CS 341: ALGORITHMS

Lecture 16: max flow

Readings: CLRS 26.2

Trevor Brown

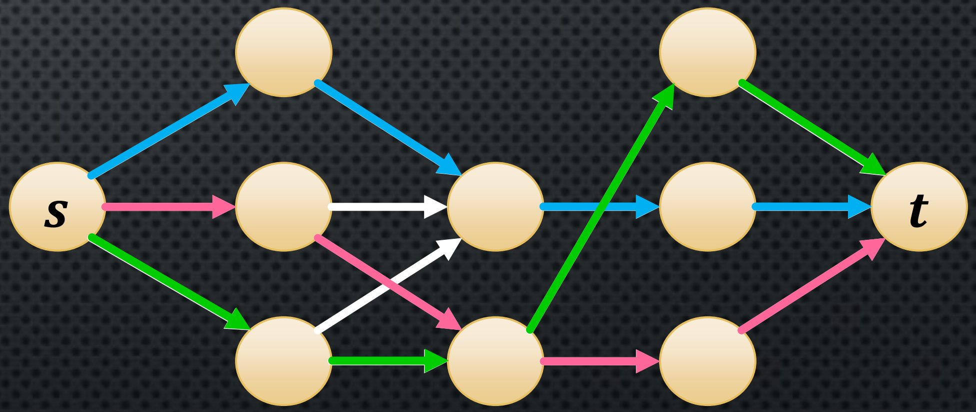
https://student.cs.uwaterloo.ca/~cs341

trevor.brown@uwaterloo.ca

FLOWS AND PATHS

FLOWS AND PATHS

- **Edge-disjoint paths** problem
- Input: digraph $G = (V, E)$ and two vertices $s, t \in V$
- Output: A maximal number of edge-disjoint paths in G
- Paths P_1 and P_2 are **edge-disjoint** if they do not share any edges

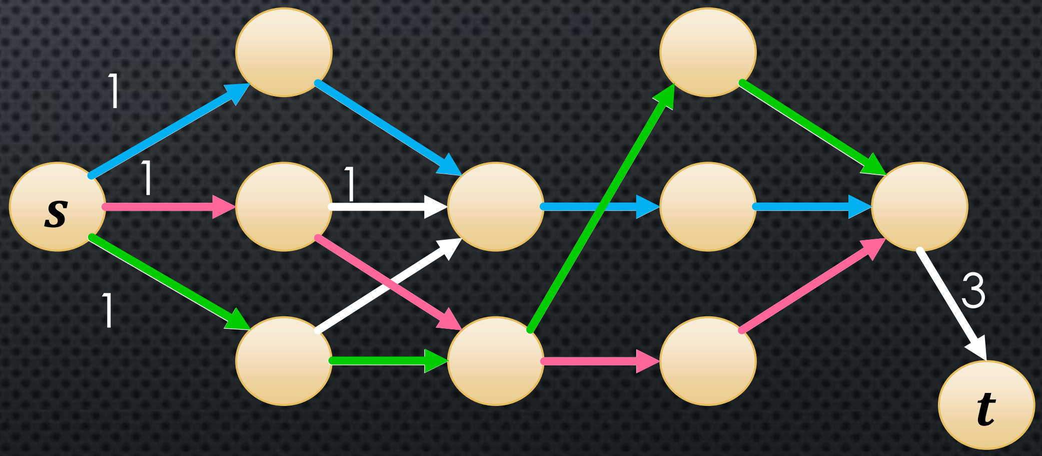


This is a special case of the **maximum flow** problem

… where the union of paths defines a **flow**

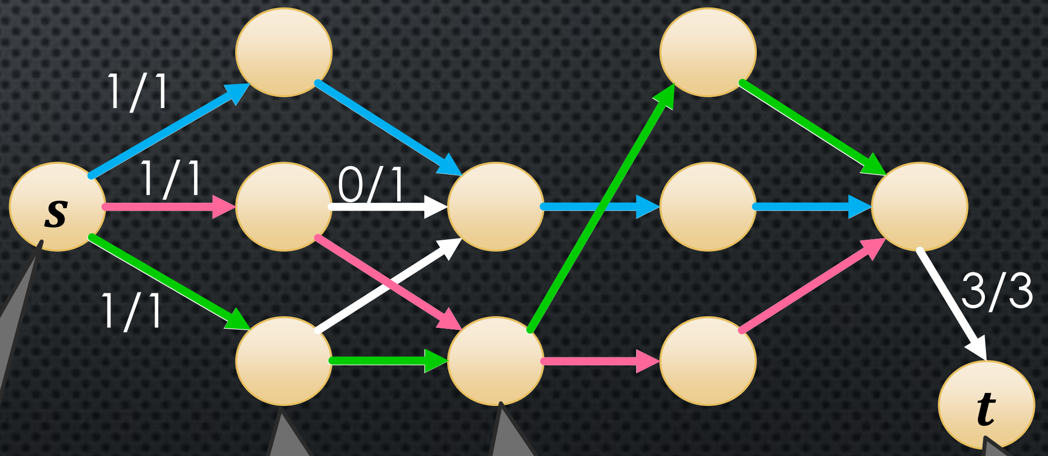
s-t FLOWS

- Let $G = (V, E)$ be a digraph
  where each edge $e \in E$ has a **capacity** $\boldsymbol{c(e) > 0}$



s-t FLOWS

- Let $G = (V, E)$ be a digraph
  where each edge $e \in E$ has a **capacity** $\boldsymbol{c(e) > 0}$
- An s-t flow assigns a number $f(e)$ to each edge satisfying:
  - Capacity constraints
    - $0 \le f(e) \le c(e)$ for each edge
  - Conservation of flow
    - $f^{in}(v) = f^{out}(v)$ for $v \notin \{s,t\}$
  - **Source** $f^{in}(s) = 0$
  - **Sink** $f^{out}(t) = 0$



