CS 341: ALGORITHMS

Lecture 13: graph algorithms IV – minimum spanning trees

Readings: see website

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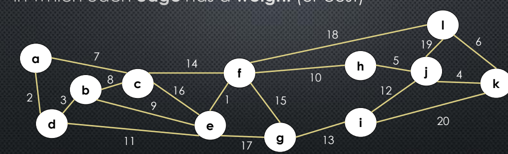
WEIGHTED UNDIRECTED GRAPH

- Consider an **undirected** graph in which each **edge** has a **weight** (or cost)

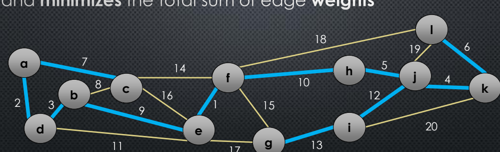
Problem can also be defined for directed graphs…



2

MINIMUM SPANNING TREE (MST)

- A tree (connected acyclic graph) that includes every node, and **minimizes** the total sum of edge **weights**



Problem can also be defined for minimum spanning **forest**. Algorithm taught here works.

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APPLICATION: INTERNET BACKBONE PLANNING

- Want to connect n cities with internet backbone links
  - Direct links possible between each pair of cities
  - Each link has a certain dollar cost (excavation, materials, distance & time, legal costs…)
  - Want to **minimize total cost**

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APPLICATION: IMAGE **SEGMENTATION** [PAPER]

break image into **regions** by colour similarity via other techniques

turn regions into nodes, and add edges between them with weights = "dissimilarity," then build MST

break MST into large, highly similar **segments**, and assign the dominant colour to each **segment**

Just for fun, don't need to know this

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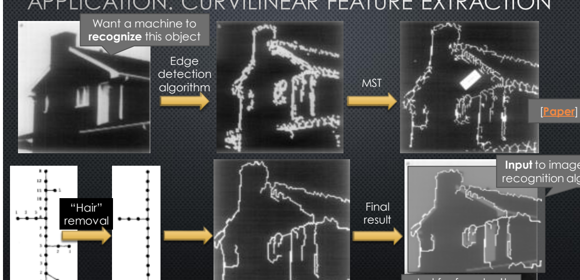
APPLICATION: CURVILINEAR FEATURE EXTRACTION



Just for fun, don't need to know this

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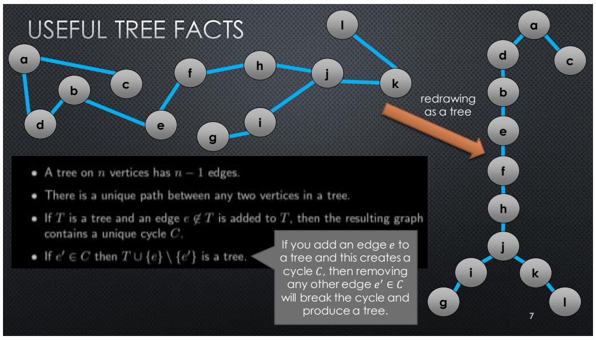
USEFUL TREE FACTS
• A tree on n vertices has n − 1 edges.
• There is a unique path between any two vertices in a tree.
• If T is a tree and an edge e ∉ T is added to T, then the resulting graph contains a unique cycle C.
• If e′ ∈ C then T ∪ {e} \ {e′} is a tree.
If you add an edge e to a tree and this creates a cycle C, then removing any other edge e′ ∈ C will break the cycle and produce a tree.
redrawing as a tree
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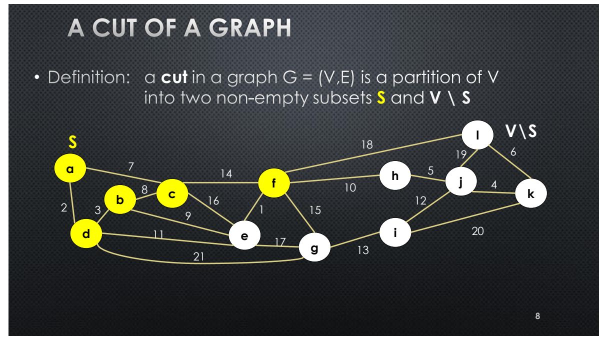
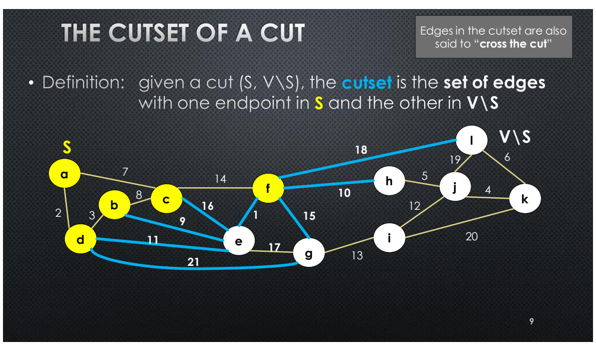
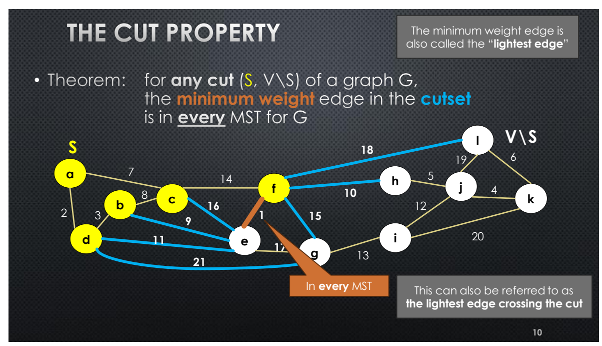

