CS 341: ALGORITHMS

Lecture 15: graph algorithms VI – all pairs shortest paths

Readings: see website

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ALL PAIRS SHORTEST PATHS (APSP) PROBLEM

Instance: *A directed graph $G = (V, E)$, and a* **weight matrix** *W, where $W[i, j]$ denotes the weight of edge ij, for all $i, j \in V$, $i \neq j$.*
Find: *For all pairs of vertices $u, v \in V$, $u \neq v$, a directed path P from u to v such that*

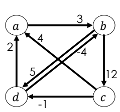
$$w(P) = \sum_{ij \in P} W[i, j]$$

is minimized.

We allow edges to have negative weights, but we assume there are no negative-weight directed cycles in G.

We use the following conventions for the weight matrix W:

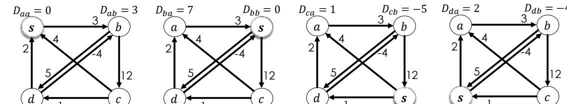
$$W[i,j] = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{if } i = j \\ \infty & \text{otherwise.} \end{cases}$$



$$\begin{array}{c|c} \textbf{from:} & \textbf{to:}\quad a \quad b \quad c \quad d \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & W[i,j] = \begin{bmatrix} 0 & 3 & \infty & \infty \\ \infty & 0 & 12 & 5 \\ 4 & \infty & 0 & -1 \\ 2 & -4 & \infty & 0 \end{bmatrix} \end{array}$$

EASY SOLUTION

Run Bellman-Ford n times, once for each possible source



Output: Matrix D of shortest path lengths

$$\begin{array}{c|c} \textbf{from:} & \textbf{to:}\quad a \quad b \quad c \quad d \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & D[i,j] = \begin{bmatrix} 0 & 3 & 15 & 8 \\ 7 & 0 & 12 & 5 \\ 1 & -5 & 0 & -1 \\ 2 & -4 & 8 & 0 \end{bmatrix} \end{array}$$

Complexity $O(n^2 m)$. (Could be $O(n^4)$.)

Can we do better?

A Dynamic Programming Approach

Suppose we successively consider paths of length $1, 2, \ldots, n-1$. Let $L_m[i,j]$ denote the minimum-weight (i,j)-path having at most m edges.

We want to compute L_{n-1}.

Base case: $L_1 = W$

General case: How to express solution in terms of **optimal solutions** to **subproblems**?

Express shortest path with **m edges** in terms of shortest path(s) with **< m edges**?

For $m \geq 2$,

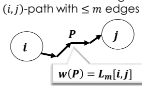
$$L_m[i,j] = \min\{L_{m-1}[i,k] + L_1[k,j] : 1 \leq k \leq n\}.$$

Problem: we don't **know** the **predecessor of j** on the optimal path P

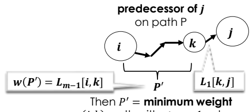
Try **all** possible predecessors k

Arguing **optimal substructure**

Let P = minimum weight (i,j)-path with $\leq m$ edges



Let k be the **predecessor of j** on path P



Then P' = **minimum weight** (i,k)-path with $\leq m-1$ edges

(or could shrink $w(P)$; contra!)

Algorithm: *FairlySlowAllPairsShortestPath*(W)

$$\begin{array}{l} L_1 \leftarrow W \\ \textbf{for } m \leftarrow 2 \textbf{ to } n-1 \\ \quad \textbf{do} \begin{cases} \textbf{for } i \leftarrow 1 \textbf{ to } n \\ \quad \textbf{do} \begin{cases} \textbf{for } j \leftarrow 1 \textbf{ to } n \\ \quad \textbf{do} \begin{cases} \ell \leftarrow \infty \\ \textbf{for } k \leftarrow 1 \textbf{ to } n \\ \quad \textbf{do } \ell \leftarrow \min\{\ell, L_{m-1}[i,k] + W[k,j]\} \\ L_m[i,j] \leftarrow \ell \end{cases} \end{cases} \end{cases} \\ \textbf{return } (L_{n-1}) \end{array}$$

Time complexity? $O(n^4)$

Space complexity is a bit subtle…

To compute L_m, only need W and L_{m-1}. No need to keep $L_2, \ldots, L_{m-2}$. So space is $O(|W| + |L_m| + |L_{m-1}|) = O(|L_m|) = O(n^2)$

Note: this is asymptotically the same as **input size** for dense graphs where $|E| \in \Theta(|V|^2)$

Home exercise: do we need to keep **both** L_m and L_{m-1}? Or can we reuse L_{m-1} directly as our L_m array, and modify it in-place?