$f^{out}(s) = 3$

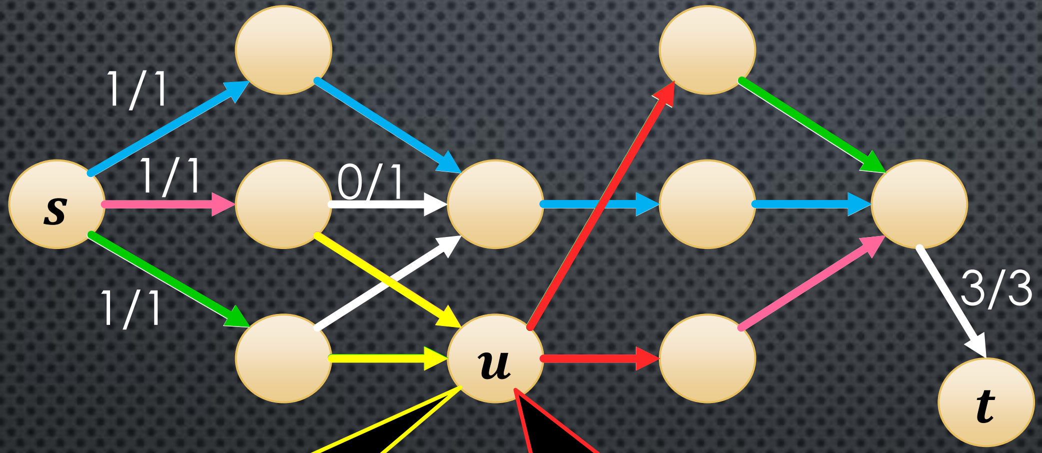
This is the **value** of the flow

$f^{in} = f^{out} = 1$

$f^{in} = f^{out} = 2$

$f^{in}(t) = 3$

DEFINING $f^{in}(u)$ AND $f^{out}(u)$



$$f^{in}(u) = \sum_{\boldsymbol{e} \text{ into } \boldsymbol{u}} f(e)$$

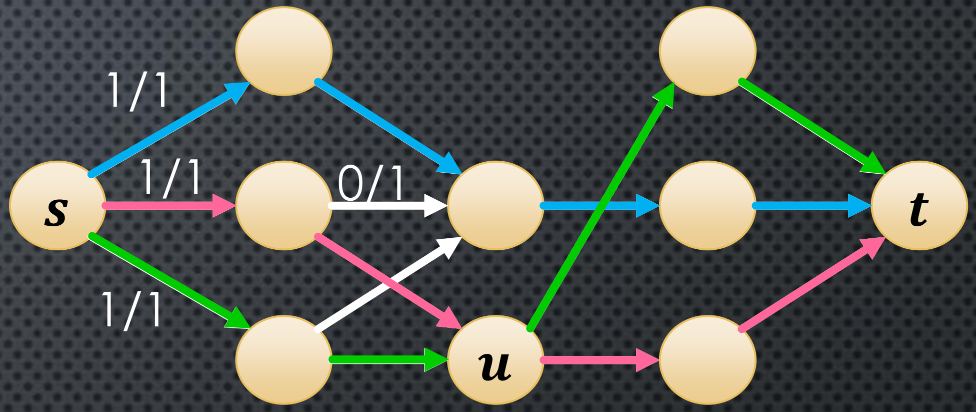
$$f^{out}(u) = \sum_{\boldsymbol{e} \text{ outof } \boldsymbol{u}} f(e)$$

MAX s-t FLOW

- Input: digraph $G = (V, E)$ with capacities $c(e)$ for $e \in E$,
  and two vertices s, t
- Output: a flow from s to t with maximum value
  i.e., that maximizes $f^{out}(s)$
- Motivation
  - Liquid flowing through pipes
    Current through electrical networks
    Internet/telephony traffic routing
  - Also useful for seemingly unrelated problems (next time)

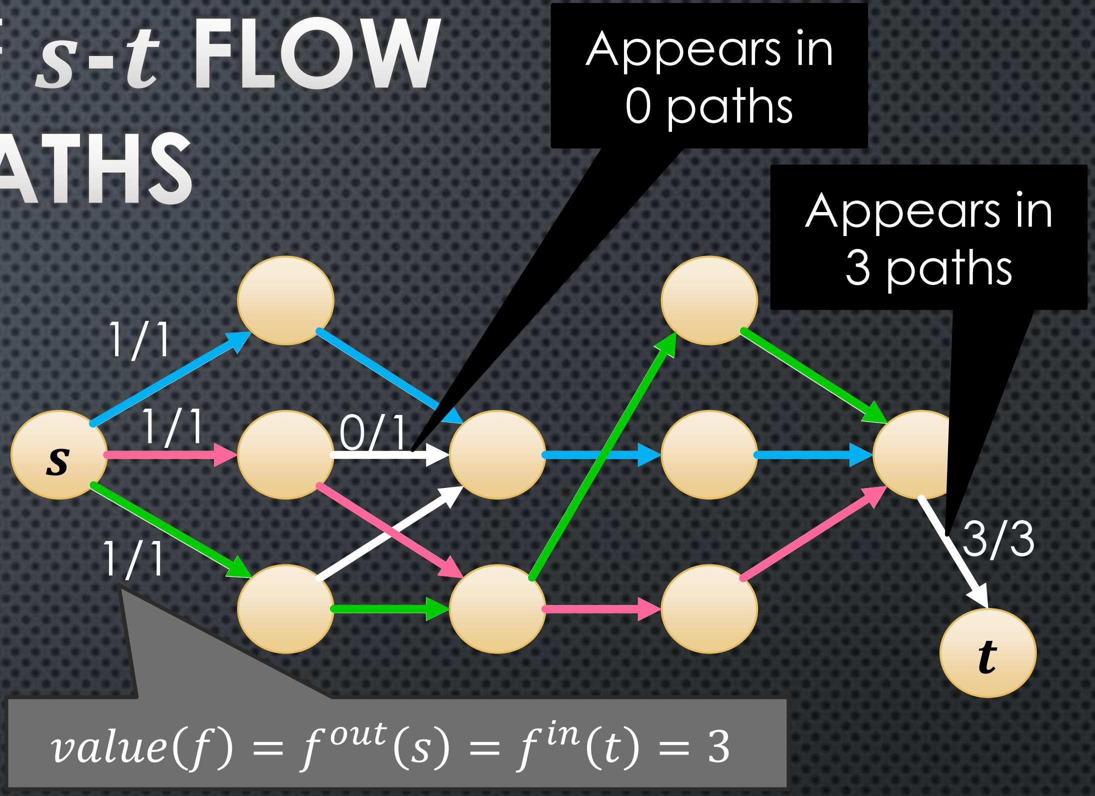
CONNECTION BETWEEN FLOWS AND PATHS

- In this example, max flow is 3
- Note max flow is limited by the sum of capacities **out of s**
    - … and **into t**
- Flows vs paths
    - a flow can always be decomposed into “capacity-disjoint” paths