A CUT OF A GRAPH
• Definition: a cut in a graph G = (V,E) is a partition of V into two non-empty subsets S and V \ S
S
V\S
8


THE CUTSET OF A CUT
Edges in the cutset are also said to "cross the cut"
• Definition: given a cut (S, V\S), the cutset is the set of edges with one endpoint in S and the other in V\S
S
V\S
9


THE CUT PROPERTY
The minimum weight edge is also called the "lightest edge"
• Theorem: for any cut (S, V\S) of a graph G, the minimum weight edge in the cutset is in every MST for G
S
V\S
In every MST
This can also be referred to as the lightest edge crossing the cut
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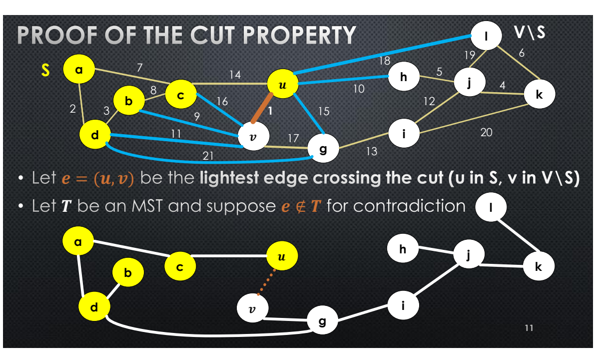
PROOF OF THE CUT PROPERTY
S
V\S
• Let e = (u, v) be the lightest edge crossing the cut (u in S, v in V\S)
• Let T be an MST and suppose e ∉ T for contradiction
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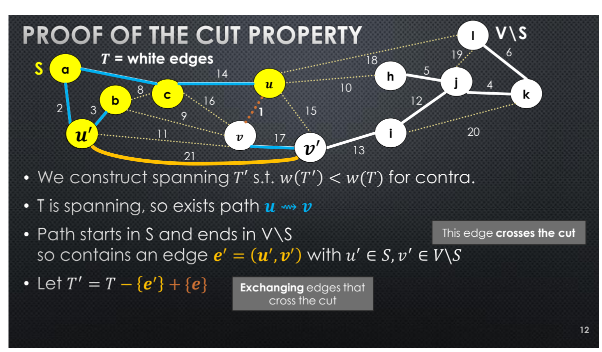
PROOF OF THE CUT PROPERTY
S
V\S
T = white edges
• We construct spanning T′ s.t. w(T′) < w(T) for contra.
• T is spanning, so exists path u ⇝ v
• Path starts in S and ends in V\S so contains an edge e′ = (u′, v′) with u′ ∈ S, v′ ∈ V\S
This edge crosses the cut
• Let T′ = T − {e′} + {e}
Exchanging edges that cross the cut
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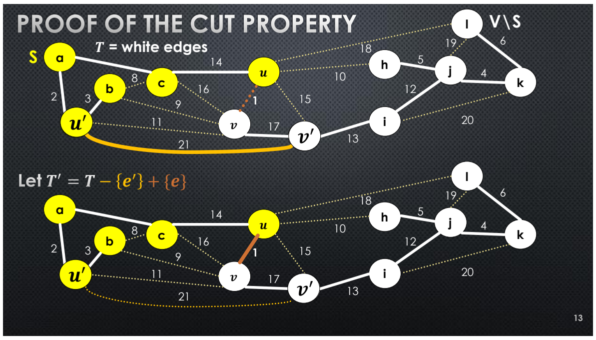
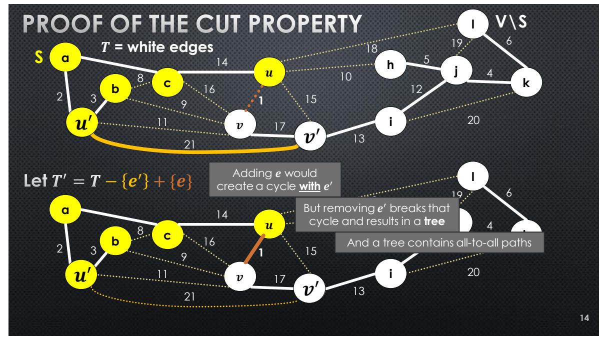