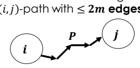
BETTER SOLUTION: SUCCESSIVE DOUBLING

The idea is to construct $L_1, L_2, L_4, \ldots L_{2^t}$, where t is the smallest integer such that $2^t \geq n-1$.

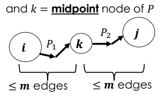
Initialization: $L_1 = W$ (as before).

Arguing **optimal** substructure

Let P = minimum weight (i,j)-path with $\leq 2m$ **edges**



and k = **midpoint** node of P



Then $P = P_1 \cup P_2$ where: P_1 is the minimum weight (i,k)-path with $\leq m$ edges and P_2 is the minimum weight (k,j)-path with $\leq m$ edges

(or else we could improve P by improving P1 or P2)

Updating: For $m \geq 1$,

$$L_{2m}[i,j] = \min\{L_m[i,k] + L_m[k,j] : 1 \leq k \leq n\}.$$

Don't know which node is midpoint of P, so try all k…

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Second Solution: Successive Doubling

Algorithm: *FasterAllPairsShortestPath*(W)

```
L_1 ← W
m ← 1
while m < n − 1
do { for i ← 1 to n
     do { for j ← 1 to n
          do { ℓ ← ∞
               for k ← 1 to n
                 do ℓ ← min{ℓ, L_m[i,k] + L_m[k,j]}
               L_2m[i,j] ← ℓ
     m ← 2m
return (L_m)
```

Complexity analysis

$O(n^3 \log n)$ runtime

$O(n^2)$ space

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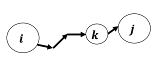
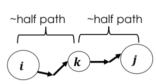
SUMMARY & WHAT'S NEXT

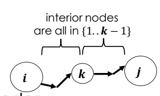
- First solution: sub-problem is a path to the **predecessor node**
  - Optimality: try all possible predecessor nodes $\boldsymbol{k}$



- Second solution: sub-problems are paths to/from the **midpoint node**
  - Optimality: try all possible midpoint nodes $\boldsymbol{k}$



- **Third solution:** sub-problems are paths in which **all interior nodes** are in $\{1..\boldsymbol{k}-1\}$
  - I.e., we **restrict paths** to using a **prefix** of all nodes
  - Optimality: try all ways to use **new node** $\boldsymbol{k}$ as an interior node



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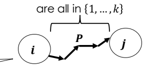
THIRD SOLUTION: FLOYD-WARSHALL

Let $\boldsymbol{D_k[i,j]}$ denote the length of the minimum-weight **path** $\boldsymbol{i \rightsquigarrow j}$ in which all **interior nodes** are in the set $\{\boldsymbol{1,\ldots,k}\}$.
We want to compute $\boldsymbol{D_n}$.

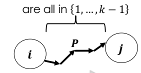
Optimal solution: interior nodes are all in $\{1,\ldots,k\}$



Let P be a min-weight (i,j)-path in which all interior nodes are in $\{1,\ldots,k\}$

Case 1: k is **not** used in P

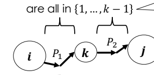
interior nodes are all in $\{1,\ldots,k-1\}$



Then $\boldsymbol{D_k[i,j] = D_{k-1}[i,j]}$

Case 2: k **is** used in P

interior nodes are all in $\{1,\ldots,k-1\}$



Then $\boldsymbol{D_k[i,j]} = D_{k-1}[i,k] + D_{k-1}[k,j]$

How can we argue k is not in either P1 or P2?

Because P would then contain a cycle, and the cycle **cannot make P shorter**

So there must be an equivalent or better P without a cycle

more formal proof in bonus slides

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FLOYD-WARSHALL ALGORITHM

- Let $D_k[i,j]$ denote the length of the minimum-weight (i,j)-path in which all interior nodes are in the set of nodes $\{1\ldots k\}$.
- Base case: $D_0 = W$
- Recurrence: $D_k[i,j] = \min\{D_{k-1}[i,j],\ D_{k-1}[i,k] + D_{k-1}[k,j]\}$

```
FloydWarshall(W[1..n, 1..n])
    D0 = copy of weight matrix W
    D1 = new n * n matrix
    Dlast = pointer to D0
    Dcurr = pointer to D1
    for k = 1..n
        for i = 1..n
            for j = 1..n
                Dcurr[i,j] = min( Dlast[i,j], Dlast[i,k] + Dlast[k,j] )
        swap pointers Dlast and Dcurr
    return Dlast
```

Time complexity?
Space complexity?