LEMMA 1: DECOMPOSITION OF s-t **FLOW** INTO CAPACITY-DISJOINT s-t **PATHS**

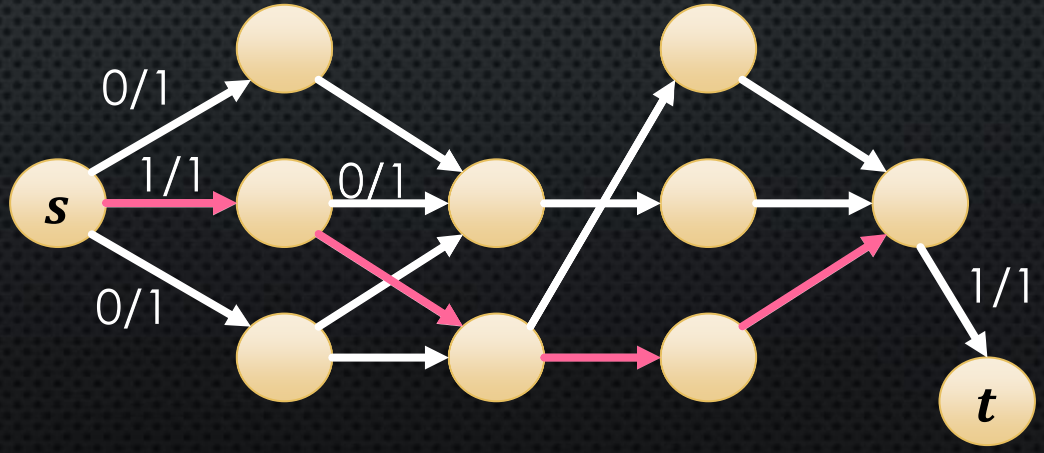
- Let f be an s-t flow where $f(e)$ is an integer for each $e \in E$, $f^{in}(s) = 0$ and $value(f) = k$
- Then there are s-t paths $P_1, P_2, \dots, P_k$ such that each **edge** $\boldsymbol{e}$ appears in $\boldsymbol{f(e)}$ of these **paths**
- Proof sketch by induction
  - Base case: when k=1 there is only one path



Appears in 0 paths

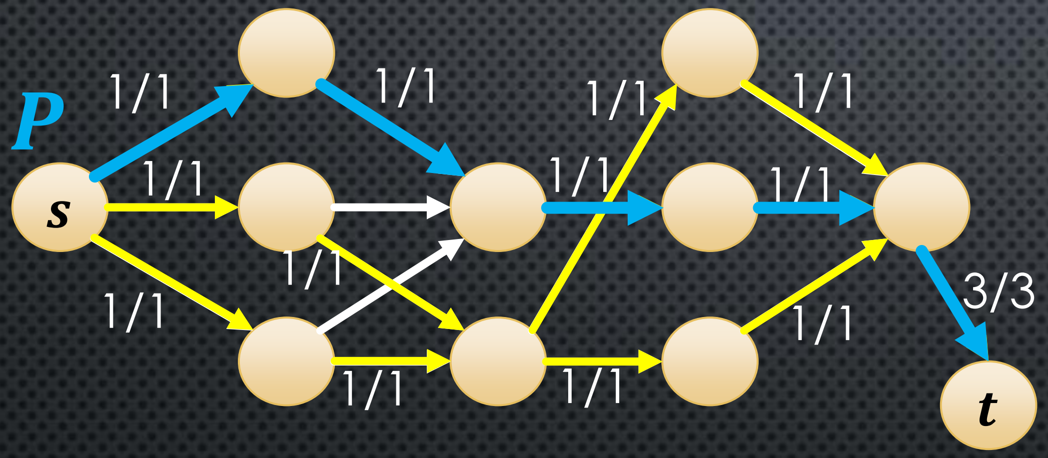
Appears in 3 paths

$value(f) = f^{out}(s) = f^{in}(t) = 3$



INDUCTIVE STEP

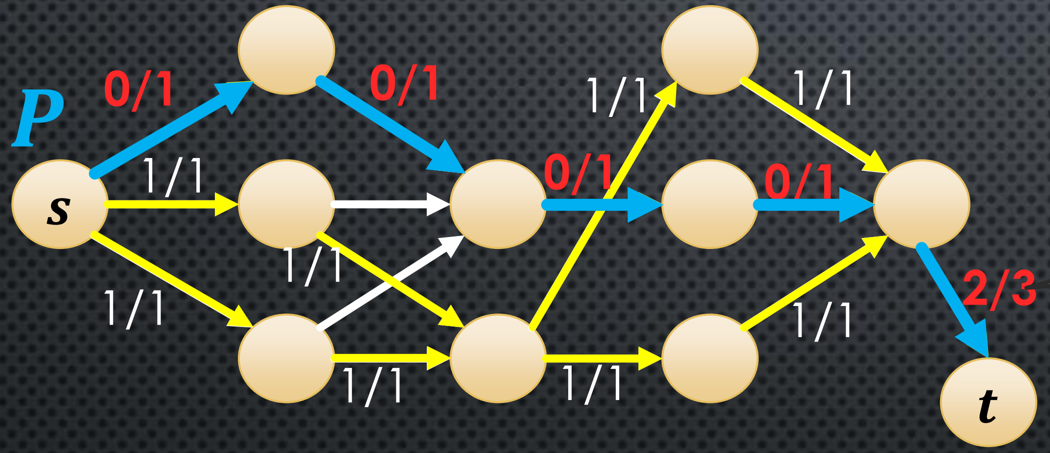
- Suppose lemma holds for $k-1$, show it holds for k (where $k \geq 2$)
- Consider the edges with **non-zero flow**



- There must exist some **s-t path P** in these edges (why?)
- **Decrease** the flow of each edge in P by 1

INDUCTIVE STEP

- Suppose lemma holds for $k - 1$, show it holds for k (where $k \geq 2$)
- Consider the edges with **non-zero flow**