PROOF OF THE CUT PROPERTY
S
V\S
T = white edges
Let $T' = T - \{e'\} + \{e\}$
13


PROOF OF THE CUT PROPERTY
S
V\S
T = white edges
Let $T' = T - \{e'\} + \{e\}$
Adding e would create a cycle with e'
But removing e' breaks that cycle and results in a tree
And a tree contains all-to-all paths
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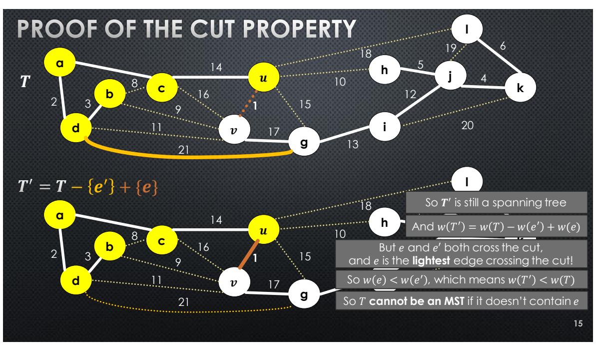
PROOF OF THE CUT PROPERTY
T
$T' = T - \{e'\} + \{e\}$
So T' is still a spanning tree
And $w(T') = w(T) - w(e') + w(e)$
But e and e' both cross the cut, and e is the lightest edge crossing the cut!
So $w(e) < w(e')$, which means $w(T') < w(T)$
So T cannot be an MST if it doesn't contain e
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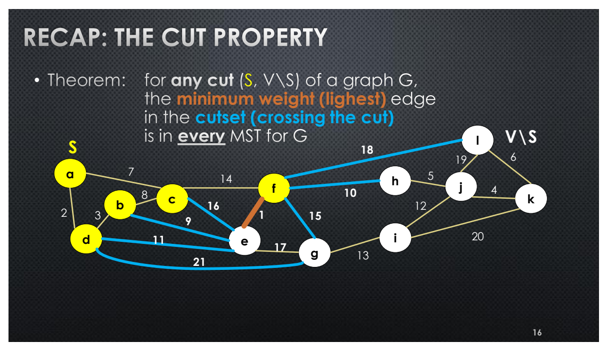
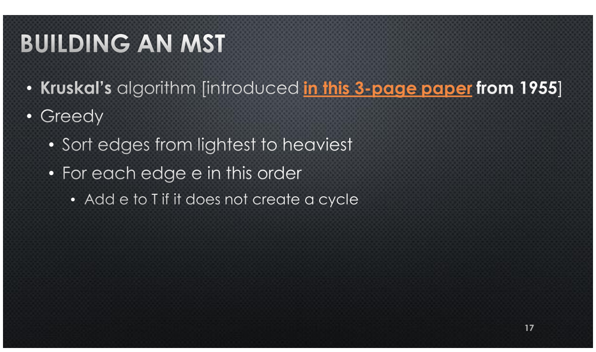

RECAP: THE CUT PROPERTY
• Theorem: for any cut (S, V\S) of a graph G,
the minimum weight (lightest) edge
in the cutset (crossing the cut)
is in every MST for G
S
V\S
16


