycle $v_1, v_2, \ldots, v_j, v_1$. Then $v_1 < v_2 < \cdots < v_j < v_1$, so a topological ordering does not exist.

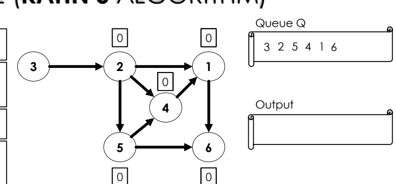
($\Leftarrow$): Suppose D is a DAG. Then the algorithm below constructs a topological ordering.


```
Kahn(adj[1..n])
    indeg[1..n] = [0, ..., 0]
    for each edge (u,v) in adj
        indeg[v] = indeg[v] + 1

    order = new list
    q = new queue containing {v : indeg[v] == 0}
    for i = 1..n
        if q.empty() return null
        v = q.dequeue()
        order.append(v)

        for each w in adj[v]
            indeg[w] = indeg[w] - 1
            if indeg[w] == 0 then q.enqueue(w)

    return order
```

- $indeg[v]$ = # of edges pointing **into** node v
- = number of **unsatisfied constraints** on v
- Nodes with $\boldsymbol{indeg}$ **0** have **no unsatisfied dependencies**
- So this step is enqueuing nodes whose dependencies are already satisfied
- q **always** contains nodes with no unsatisfied dependencies (indeg 0)
- No such order!
- Add v to the topological order
- Remove v's out edges. If we have now satisfied all dependencies for some w, add w to the queue also.

EXAMPLE (**KAHN'S** ALGORITHM)

- Compute **indegree** for all vertices
- For each node u
  For each w in adj(u)
  w.deg = w.deg+1
- vertices with indeg 0 go into the queue
- Until Q is empty: pop, output that element, decrement its neighbours, enqueue new indeg 0's



Running time with adjacency lists?

```
Kahn(adj[1..n])
    indeg[1..n] = [0, ..., 0]
    for each edge (u,v) in adj
        indeg[v] = indeg[v] + 1

    order = new list
    q = new queue containing {v : indeg[v] == 0}
    for i = 1..n
        if q.empty() return null
        v = q.dequeue()
        order.append(v)

        for each w in adj[v]
            indeg[w] = indeg[w] - 1
            if indeg[w] == 0 then q.enqueue(w)

    return order
```

- $O(n)$
- $O(n+m)$ total work over all iterations
- $O(n)$ iterations
- $O(1)$ per check
- $O(1)$
- $O(\deg(v))$ per iteration i
- $\sum_{v \in V} \deg(v) \in \boldsymbol{O(n+m)}$ **total work over all nodes** v

Total $O(n+m)$

BONUS SLIDES

SCC: HOW ABOUT A DIFFERENT ORDERING?

- Rather than doing DFS in the **reverse** graph in order of **decreasing** finish times
- Why not do DFS in the **original** graph in order of **increasing** finish times?
- Exercise: does this work?

SCC: HOW ABOUT A DIFFERENT ORDERING?

- Why not do DFS in the **original** graph in order of **increasing** finish times?

Doesn't work!

Output depends where first DFS starts…

If first DFS starts at c, then…

DFSVisit(b) would reach two SCCs.