Removing path P with flow 1 changes flow value from k to $k - 1$

Every vertex still satisfies conservation of flow

So this is an **s-t flow** with value $k - 1$

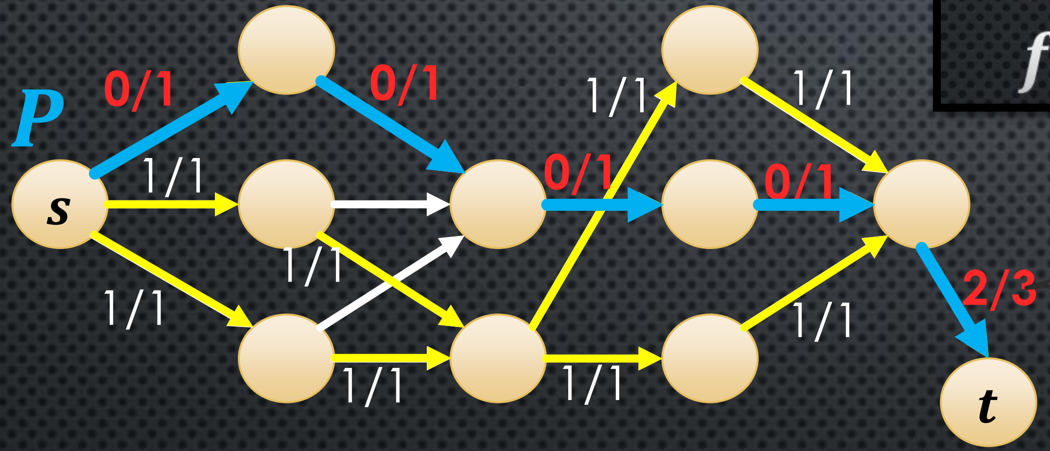
So the inductive hypothesis applies…

- There must exist some **s-t path P** in these edges (
- **Decrease** the flow of each edge in P by 1

INDUCTIVE STEP

Lemma

- Let f be an s-t flow where $f(e)$ is an integer for each $e \in E$, $f^{in}(s) = 0$ and $value(f) = k$
- Then there are s-t paths $P_1, P_2, \ldots, P_k$ such that each **edge e** appears in $f(e)$ of these **paths**



Removing path P with flow 1 changes flow value from k to $k-1$

Every vertex still satisfies conservation of flow

So this is an **s-t flow** with value $k-1$

So the inductive hypothesis applies…

So, there are **s-t paths $P_1, P_2, \ldots, P_{k-1}$** such that each edge e appears in $f(e)$ of these paths

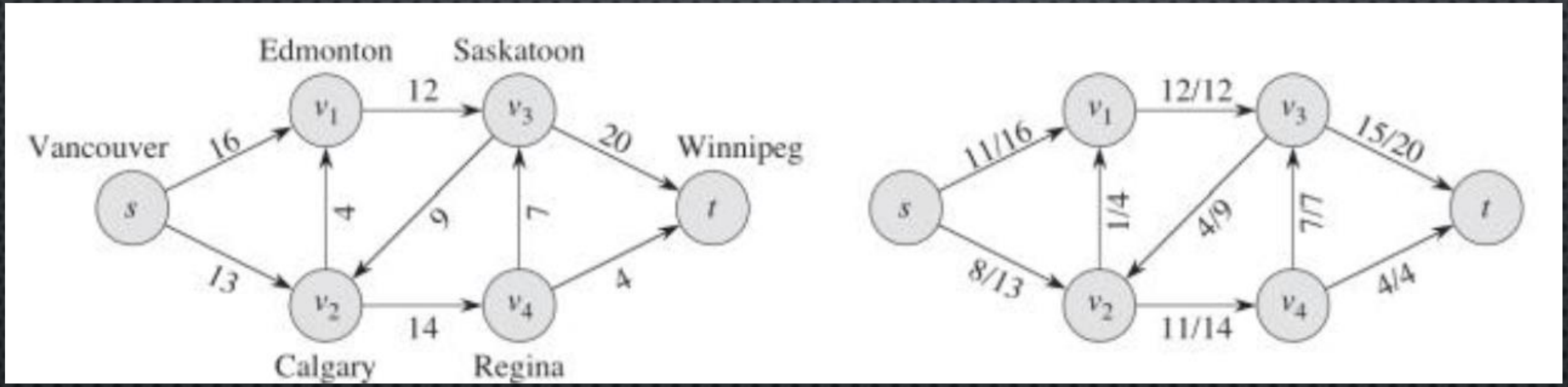
And by adding P we can obtain k such paths

EXAMPLE APPLICATION OF LEMMA 1

- Given a flow of value k where $f(e) \in \{0,1\}$ for all $e \in E$
- The lemma says the flow f can be decomposed into k edge-disjoint paths
- So if our goal is to find k edge-disjoint paths we can just focus in finding such a flow instead
  - (so we don't need to worry about which edges belong to which paths during the algorithm)
- Can extract paths from such a flow by repeatedly doing: BFS on the non-zero flow edges, identifying an s-t path, and decrementing the flows along that path

HOME EXERCISE

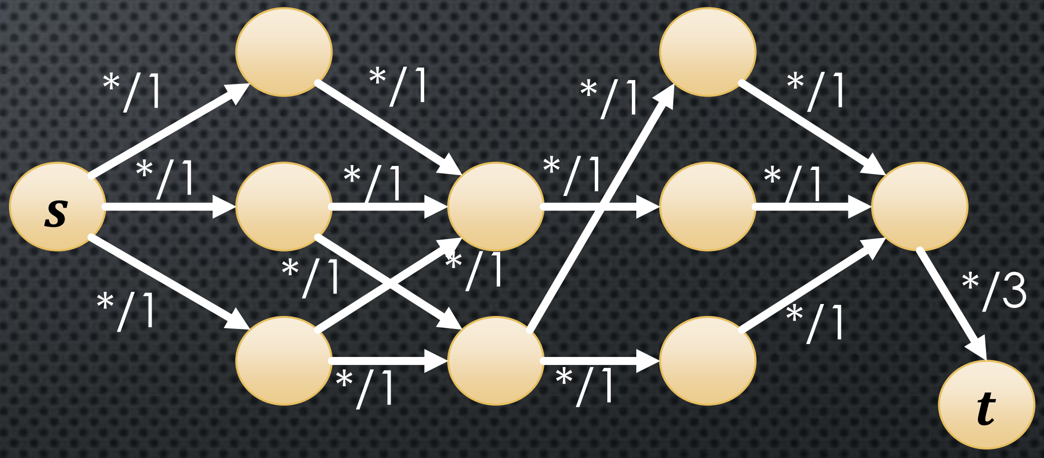
- Find a decomposition of the following flow into capacity-disjoint paths