BUILDING AN MST
• Kruskal's algorithm [introduced in this 3-page paper from 1955]
• Greedy
• Sort edges from lightest to heaviest
• For each edge e in this order
• Add e to T if it does not create a cycle
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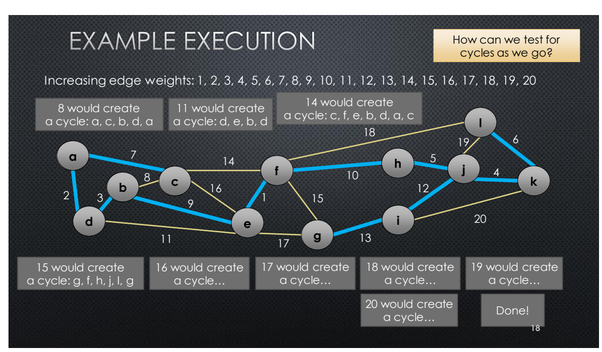
EXAMPLE EXECUTION
How can we test for cycles as we go?
Increasing edge weights: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
8 would create a cycle: a, c, b, d, a
11 would create a cycle: d, e, b, d
14 would create a cycle: c, f, e, b, d, a, c
15 would create a cycle: g, f, h, j, l, g
16 would create a cycle...
17 would create a cycle...
18 would create a cycle...
19 would create a cycle...
20 would create a cycle...
Done!
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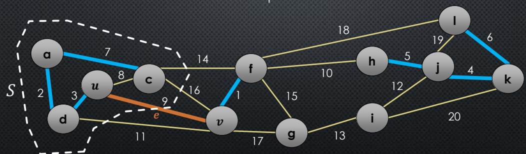
PROOF

- Let T be partial spanning tree just before adding $e = (u, v)$, the lightest edge that does not create a cycle
- Let S be the connected component of T that contains u



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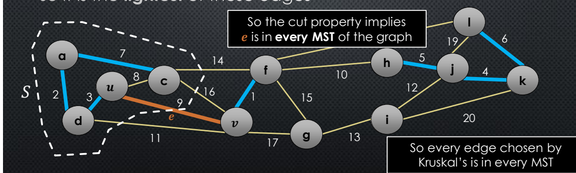
PROOF

- Note $e = (u, v)$ crosses the cut $(S, V \setminus S)$ or it would create a cycle
- Out of all edges crossing the cut, e is considered first, so it is the **lightest** of these edges



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IMPLEMENTING KRUSKAL'S

- Sort edges from lightest to heaviest
- For each edge e in this order
  - Add e to T if it **does not create a cycle**

How can we determine whether adding e would create a cycle?

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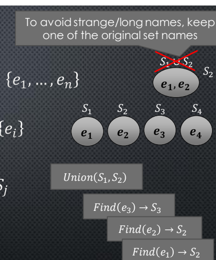
UNION FIND

- Represents a **partition** of set $S = \{e_1, \dots, e_n\}$ into **disjoint subsets**
  - Initially n disjoint subsets $S_i = \{e_i\}$
- Operations
  - $Union(S_i, S_j)$ replaces S_i and S_j by their union $S_i \cup S_j$
  - $Find(e_i)$ returns the **label** of the set containing e_i



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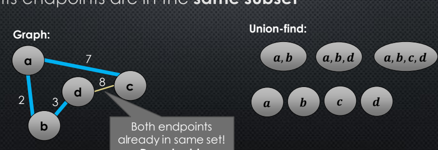
KRUSKAL'S USING UNION-FIND

- Each graph node is initially in its own subset
- Add an edge → union two subsets
- An edge **creates a cycle IFF** its endpoints are in the **same subset**



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PSEUDOCODE FOR KRUSKAL'S USING UNION-FIND

```
Kruskal(V[1..n], E[1..m])
    sort E[1..m] in increasing order by weight
    uf = new UnionFind data structure
    mst = new List
    for j = 1..m
        set_a = uf.find(E[j].source)
        set_b = uf.find(E[j].target)
        if set_a != set_b
            mst.add(E[j])
            uf.merge(set_a, set_b)
    return mst
```

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TIME COMPLEXITY?

```
Kruskal(V[1..n], E[1..m])
    sort E[1..m] in increasing order by weight
    uf = new UnionFind data structure
    mst = new List
    for j = 1..m
        set_a = uf.find(E[j].source)
        set_b = uf.find(E[j].target)
        if set_a != set_b
            mst.add(E[j])
            uf.merge(set_a, set_b)
    return mst
```

Need to know runtime for union find…

For an efficient union-find algorithm (with union by rank and path compression), we get a total running time for Kruskal's algorithm of $O(\alpha(m+n)(m+n))$, where $\alpha(x)$ is the inverse Ackermann function. For all practical x, $\alpha(x) \le 5$, so this is ***pseudo*-linear**.