This returns **distances**.
Can reconstruct paths from this.

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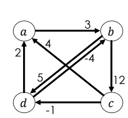
EXAMPLE



$$D_0 = \begin{pmatrix} 0 & 3 & \infty & \infty \\ \infty & 0 & 12 & 5 \\ 4 & \infty & 0 & -1 \\ 2 & -4 & \infty & 0 \end{pmatrix} \quad D_1 = \begin{pmatrix} 0 & 3 & \infty & \infty \\ \infty & 0 & 12 & 5 \\ 4 & \boxed{7} & 0 & -1 \\ 2 & -4 & \infty & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & 3 & \boxed{15} & \boxed{8} \\ \infty & 0 & 12 & 5 \\ 4 & 7 & 0 & -1 \\ 2 & -4 & \boxed{8} & 0 \end{pmatrix} \quad D_3 = \begin{pmatrix} 0 & 3 & 15 & 8 \\ \boxed{16} & 0 & 12 & 5 \\ 4 & 7 & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix}$$

$$D_4 = \begin{pmatrix} 0 & 3 & 15 & 8 \\ \boxed{7} & 0 & 12 & 5 \\ \boxed{1} & \boxed{-5} & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix}$$

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STABLE MATCHING PROBLEM

(SOLVED WITH A GREEDY GRAPH ALGORITHM)

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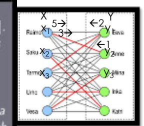
Problem 4.6

Stable Matching

Instance: *Two sets of size n say $X = [x_1, \ldots, x_n]$ and $Y = [y_1, \ldots, y_n]$. Each x_i has a* **preference ranking** *of the elements in Y, and each y_i has a preference ranking of the elements in X. $pref(x_i, j) = y_k$ if y_k is the j-th favourite element of Y of x_i; and $pref(y_i, j) = x_k$ if x_k is the j-th favourite element of X of y_i.*

Find: *A* **matching** *of the sets X and Y such that there does not exist a pair (x_i, y_j) which is not in the matching, but where x_i and y_j prefer each other to their existing matches. A matching with this this property is called a* **stable matching**.

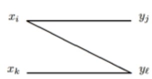


Real-world examples (1950s):

- Matching medical interns to hospitals.
- Matching organs to patients requiring transplants

The 2012 Nobel Prize in economics was awarded to Roth and Shapley for their work in the "theory of stable allocation and the practice of market design".

An example of an instability: Suppose x_i is matched with y_j, x_k is matched with y_ℓ, x_i prefers y_ℓ to y_j, and y_ℓ prefers x_i to x_k.



Overview of the Gale-Shapley Algorithm

Elements of X propose to elements of Y.

If y_j accepts a proposal from x_i, then the pair $\{x_i, y_j\}$ is **matched**.

An unmatched y_j must accept a proposal from any x_i.

If $\{x_i, y_j\}$ is a matched pair, and y_j subsequently receives a proposal from x_k, where y_j prefers x_k to x_i, then y_j accepts and the pair $\{x_i, y_j\}$ is replaced by $\{x_k, y_j\}$.

If $\{x_i, y_j\}$ is a mathced pair, and y_j subsequently receives a proposal from x_k, where y_j prefers x_i to x_k, then y_j rejects and nothing changes.

A matched y_j never becomes unmatched.

An x_i might make a number of proposals (up to n); the order of the proposals is determined by x_i's preference list.

Algorithm: *Gale-Shapley*$(X, Y, pref)$

$Match \leftarrow \emptyset$ — Keeps track of **current** matches

while there exists an unmatched x_i — Termination is not so obvious…

do
- let y_j be the next element in x_i's preference list — Propose to **most desired** y
- **if** y_j is not matched — Unmatched y_j accepts **any** proposal
- **then** $Match \leftarrow Match \cup \{x_i, y_j\}$
- **else**
  - suppose $\{x_k, y_j\} \in Match$ — Matched y_j considers **upgrading**
  - **if** y_j prefers x_i to x_k
  - **then** $Match \leftarrow Match \setminus \{x_k, y_j\} \cup \{x_i, y_j\}$
    comment: x_k is now unmatched

return $(Match)$

EXAMPLE:

Suppose we have the following preference lists:

$$x_1 : y_2 > y_3 > y_1 \qquad y_1 : x_1 > x_2 > x_3$$
$$x_2 : y_1 > y_3 > y_2 \qquad y_2 : x_2 > x_3 > x_1$$
$$x_3 : y_1 > y_2 > y_3 \qquad y_3 : x_3 > x_2 > x_1$$