FLOWS AND CUTS

UPPER BOUNDING THE MAX FLOW FOR G

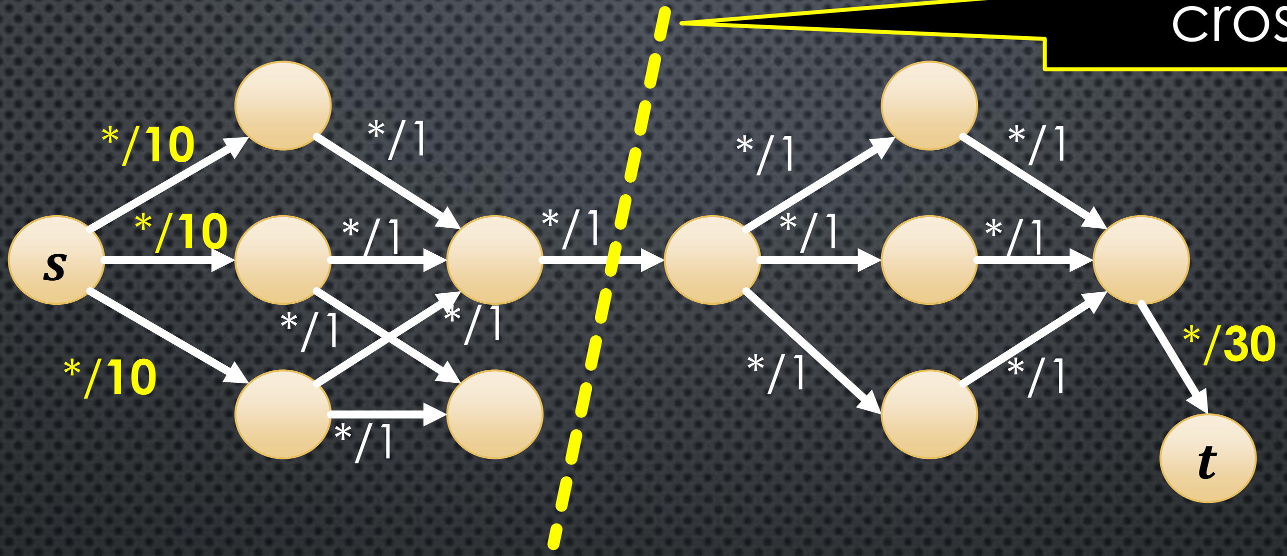
- What is a good upper bound on the value of a flow?
  - And how do we know a flow is maximal?
- Trivial upper bound
  - Sum of capacities of all edges
- Slightly better
  - $\min\left(c^{out}(s), c^{in}(t)\right)$
    where $c^{out}(s) = \sum_{e \text{ outof } s} c(e)$ and $c^{in}(t) = \sum_{e \text{ into } t} c(e)$
  - Tightly bounds max flow in this case…



$\sum c(e) = 17$

But max flow is 3…

BUT WHAT ABOUT THIS CASE?



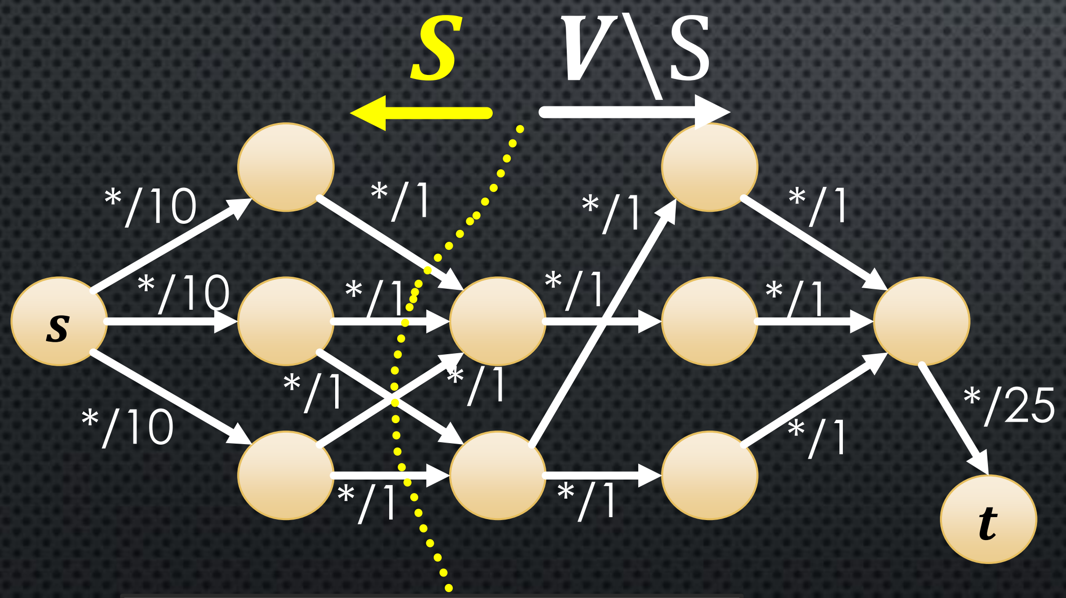
Looks like an edge crossing a cut…

Estimate using $\min\left(c^{out}(s), c^{in}(t)\right) = 30$

But real answer is 1…

DEFINITIONS: AN *s-t* **CUT** AND ITS **CAPACITY**

- An s-t cut is a partition $(S, V\backslash S)$ where $s \in S$ and $t \in V\backslash S$
  - i.e., the partition separates s and t



(Recall S does not need to be connected)