A simpler implementation with union-by-rank only yields $O(m \log n)$

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OTHER NOTABLE MST ALGORITHMS

- Prim's algorithm
  - Incrementally extend a tree T into an MST, by:
  - Initializing T to contain any arbitrary node in G
  - Repeatedly selecting the lightest edge that crosses cut (T, V\T)
  - Visualization: https://www.cs.usfca.edu/~galles/visualization/Prim.html
- Borůvka's algorithm
  - Like Kruskal (merging components), but with **phases**
  - In each phase, select an outgoing edge for **every** component, and add **all** edges found in the phase

Use priority queue to store **outgoing** edges from T (and repeatedly extract the minimum weight one)

There is also a fast **parallel hybrid** of Prim and Borůvka

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A FUN APPLICATION: MAZE BUILDING

- Create grid graph with
- edges up/down/left/right
- **Randomize** edge **weights** then run Kruskal's

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VISUALIZING KRUSKAL'S (WITHOUT PATH COMPRESSION)

- https://www.cs.usfca.edu/~galles/visualization/Kruskal.html

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BONUS SLIDES

- Kruskal's proof via exchange argument instead

- Implementing union-find efficiently

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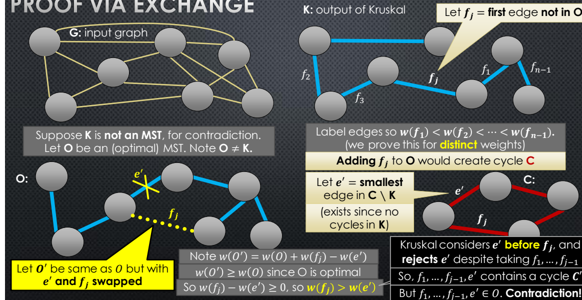
PROOF VIA EXCHANGE



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UNION FIND IMPLEMENTATION

- Suppose we are partitioning set $\{1, \ldots, n\}$ into **subsets** $S_1, \ldots, S_n$
- Represent the partition as a **forest** of **trees**
  - Initially one single-node tree per subset
  - Each node has a **parent pointer**
- $Find(i)$ returns the **root** of the tree containing **element** i
- $Union(i, j)$ makes one root the parent of the other

Union-find forest (physical):

$parent$	2	4	**4**	4
	1	2	3	4

Union-find forest (logical):

Let's union the ***sets*** *containing* ***elements*** *1 and 2*
$find(1) \rightarrow 1, \quad find(2) \rightarrow 2$
$Union(1,2): \; parent[1] = 2$

How about elements 4 and 1?
$find(4) \rightarrow 4, \quad find(1) \rightarrow 2$
$Union(4,2): \; parent[2] = 4$

How about elements 3 and 1?
$find(3) \rightarrow 3, \quad find(1) \rightarrow 4$
$Union(3,4): \; parent[3] = 4$

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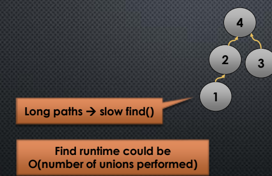
PROBLEM: SLOW FIND()



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UNION-FIND WITH **UNION BY RANK**

- Keep track of **heights** of trees
- Make **root with greater height** be the **parent**
  - Union of two trees with height h has height $h + 1$
  - Union of tree with height h and tree with height $< h$ has height h
- **Runtime** with union by rank?

Union-find forest:

Let's union the ***sets*** *containing* ***elements*** *1 and 2*
$find(1) \rightarrow 1, \quad find(2) \rightarrow 2$
$Union(1,2)$: **same height →** $parent[1] = 2$

How about elements 4 and 1?
$find(4) \rightarrow 4, \quad find(1) \rightarrow 2$
$Union(4,2)$: **2's height is greater →** $parent[4] = 2$

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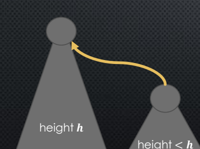
RUNTIME OF UNION BY RANK

- Can prove the following **lemma** by induction:
  - Each tree of height h contains at least 2^h nodes

Case 1: trees of different height



By I.H.,
left tree already has $\geq 2^h$ nodes.
So result has height h and $\geq 2^h$ nodes

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RUNTIME OF UNION BY RANK

- Can prove the following **lemma** by induction:
  - Each tree of height h contains at least 2^h nodes

Case 2: trees of same height

By I.H.,
each tree has $\geq 2^h$ nodes.
Result has height $h + 1$ and $\geq 2^h + 2^h$ nodes

And $2^h + 2^h = 2^{h+1}$. QED

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RUNTIME OF UNION BY RANK

- How does the **lemma** help?
  - Each tree of height h contains at least 2^h nodes
- There are only $\boldsymbol{n}$ **nodes** in the graph
  - So **height** is at most $\mathbf{log}\, \boldsymbol{n}$
  - (Lemma: a tree of height $\log n$ contains at least $2^{\log n}$ nodes and $2^{\log n} = n$)
- So the longest path in the union-find forest is $\log n$
  - So all union-find operations run in $\Theta(\log n)$ time!