The *Gale-Shapley algorithm* could be executed as follows:

proposal	result	*Match*
x_1 proposes to y_2	y_2 accepts	$\{x_1, y_2\}$
x_2 proposes to y_1	y_1 accepts	$\{x_1, y_2\}, \{x_2, y_1\}$
x_3 proposes to y_1	y_1 rejects	
x_3 proposes to y_2	y_2 accepts	$\{x_3, y_2\}, \{x_2, y_1\}$
x_1 proposes to y_3	y_3 accepts	$\{x_3, y_2\}, \{x_2, y_1\}, \{x_1, y_3\}$

Proof of Correctness

First we need to show that the algorithm always terminates, i.e., it is impossible that an unmatched x_i has proposed to every y_j.

Termination of the algorithm: Once an element of Y is matched, they are never unmatched. If x_i has proposed to every y_j, then every y_j is matched. But then every element of X is matched, which is a contradiction.

So the algorithm terminates, and each $\boldsymbol{x_i}$ is matched with some $\boldsymbol{y_j}$

Need to argue the matching is **stable** (i.e., optimal)

That is, no $\boldsymbol{x_i}$ and $\boldsymbol{y_j}$ prefer **each other more** than their current partners

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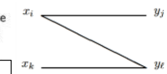
To prove that the algorithm terminates with a stable matching: Suppose there is an instability: x_i is matched with y_j, x_k is matched with y_ℓ, x_i prefers y_ℓ to y_j and y_ℓ prefers x_i to x_k.



Observe: x_i proposes to $\boldsymbol{y_\ell}$ **before** proposing to y_j

There three cases to consider:

(1) y_ℓ rejected x_i's proposal.

(2) y_ℓ accepted x_i's proposal, but later accepted another proposal.

(3) y_ℓ accepted x_i's proposal, and did not accept any subsequent proposal.

Implies y_ℓ already matched with someone better than x_i

And y_ℓ can only change to even **better** partners, so y_ℓ's current partner is better than x_i

Contradicts our assumption that this instability exists!

Then y_ℓ should end up matched with x_i. Contradiction!

Other proposal must be to someone **better**. Contradiction!

All three cases are impossible, so assumption is wrong. There **cannot** be an **instability**!

COMPLEXITY

It is obvious that the number of iterations is at most n^2 since every x_i proposes at most once to every y_j.

The average number of iterations is $\Theta(n \log n)$ (but we will not prove this).

But how much **time** does it take **per iteration**?

Depends on how we **implement** the algorithm…

Algorithm: *Gale-Shapley*$(X, Y, pref)$
$Match \leftarrow \emptyset$
while there exists an unmatched x_i
do
  let y_j be the next element in x_i's preference list
  if y_j is not matched
    then $Match \leftarrow Match \cup \{x_i, y_j\}$
    else
      suppose $\{x_k, y_j\} \in Match$
      if y_j prefers x_i to x_k
        then $Match \leftarrow Match \backslash \{x_k, y_j\} \cup \{x_i, y_j\}$
          comment: x_k is now unmatched
return $(Match)$

Maintain a **queue** of unmatched x elements

Simple **list** of preferences

Want to know **who** y_j is matched with

Maintain **arrays** of matches. If x_i and y_j are matched then $\boldsymbol{M_x[i] = j}$ and $\boldsymbol{M_y[j] = i}$ (Initially $M_x[i], M_y[i] = 0$)

Want to **quickly** look up y_j's **rank** for x_i and x_k

Construct an **array** $\boldsymbol{R[j, i]}$ containing the **rank** of $\boldsymbol{x_i}$ in $\boldsymbol{y_j}$'s preference list

I.e., want $\boldsymbol{R[j, i] = k}$ if x_i is y_j's ***k*-th favourite** partner

$O(n^2)$ preprocessing

Allows comparing two ranks in $O(1)$ time!

So, we get $O(1)$ time per iteration, and $O(n^2)$ time in total

Exercise: try writing pseudocode for this implementation

FORMULATING GRAPH PROBLEMS

Graphs are a very important formalism in computer science.

Efficient algorithms are available for many important problems:

- exploration,
- shortest paths,
- minimum spanning trees, etc.

If we formulate a problem as a graph problem, chances are that an efficient non-trivial algorithm for solving the problem is known.

Some problems have a natural graph formulation.