Let $\delta^{out}(S)$ be the set of edges **directed out from S**

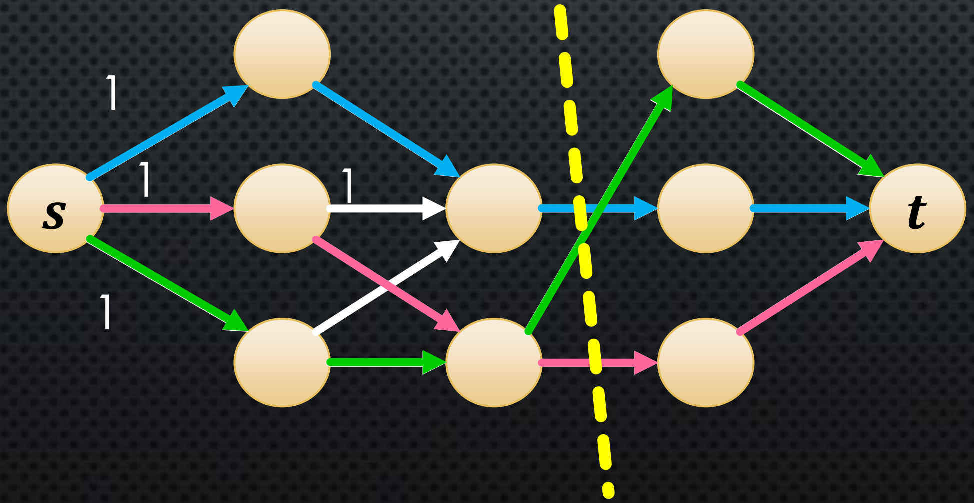
$$\delta^{out}(S) = \{(u, v) \in E : u \in S, v \in V\backslash S\}$$

The **capacity** of the cut is the sum of the capacities of these edges

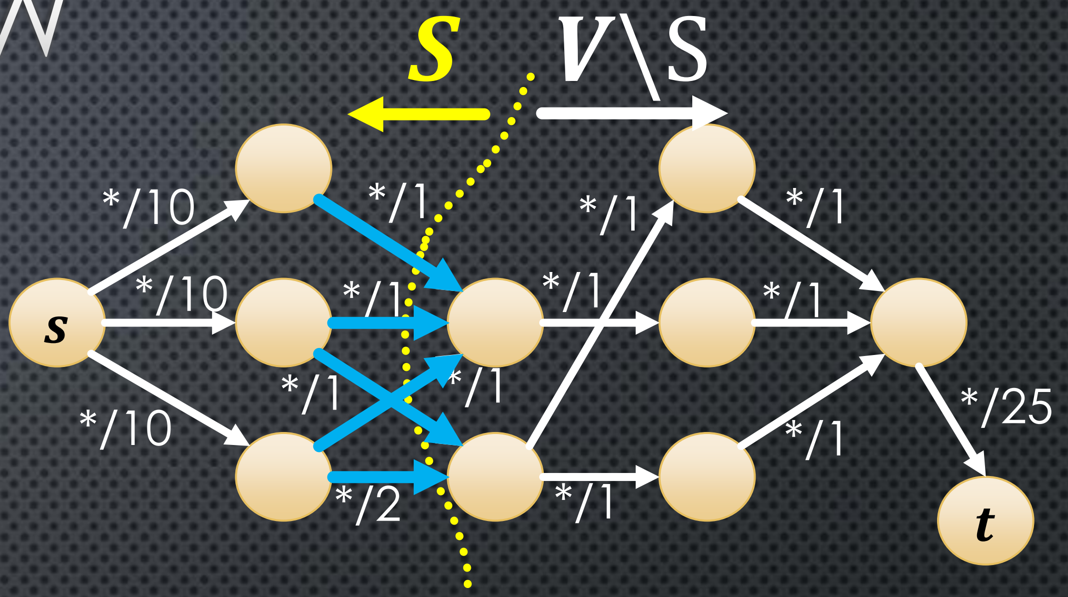
$$c^{out}(S) = \sum_{e \in \delta^{out}(S)} c(e)$$

UPPER BOUNDING EDGE-DISJOINT PATHS BY S-T CUT

- For the edge-disjoint paths problem, where $c(e) = 1$ for all e, cut capacity is just the **number of edges crossing the cut**
- If an s-t cut S has at most k edges crossing the cut, then are at most k edge-disjoint s-t paths, since each s-t path has an edge crossing the cut



GENERALIZING TO MAX FLOW



Lemma 2: if an s-t cut S has capacity k, the value of every flow must be $\leq k$

- Proof sketch: for contra assume a flow with value $k' > k$
- By earlier lemma, a flow with value k' can be decomposed into k' capacity-disjoint paths each w/flow 1
- Each such path crosses the cut, and consumes one unit of the cut's capacity (up to k' in total)
- But the cut's capacity is only k, so the paths are not capacity-disjoint! Contradiction.

COROLLARY: MAX FLOW ≤ MIN s-t CUT

- Recall lemma 2: if an s-t cut S has capacity k,
  the value of every flow must be $\leq k$
- This holds for **any** s-t cut
- Including the s-t cut S with the minimum capacity
- So, **max s-t flow ≤ min capacity over all possible s-t cuts**

- In fact, it turns out max flow is **exactly** the min cut capacity
  - So we can solve max flow by finding a min cut…

MIN s-t CUT PROBLEM

- Input: digraph $G = (V, E)$ with capacities $c(e) > 0$ for $e \in E$, and two vertices s, t
- Output: an s-t cut S with minimal capacity $c^{out}(S)$
- This is a natural and useful problem on its own, and we will see some other interesting applications soon…

MAX-FLOW MIN-CUT THEOREM

- **Theorem 3:** every max s-t flow has value equal to the capacity of a min s-t cut
- One of the most beautiful and important results in combinatorial optimization and graph theory
- Diverse applications in CS and math
- We give an **algorithmic proof** of this theorem
  - (showing that one algorithm solves both max-flow and min-cut at the same time)