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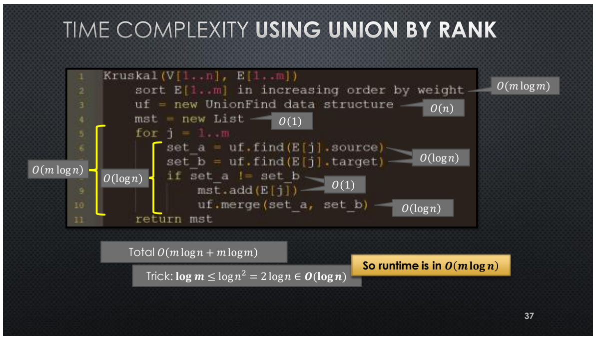
TIME COMPLEXITY USING UNION BY RANK
Kruskal(V[1..n], E[1..m])
sort E[1..m] in increasing order by weight
uf = new UnionFind data structure
mst = new List
for j = 1..m
set_a = uf.find(E[j].source)
set_b = uf.find(E[j].target)
if set_a != set_b
mst.add(E[j])
uf.merge(set_a, set_b)
return mst
$O(m \log m)$
$O(n)$
$O(1)$
$O(\log n)$
$O(m \log n)$
$O(\log n)$
$O(1)$
$O(\log n)$
Total $O(m \log n + m \log m)$
Trick: $\log m \le \log n^2 = 2 \log n \in O(\log n)$
So runtime is in $O(m \log n)$
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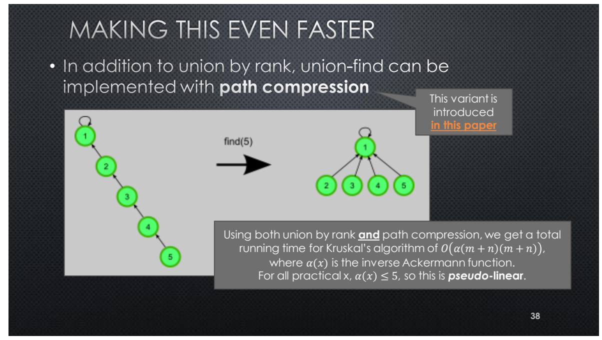
MAKING THIS EVEN FASTER
• In addition to union by rank, union-find can be implemented with path compression
This variant is introduced in this paper
find(5)
1
2
3
4
5
1
2
3
4
5
Using both union by rank and path compression, we get a total running time for Kruskal's algorithm of $O(\alpha(m+n)(m+n))$, where $\alpha(x)$ is the inverse Ackermann function.
For all practical x, $\alpha(x) \le 5$, so this is pseudo-linear.
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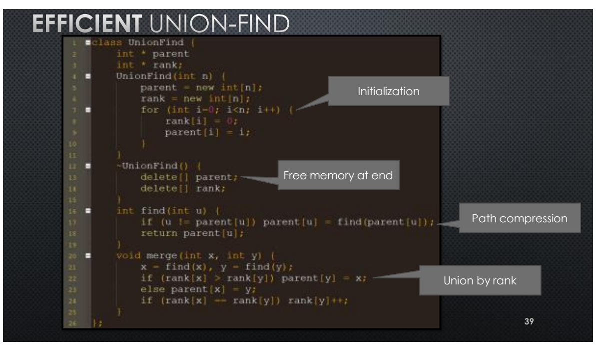
EFFICIENT UNION-FIND
class UnionFind {
int * parent
int * rank;
UnionFind(int n) {
parent = new int[n];
rank = new int[n];
for (int i=0; i<n; i++) {
rank[i] = 0;
parent[i] = i;
}
}
~UnionFind() {
delete[] parent;
delete[] rank;
}
int find(int u) {
if (u != parent[u]) parent[u] = find(parent[u]);
return parent[u];
}
void merge(int x, int y) {
x = find(x), y = find(y);
if (rank[x] > rank[y]) parent[y] = x;
else parent[x] = y;
if (rank[x] == rank[y]) rank[y]++;
}
};
Initialization
Free memory at end
Path compression
Union by rank
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