- For others we need to choose a less intuitive graph formulation.
- Some problems that do not seem to be graph problems at all can be formulated as such.

The RootBear Problem:

Suppose we have a canyon with perpendicular walls on either side of a forest.

- We assume a north wall and a south wall.

Viewed from above we see the A&W RootBear attempting to get through the canyon.

- We assume trees are represented by points.
- We assume the bear is a circle of given diameter d.
- We are given a list of coordinates for the trees.

Find an algorithm that determines whether the bear can get through the forest.

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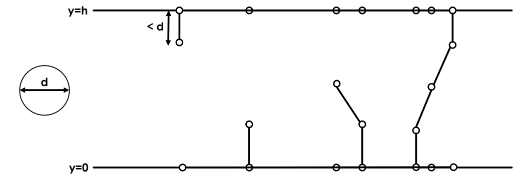
For each input point (x,y): **add vertices** (x,0), (x,h), (x,y) to V

For all pairs of vertices u, v in V: if dist(u,v) < d, **add edge** uv

Also add edges between **all** vertices **on each canyon wall**



Bear **cannot** get through the canyon if North and South walls are **connected**

Test connectivity using BFS from any point on the North wall, and checking if any point on the South wall is visited.

Exercise: what if each tree had radius r**?**

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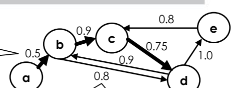
Reliable network routing:

- Suppose we have a computer network with many links.
- Every link has an assigned reliability.
  - The reliability is a probability between 0 and 1 that the link will operate correctly.
- Given nodes u and v, we want to choose a route between nodes u and v with the highest reliability.
  - The reliability of a route is a product of the reliabilities of all its links.

Reliability of path a->b->c->d: 0.5 * 0.9 * 0.75 = **0.3375**



Higher reliability via path **a->b->d**: 0.5 * 0.8 = **0.4**

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Can we turn this into a **shortest path** problem?

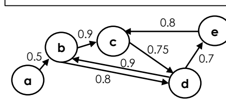
Problem 1: need **product** of weights **not sum**

Use **logs** to turn product of weights into a **sum**.

Recall: $\log xy = \log x + \log y$. So $\log \prod w = \sum \log w$.

$$\log \prod \frac{1}{w} = \log \frac{1}{\prod w} = \log 1 - \log \prod w = -\log \prod w$$

$= -\sum \log w = \sum(-\log w)$. ← Want to minimize this!

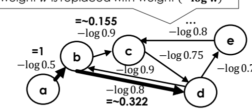


Shortest path **minimizes** a **sum** of weights $\sum w$

Problem 2: want to **maximize** the product

A path P has **maximum** $\prod w$
IFF it has **maximum** $\log \prod w$
IFF it has **minimum** $\log \prod \frac{1}{w}$

Solution: create a new graph where each weight w is replaced with weight $(-\log w)$



if $w \le 1$ then $\log w \le 0$ so $(-\log w) \ge 0$

So we can use Dijkstra!

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BONUS SLIDES

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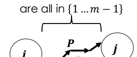
A MORE FORMAL OPTIMALITY ARGUMENT
FOR YOUR NOTES

By induction: **suppose $D_{m-1}[i,j]$ is correct** for all i, j. We show $D_m[i,j]$ is correct.

(Base case $D_0[i,j]$ is left as an exercise)

Case 1: m is **not** used in P

interior nodes are all in $\{1 \dots m-1\}$



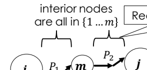
Then $w(P) = D_{m-1}[i,j]$ by I.H., and $D_m[i,j] = D_{m-1}[i,j]$

Case 2: m **is** used in P

interior nodes are all in $\{1 \dots m\}$

Reduce P_1, P_2 to **subproblems**

but what if $m \in P_1, P_2$?



Claim: $\exists$ optimal path $P' = P_1', m, P_2'$ such that P_1' and P_2' have all interior nodes in $\{1 \dots m-1\}$

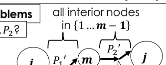
(details in slide notes)

(If m appears twice in P, it creates a cycle which can be removed to get P' with same or better weight)

Consider P'

Let P be a min-weight (i,j)-path in which all interior nodes are in $\{1 \dots m\}$

all interior nodes in $\{1 \dots m-1\}$



By I.H., $w(P_1') = D_{m-1}[i,m]$

and $w(P_2') = D_{m-1}[m,j]$

And $w(P_1') + w(P_2') = D_{m-1}[i,m] + D_{m-1}[m,j] = D_m[i,j]$

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