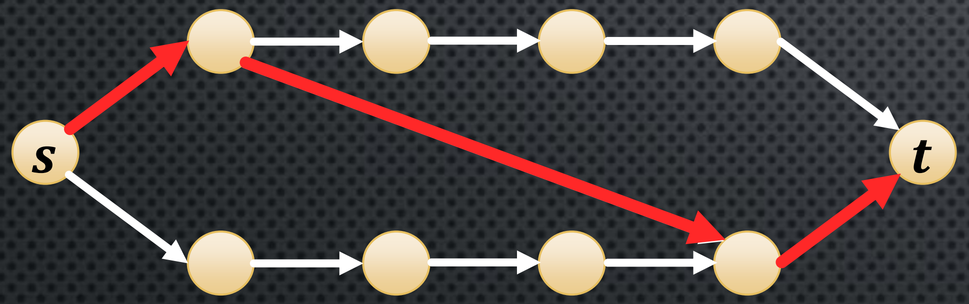
FORD-FULKERSON METHOD

Algorithm development

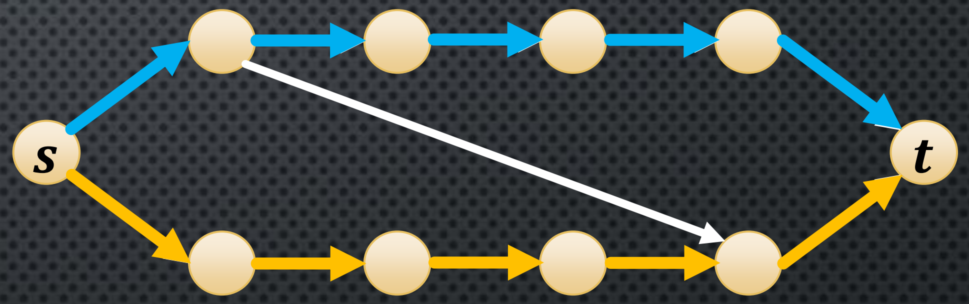
(mixed together with proof of max-flow min-cut theorem)

NAÏVE ALGORITHM ATTEMPT

- For simplicity, try edge-disjoint path problem first (unit capacities)
- Greedy idea: find a shortest s-t path (to use few edges), then repeat on the remaining edges



greedy solution



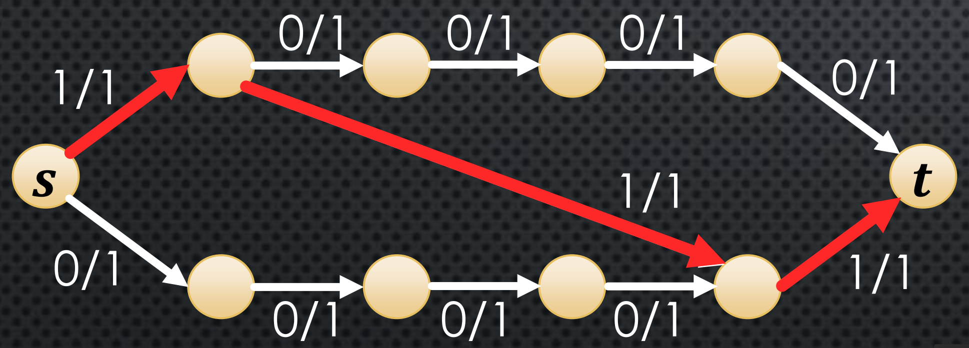
optimal solution

- Difficult for greedy is to **decide** on a path **permanently**
- Unclear how to find a path that **belongs** in the optimal solution

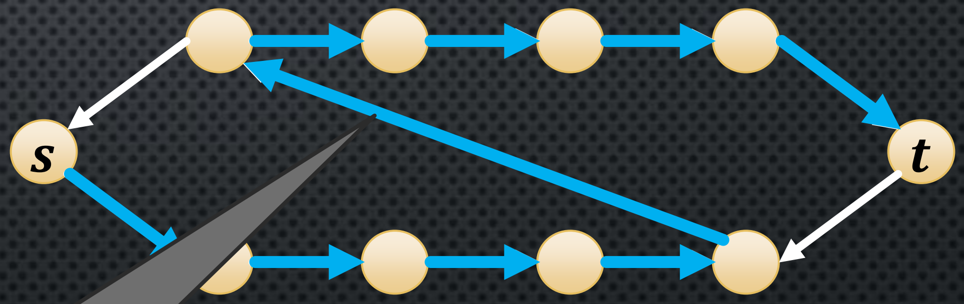
FORD-FULKERSON METHOD

Same Ford as in Bellman-Ford :)

- Ford-Fulkerson is a more general "local search" algorithm which can **undo** previous decisions to improve the flow
- Greedy flow can be improved by "pushing back" some flow using an **augmenting path** through a **residual graph**



greedy flow



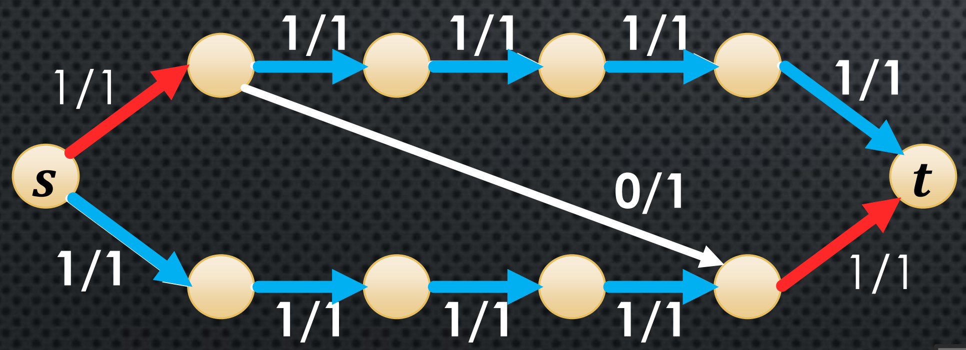
Pushes back the flow on this edge (negating its flow)

"augmenting path"

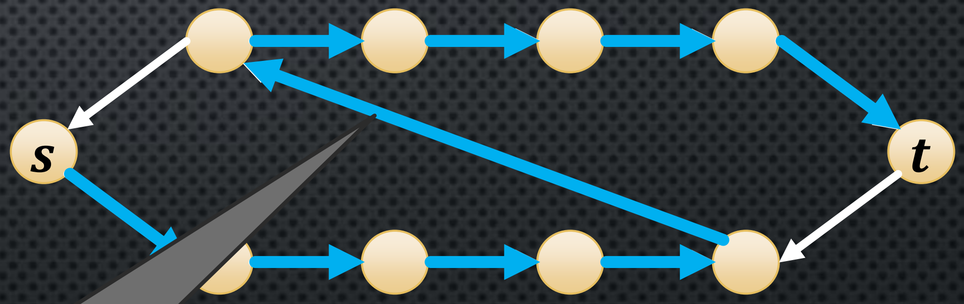
FORD-FULKERSON METHOD

Same Ford as in Bellman-Ford :)

- Ford-Fulkerson is a more general "local search" algorithm which can **undo** previous decisions to improve the flow
- Greedy flow can be improved by "pushing back" some flow using an **augmenting path** through a **residual graph**



improved flow



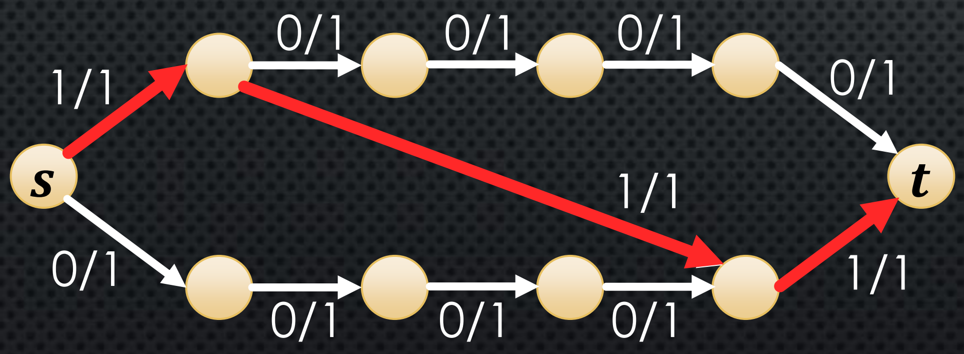
Pushes back the flow on this edge (negating its flow)

"augmenting path"

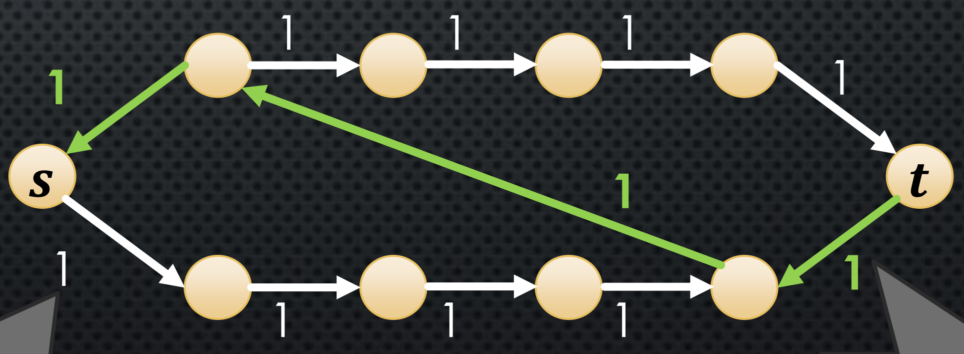
So, what's the residual graph, how do we find an augmenting path, and how do we improve the flow?

RESIDUAL GRAPH

- A **residual graph** $\boldsymbol{R_f}$ is defined for a **given flow** $\boldsymbol{f}$ and **graph** $\boldsymbol{G}$
- R_f has the same vertices as G
- For each edge $e = uv$ in G,
  - If $f(\mathrm{e}) < c(e)$, then R_f contains a **forward** edge (u, v) with the **remaining capacity** $\boldsymbol{c(e) - f(e)}$
  - If $f(e) > 0$, then R_f contains a **backwards** edge $\boldsymbol{(v, u)}$ with **capacity** $\boldsymbol{f(e)}$ representing flow that could be "pushed back"



greedy flow $\boldsymbol{f}$



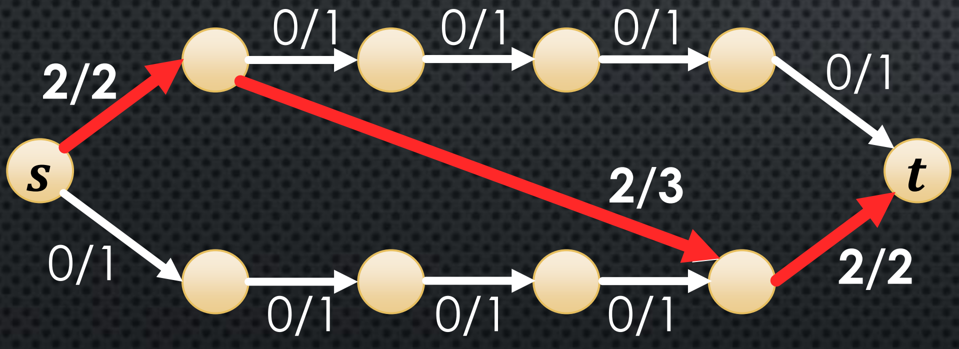
Forward edge:
remaining capacity

residual graph for this flow

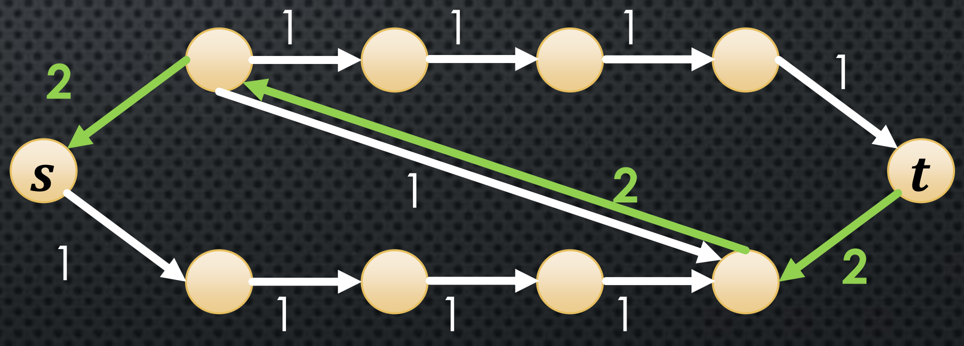
Backwards edge:
can **undo** flow

ANOTHER EXAMPLE RESIDUAL GRAPH

- Recall: for each edge $e = uv$ in G,
  - If $f(e) < c(e)$, then R_f contains a **forward** edge (u, v) with the **remaining capacity** $\boldsymbol{c(e) - f(e)}$
  - If $f(e) > 0$, then R_f contains a **backwards** edge (v, u) with **capacity** $\boldsymbol{f(e)}$ representing flow that could be "pushed back"



flow f with value 2



residual graph for this flow